Oxford MAT Livestream 2026

Compiled by `www.CasperYC.club`

Contents

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TMUA Specification (April 2025, Section 1 §MM1)

- Laws of indices for all rational exponents.
- Use and manipulation of surds. Simplifying expressions that contain surds, including rationalising the denominator.
- Quadratic functions and their graphs; the discriminant of a quadratic function; completing the square; solution of quadratic equations.
- Simultaneous equations: analytical solution by substitution, e.g. of one linear and one quadratic equation.
- Solution of linear and quadratic inequalities
- Algebraic manipulation of polynomials, including:
  - expanding brackets and collecting like terms
  - factorisation and simple algebraic division (by a linear polynomial, including those of the form $ax + b$, and by quadratics, including those of the form $ax^2 + bx + c$)
  - use of the Factor Theorem and the Remainder Theorem
- Qualitative understanding that a function is a many-to-one (or sometimes just a one-to-one) mapping.
- Familiarity with the properties of common functions, including $f(x) = \sqrt{x}$ (which always means the "positive square root") and $f(x) = |x|$.

Revision

- $a^m a^n = a^{m+n}$ for any positive real number a and any rational numbers m and n.
- $(a^m)^n = a^{mn}$ for any positive real number a and any rational numbers m and n.
- $a^{-n} = \dfrac{1}{a^n}$ for any positive real number a and any real number n.
- $a^0 = 1$ for any non-zero real number a.
- (Rationalising the denominator) It's sometimes helpful to simplify an expression like $\dfrac{a}{\sqrt{b}}$ to $\dfrac{a\sqrt{b}}{b}$, or an expression like $\dfrac{a}{b+\sqrt{c}}$ like this

$$\frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})} = \frac{a(b-\sqrt{c})}{b^2-c}.$$

- (Completing the square) If $a \neq 0$ then we can write ax^2+bx+c in the form $a(x+r)^2+p$ because
$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a}\right).$$
This is helpful if we're trying to prove that the quadratic is non-negative for all values of x, because anything squared is non-negative.

- Once we've written the quadratic in the form $a(x+r)^2+p$, we know that there is a vertical line of symmetry at $x=-r$, because the values of the quadratic at $-r \pm D$ will both be aD^2+p, for any distance D away from the line of symmetry.

- The discriminant of the quadratic $ax^2+bx+c=0$ is equal to b^2-4ac. If the discriminant is positive then the quadratic has two real solutions. If the discriminant is zero then there's one (repeated) real solution. If the discriminant is negative then there are no real solutions.

- (The quadratic formula) If $b^2-4ac \geqslant 0$, then the solution(s) of $ax^2+bx+c=0$ are given by the quadratic formula
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

- If $b^2-4ac \geqslant 0$, then ax^2+bx+c can be written as $a(x-\alpha)(x-\beta)$, where α and β are real numbers called the "roots", with values given by the quadratic formula above.

- (Difference of two squares) The expression x^2-a^2 factorises as $(x-a)(x+a)$.

- Given two linear equations for x and y like $2x+3y=7$ and $3x+5y=9$, we can solve by rearranging the first for x and substituting that into the second equation, then rearranging the resulting equation for y, before back-substituting for x.

- We can add and subtract from each side of an inequality. For example, if $x<2$ then $x+3<2+3$, and if $y>-3$ then $y-5>-3-5$.

- We can multiply each side of an inequality by a fixed number, but if we multiply by a negative number then the direction of the inequality changes. For example, $-6<-3$, but if we multiply both sides by $-\frac{1}{3}$ then we need to flip the sign to get the true statement $2>1$.

- Squaring each side of an inequality is a bit like multiplying, and we need to be careful. If both sides of an inequality are positive, then squaring each side preserves the inequality, because x^2 is an increasing function when $x>0$.

- If we have a quadratic inequality like $(x+3)(x-5)<0$, then we can consider the signs of the terms. In this case the inequality is satisfied for $-3<x<5$.

- A polynomial is an expression of the form $a_0+a_1x+a_2x^2+\cdots+a_kx^k$ for some whole number $k \geqslant 0$, and with real numbers $a_0, a_1, \ldots, a_k$, known as the coefficients.

- The degree of a polynomial is the highest exponent of x, so the degree of any quadratic is 2, the degree of $x^{17}-x^{12}$ is 17, and the degree of $100x+\pi x^3-x$ is 3, for example. The degree of a constant polynomial ($y=a_0$) is zero.

- (Polynomial division) Given polynomials f and g where the degree of f is at least as large as the degree of g, we can divide f by g. By this I mean that we can find polynomials q (called the quotient) and r (called the remainder) such that $f(x) = q(x)g(x) + r(x)$, and such that the degree of r is strictly less than the degree of g.

  If g has degree 1 then this remainder r will just be a number, but in general it's a polynomial.

  Practically, we might find q and r by comparing coefficients on each side of the identity. I like to start with the highest exponent and work downwards.

  Example: divide $3x^3 - 2x^2 + x + 10$ by $x^2 + 4x + 3$.

  I'm looking for q and r such that $3x^3 - 2x^2 + x + 10 = q(x)\left(x^2 + 4x + 3\right) + r(x)$, with the degree of r less than 2 (so just a linear polynomial). Looking at the highest exponent on the left, I can see that the degree of q must be 1. So perhaps I'll write $q(x) = Ax + B$ and $r(x) = Cx + D$ and solve for those coefficients. Rather than writing out everything in terms of A, B, C, and D all at once, I'm going to start with the highest exponent and work my way down.

  The coefficient of x^3 on the right will be A, and I want this to be 3 to match the coefficient of x^3 on the left. Let's write that down; I have something like

  $$3x^3 - 2x^2 + x + 10 = (3x + B)\left(x^2 + 4x + 3\right) + Cx + D$$

  and it's time to think about the coefficient of x^2. On the left that's -2 and on the right it's currently 12 from the 3×4 that I've already got... plus $B \times 1$. I should take $B = -14$. I've run out of terms in the quotient q, so it's time to use the remainder r to mop up any leftover terms.

  The coefficient of x on the left is 1, but on the right so far I have $3 \times 3 - 14 \times 4 = -47$. So I should add $48x$ via a term in the remainder. Similarly, the constant coefficient is currently -42 on the right-hand side, but I want it to be 10 to match the left-hand side, so I'll add 52.

  $$3x^3 - 2x^2 + x + 10 = (3x - 14)\left(x^2 + 4x + 3\right) + 48x + 52.$$

  With practice, it's possible to do this without ever writing the letters A, B, C, or D.

- (Factor Theorem) If $p(a) = 0$ for a polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$. This means that $p(x) = (x - a)q(x)$ for some polynomial $q(x)$. Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

- If we know that $p(a) = 0$ and we want to actually find $q(x)$ such that $p(x) = (x - a)q(x)$, then we could use polynomial division, as demonstrated above.

- (Remainder Theorem) If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$. This means that $p(x) = (x - a)q(x) + p(a)$ for some polynomial $q(x)$.

- When sketching the graph of $y = ax^2 + bx + c$, we should consider whether a is positive or negative, whether the quadratic has any roots (and if so, where), and where it crosses the y-axis.

- Sometimes a function which is not a quadratic might secretly be a quadratic in a different variable. For example, $y = e^{2x} + e^{x+3} - 1$ is not a quadratic, but if we write $u = e^x$ then we have $y = u^2 + e^3 u - 1$, which is a quadratic. This is sometimes called "changing variable" or "finding the hidden quadratic".

- A "one-to-one" function has the property that no two distinct inputs give the same output. At university (but not on TMUA), we might use the adjective "injective" to describe such a function.

  Example: $f(x) = x^3$ is a one-to-one function because, for any two inputs x_1 and x_2, if $(x_1)^3 = (x_2)^3$ then $x_1 = x_2$.

  Example: $f(x) = x^2 + 1$ is not a one-to-one function because $(-3)^2 = (3)^2$. Note that I only have to give one counter-example.

- Sometimes people say "many-to-one" to describe a function that (might) have multiple inputs that give the same output.

- Functions cannot be "one-to-many". For each input, there must be exactly one output.

- The function $f(x) = \sqrt{x} = x^{1/2}$ is defined for $x \geqslant 0$ as the non-negative number that squares to x. It's called the square root of x.

- The function $|x|$ just means x if $x \geqslant 0$, and it means $-x$ if $x < 0$. It's called the absolute value of x. It's the distance between x and 0 on the number line.

Revision Questions

1. Given the polynomial $p(x) = 2x^3 - 5x^2 + 7x - 3$, evaluate $p(2)$.
2. Find a positive number x which satisfies $x^2 = x + 1$.
3. What is the discriminant of the quadratic $2x^2 + 5x + 1$?
4. For which values of k does $x^2 - x + k = 0$ have exactly two real solutions?
5. For which values of k does $x^4 - x^2 + k = 0$ have exactly two real solutions?
6. How many real solutions does $x^2 + bx + 1 = 0$ have? Find different cases in terms of b.
7. Write $x^2 + 4x + 3$ in the form $(x + a)^2 + b$.
8. Find the maximum or minimum value of the function $f(x) = -2x^2 + 8x + 5$ by completing the square. Is this value a maximum or a minimum?
9. Let $p(x) = x^3 - 13x^2 - 65x - 51$. Check that $p(17) = 0$. Factorise $p(x)$.
10. Divide $2x^3 + 5x^2 - 5x - 19$ by $x^2 + 4x + 3$.
11. Divide $p(x) = x^3 - 7x^2 + 15x + 1$ by $x - 3$. Check that the remainder matches the value of $p(3)$.
12. If $(x - 2)$ is a factor of $p(x)$, what is the value of $p(2)$?
13. Show that $(x - 2)$ is a factor of $f(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$, and factorise $f(x)$.
14. Find all the roots of the polynomial $p(x) = x^3 - 6x^2 + 11x - 6$.
15. Determine whether $(x - 3)$ is a factor of $p(x) = 2x^4 - 11x^3 + 19x^2 - 20x + 15$.
16. In each of the following cases, choose a variable u in terms of x to make the function into a quadratic in u. There might be more than one sensible choice of u in each case.
    - $y = 2x^6 + x^3 + 1$,
    - $y = x + \sqrt{2x}$,
    - $y = 3e^{-3x} + 6e^{-6x}$,
    - $y = \dfrac{1 + x}{(1 - x)^2}$.
17. Given that the polynomial $q(x)$ has roots 2, -3, and 1, what could $q(x)$ be? Write down at least two possible polynomials, with different degrees.
18. Given that the polynomial $v(x) = x^3 + 2x^2 + ax + b$ has a double root at $x = 1$, find the values of a and b.
19. Solve the simultaneous equations $x + 4y = 1$ and $2x - y = 3$.
20. Solve the simultaneous equations $x^2 + 2x + xy + y^2 = 5$ and $x + y = 2$.

21. Solve the simultaneous equations $x^2 + y = 2$ and $x + y^2 = 2$.
22. For which values of x is it true that $x^2 + 4x + 3 > 0$?
23. Given $-2 < a < 1$, what can you say about a^2?
24. Given $a < b$ and $c < d$, what (if anything) can you say about the relationship between ac and bd? What (if anything) can you say if you're also told that $a > 0$ and $c > 0$?

TMUA Questions

TMUA 2020 Paper 1 Question 3

Find the complete set of values of x for which

$$(x+4)(x+3)(1-x) > 0 \quad \text{and} \quad (x+2)(x-2) < 0$$

A $1 < x < 2$

B $-2 < x < 1$

C $-2 < x < 2$

D $x < -2$ or $x > 1$

E $x < -4$ or $x > 2$

F $x < -4$ or $-3 < x < 1$

G $-4 < x < -2$ or $x > 1$

[See the next page for hints]

TMUA 2020 Paper 1 Question 9

The quadratic expression $x^2 - 14x + 9$ factorises as $(x-\alpha)(x-\beta)$, where α and β are positive real numbers.
Which quadratic expression can be factorised as $(x-\sqrt{\alpha})(x-\sqrt{\beta})$?

A $x^2 - \sqrt{10}\,x + 3$

B $x^2 - \sqrt{14}\,x + 3$

C $x^2 - \sqrt{20}\,x + 3$

D $x^2 - 178x + 81$

E $x^2 - 176x + 81$

F $x^2 + 196x + 81$

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 3

- Start by thinking about the inequalities separately.
- Sketch the polynomial on the left-hand side of each inequality. What does the cubic look like? There are no repeated roots, so at each root, the sign of the polynomial (> 0 or < 0) changes.
- Once you've understood the values of x for which the two inequalities separately hold, it's time to think about the word "and" in the middle of the question. I drew a little table to keep track of the different overlapping ranges of x. I could imagine drawing and labelling a number line.

TMUA 2020 Paper 1 Question 9

- You don't actually need to solve for α and β to do this question (but you can if you like). I started to do it that way, but got bored at roughly the point where I needed to work out a square root of a complicated expression.
- Instead, you could multiply out these factorised expressions; if we can find the coefficients of this new quadratic expression then we're done.
- You might not immediately know the coefficients of this quadratic expression, but I think that with some work, you do know the squares of these coefficients.

TMUA 2020 Paper 1 Question 20
For how many values of a is the equation

$$(x-a)(x^2-x+a)=0$$

satisfied by exactly two distinct values of x?

A 0

B 1

C 2

D 3

E 4

F more than 4

[See the next page for hints]

TMUA 2021 Paper 1 Question 8
The line $y=2x+3$ meets the curve $y=x^2+bx+c$ at exactly one point.
The line $y=4x-2$ also meets the curve $y=x^2+bx+c$ at exactly one point.
What is the value of $b-c$?

A -9

B -5.5

C -1

D 5

E 6

F 14

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 20

- The question doesn't use these words, but it's about repeated roots.
- The phrase "distinct values of x" in this question does not mean "distinct expressions in terms of a for x". So if your roots turn out to be $a + 6$ and a^2 then those might be distinct values, but they might not be (because a might be 3).
- It might help to change the letter a to y and draw the graph for the points in the (x, y)-plane for which the equation holds.
- Don't ignore the $(x - a)$ term.

TMUA 2021 Paper 1 Question 8

- The question asks for $b - c$. At the start of the question, I can't guess whether I'll solve for b and c separately and then calculate b and c, or if I'll find $b - c$ directly along the way, without needing to find b and c individually. I'll keep an eye out for $b - c$ along the way.
- The phrase "exactly one" reminds me of repeated roots... but what's the equation?
- There are two lines so we will have two equations. This is good news, because there are two unknowns, b and c. Well, actually, there are lots more things that I don't know (like the positions of the points where the lines meet the curve), but I can see that if I know b and c then I'm done.

TMUA 2021 Paper 2 Question 16

p and q are real numbers, and the equation

$$x\,|x| = px + q$$

has exactly k distinct real solutions for x.
Which one of the following is the complete list of possible values for k?

A $0, 1, 2$

B $0, 1, 2, 3$

C $0, 1, 2, 3, 4$

D $0, 2, 4$

E $1, 2, 3$

F $1, 2, 3, 4$

[See the next page for hints]

TMUA 2022 Paper 1 Question 13

Given that

$$\left(a^3 + \frac{2}{b^3}\right)\left(\frac{2}{a^3} - b^3\right) = \sqrt{2}$$

where a and b are real numbers, what is the least value of ab?

A $-\sqrt{2}$

B $\sqrt{2}$

C $-2\sqrt{2}$

D $2\sqrt{2}$

E $-\dfrac{\sqrt{2}}{2}$

F $\dfrac{\sqrt{2}}{2}$

G $-2^{\frac{1}{6}}$

H $2^{\frac{1}{6}}$

Hints

TMUA 2021 Paper 2 Question 16

- Sketch the function $y = x\,|x|$.
- The function $y = px + q$ represents a straight line, and since we don't know the values of p and q it could be any straight line (technical note; not a vertical line because we're told that p is real).
- So we're supposed to think of ways (if possible) that a line could cross the function $y = x\,|x|$ in exactly 0 or 1 or 2 or 3 or 4 places. Some of those cases are impossible, unless the answer is C.
- We could consider easy cases like "$p = 0$ and $q = 1$" or "$p = 1$ and $q = 0$" to see what happens.
- If we can imagine a case with particular behaviour, then that's probably good enough (I might not find precise values of p and q that produce that behaviour).

TMUA 2022 Paper 1 Question 13

- Whenever I see something like a^3 and a^{-3}, I'm hoping that they'll cancel each other out.
- Therefore, I'm willing to consider multiplying out the brackets in this question.
- The question asks for ab. It's not obvious if we'll solve for a and b separately, or just solve for ab. That said, we only have one equation, so I'm doubting my ability to find values of two unknowns.
- The question asks for the "least value" of ab. It's not obvious if we'll find all possible solutions for ab and select the least, or whether there will be some shortcut that means we only find the value we want. Probably I'm overthinking it and I should now get on with the question.
- a doesn't appear in the equation except as a^3. So perhaps the cube $(ab)^3$ will make an appearance before ab does.

TMUA Specification (April 2025, Section 1 §MM2)

- Sequences, including those given by a formula for the n^{th} term and those generated by a simple recurrence relation of the form $x_{n+1} = f(x_n)$.
- Arithmetic series, including the formula for the sum of the first n natural numbers.
- The sum of a finite geometric series.
- The sum to infinity of a convergent geometric series, including the use of $|r| < 1$.
- Binomial expansion of $(1 + x)^n$ for positive integer n, and for expressions of the form $(a + f(x))^n$ for positive integer n and simple $f(x)$.
- The notations $n!$ and $\binom{n}{r}$.

Revision

- A sequence a_n might be defined by a formula for the n^{th} term like $a_n = n^2 - n$.
- A sequence a_n might be defined with a "recurrence relation" like $a_{n+1} = f(a_n)$ for $n \geqslant 0$, if we're given the function $f(x)$ and also given a value for the first term a_0. (I'm counting from zero, so my "first term" is a_0, but it's also common to start with a_1, in which case the recurrence relation could be given as $a_{n+1} = f(a_n)$ for $n \geqslant 1$).
- The sum of the first n terms of a sequence a_k can be written with the notation $\displaystyle\sum_{k=0}^{n-1} a_k$ (if the first term is a_0) or $\displaystyle\sum_{k=1}^{n} a_k$ (if the first term is a_1).
- An arithmetic sequence (also known as an arithmetic progression) is a sequence where the difference between terms is constant. The terms can be written as a, $a + d$, $a + 2d$, $a + 3d$, $\ldots$, where a is the first term and d is the common difference.
- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is $\dfrac{n}{2}\,(2a + (n-1)d)$, which you can remember as "first term plus last term, times the number of terms, divided by two" or $\dfrac{n}{2}(a + l)$, writing l for the last term.
- In particular, the natural numbers $1, 2, 3, \ldots, n$ are an arithmetic sequence, and the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.
- A geometric sequence (also known as a geometric progression) is a sequence where the ratio between consecutive terms is constant. The terms can be written as a, ar, ar^2, ar^3, $\ldots$ where a is the first term and r is the common ratio.

- The sum of the first n terms of a geometric sequence with first term a and common ratio r is $\dfrac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of the first n terms of a geometric series by $(1-r)$,

$$(1-r)(a+ar+\cdots+ar^{n-1}) = (a-ar)+(ar-ar^2)+\cdots+(ar^{n-1}-ar^n) = a-ar^n.$$

- For a geometric sequence a_n, the sum to infinity is written as $\displaystyle\sum_{k=0}^{\infty} a_k$. If the common ratio r satisfies $|r|<1$ then this is equal to $\dfrac{a}{1-r}$. If $|r| \geqslant 1$ then this sum to infinity does not converge (it does not approach any particular real number).

- (Binomial Theorem) If n is a positive whole number then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{r}x^r y^{n-r} + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

  where $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ and $n!$ means $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ for a whole number n.

- In the Binomial Theorem, x and y could be real numbers or could even be functions, so you could use the Binomial Theorem on expressions like $(1+x)^n$ or even $(a+f(x))^n$.

- A periodic sequence has terms that repeat in a cycle; there is some L such that for all $n \geqslant 0$, we have $a_{n+L} = a_n$. The order of a periodic sequence is the minimum such L.

Revision Questions

1. A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.
2. A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.
3. A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?
4. A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.
5. A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2.
6. When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.
7. When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \dots$ converge? Simplify it in the case that it converges.
8. If the first term of an arithmetic progression is 5 and the common difference is 3, what is the 15$^{\text{th}}$ term?
9. The sum of the first k terms of an arithmetic progression is equal to the sum of the next k terms. What can you deduce?
10. If the sum of the first n terms of an arithmetic progression is $3n^2 + 5n$, what is the n^{th} term?
11. What is the sum of the first 100 positive even integers (starting at 2)?
12. The first term of a geometric progression is 3 and the third term is 27. Find two possibilities for the sum of the first 5 terms.
13. A sequence is defined by $a_0 = 3$ and then for $n \geqslant 1$ a_n is the sum of all previous terms. Find a_n in terms of n for $n \geqslant 1$.
14. A sequence is defined by $C_0 = 1$ and then for $n \geqslant 0$,
$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$
Find C_1 and C_2 and C_3 and C_4.
15. Expand $(2x + 3y)^3$
16. What is the coefficient of the x^2 term in the polynomial $w(x) = (3x - 1)^4$?
17. What is the sum of the coefficients of the polynomial $(x + 2)^3$? What about $(x + 2)^{300}$?

TMUA Questions

TMUA 2020 Paper 1 Question 4

The 1$^{\text{st}}$, 2$^{\text{nd}}$ and 3$^{\text{rd}}$ terms of a geometric progression are also the 1$^{\text{st}}$, 4$^{\text{th}}$ and 6$^{\text{th}}$ terms, respectively, of an arithmetic progression.
The sum to infinity of the geometric progression is 12.
Find the 1$^{\text{st}}$ term of the geometric progression.

A 1
B 2
C 3
D 4
E 5
F 6

[See the next page for hints]

TMUA 2020 Paper 2 Question 14

An arithmetic sequence T has first term a and common difference d, where a and d are non-zero integers.
Property P is:

> **For some** positive integer m, the sum of the first m terms of the sequence is equal to the sum of the first $2m$ terms of the sequence.

For example, when $a = 11$ and $d = -2$, the sequence T has property P, because

$$11 + 9 + 7 + 5 = 11 + 9 + 7 + 5 + 3 + 1 + (-1) + (-3)$$

i.e. the sum of the first 4 terms equals the sum of the first 8 terms.
Which of the following statements is/are **true**?

I For T to have property P, it is **sufficient** that $ad < 0$.
II For T to have property P, it is **necessary** that d is even.

A neither of them
B I only
C II only
D I and II

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 4

- It's usually a good idea to give names to unknown quantities. For this question, you might write a and r and d for the unknown first term of the geometric progression, the common ratio of the geometric progression, and the common difference of the arithmetic progression.
- Take care with the arithmetic progression, where we're asked to skip terms. Is the sixth term going to be $a + 5d$ or $a + 6d$ or $a + 7d$?
- When I'm given multiple pieces of information that each lead to an equation, I like to re-write those equations all in one place, each on its own row, with the equals signs aligned. It doesn't affect the mathematics to line them up, of course, but I think it does help me to "see" what I might do next.
- When you're faced with some complicated simultaneous equations, focus on what you want to know, and try to eliminate the rest. In this question, we want to know the first term of the geometric progression.
- There are multiple routes through the algebra for this question.
- If your answer doesn't match any of the options, look back through your rough work instead of starting again.

TMUA 2020 Paper 2 Question 14

- Another way to say "For T to have property P, it is sufficient that $ad < 0$" that's more emotive is "no matter what the values of a and d are, if their product is negative, then I absolutely promise you that property P holds". Is that promise true?
- Another way to say "For T to have property P, it is necessary that d is even" that's more emotive is "I've checked every single case where property P holds, and in every case the number d is always even". Do you believe that case-check has been done correctly?
- Note that if both statements are true, then whenever $ad < 0$, property P must hold, and therefore d must be even. That's clearly nonsense. Negative numbers can be odd! So there's something else going on here.
- Now that we've thought about the statements, we should engage with the arithmetic sequence part of the question. What is the sum of the first m terms of the sequence?

TMUA 2020 Paper 2 Question 15
Which one of the following is a **necessary and sufficient** condition for

$$\sum_{k=1}^{n} \sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

to be true?

A $n = 1$
B n is a multiple of 3
C n is a multiple of 6
D n is 1 more than a multiple of 3
E n is 1 more than a multiple of 6
F n is 1 more than a multiple of 6 or n is 2 more than a multiple of 6

[See the next page for hints]

TMUA 2021 Paper 1 Question 13
The function f is such that, for every integer n

$$\int_{n}^{n+1} \mathrm{f}(x)\,\mathrm{d}x = n + 1$$

Evaluate

$$\sum_{r=1}^{8} \left(\int_{0}^{r} \mathrm{f}(x)\,\mathrm{d}x \right)$$

A 36
B 84
C 120
D 165
E 204
F 288

[See the next page for hints]

Hints

TMUA 2020 Paper 2 Question 15

- This question is in radians. You can tell because there's no degree sign $\circ$.
- Most of the options refer to multiples of 3 or 6. Those values would make the fraction inside the brackets π or 2π, and I know that $\sin x$ is periodic with period 2π. So maybe this sequence is periodic too.
- We could just evaluate this for small values of n and see what happens. Because it's a sum over more and more terms, I think I would keep track of the values of $\sin\left(\frac{k\pi}{3}\right)$ and a running total, working from $k = 1$ up to $k = 6$.

TMUA 2021 Paper 1 Question 13

- Do not try to "solve" for the function f. Do not try to guess a function that works. That's not what's going on with this question.
- What's the relationship between $\displaystyle\int_n^{n+1} \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^{r} \mathrm{f}(x)\,\mathrm{d}x$?
- Check that relationship makes sense when $r = 0$.
- The number 8 is not that large. Just do it? (Sometimes I don't have a fancy summation formula... it's time to roll up my sleeves and actually add some numbers together!)
- Like the previous question, I would use a table with a running total here.

TMUA 2022 Paper 1 Question 8

A geometric sequence has first term a and common ratio r, where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n.

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k.

What is the smallest possible value of k?

A 2^{10}

B 2^{20}

C 2^{30}

D $\dfrac{2^{10}}{2^{10}-1}$

E $2^{10}(2^{10}-1)$

[See the next page for hints]

TMUA 2022 Paper 2 Question 8

A selection, S, of n terms is taken from the arithmetic sequence $1, 4, 7, 10, \ldots, 70$.

Consider the following statement:

$(*)$ There are two distinct terms in S whose sum is 74.

What is the smallest value of n for which $(*)$ is **necessarily** true?

A 12

B 13

C 14

D 21

E 22

F 23

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 8

- Note that we're not asked to solve for a, the first term of the geometric sequence. Can you see why?
- Use your knowledge of geometric sequences to convert the equation $S_{30} - S_{20} = kS_{10}$ into an equation involving a and r.
- Look for cancellation, look for simplification. You could make a substitution in the equation, if you see the same expression turn up lots of times, if you like.
- My equation boils down to something very simple involving k and r. At that point I look back to the end of the question to check what I'm being asked. It says "smallest possible value of k", which has got nothing to do with my equation. Have I missed something? Then I look back up at the start of the question to see if there are any of what I call terms and conditions for the constants k and r. There are lots of terms and conditions at the top of the question! I'd forgotten about those while I was doing the algebra. Good thing I checked.

TMUA 2022 Paper 2 Question 8

- There is an important word in $(*)$ that you might miss the first time you read it.
- The selection S could be 4 and 70. Then there are two distinct terms in S whose sum is 74. That's just 2 terms! Why isn't the answer 2?
- Note that we're not told how this selection was done; when we consider whether $(*)$ is necessarily true, we're supposed to consider all possible selections.

TMUA Specification (April 2025, Section 1 §MM3)

- Equation of a straight line, including:
  - $y - y_1 = m(x–x_1)$
  - $ax + by + c = 0$
- Conditions for two straight lines to be parallel or perpendicular to each other.
- Finding equations of straight lines given information in various forms.
- Coordinate geometry of the circle, using the equation of a circle in the forms:
  - $(x - a)^2 + (y - b)^2 = r^2$
  - $x^2 + y^2 + cx + dy + e = 0$
- Use of the following circle properties:
  - The perpendicular from the centre to a chord bisects the chord.
  - The tangent at any point on a circle is perpendicular to the radius at that point.
  - The angle subtended by an arc at the centre of a circle is twice the angle subtended by the arc at any point on the circumference.
  - The angle in a semicircle is a right angle.
  - Angles in the same segment are equal.
  - The opposite angles in a cyclic quadrilateral add to 180°.
  - The angle between the tangent and chord at the point of contact is equal to the angle in the alternate segment.

There is also a part of the TMUA Specification listing content that test-takers are expected to recall from GCSE or equivalent. For most of these worksheets, we haven't felt the need to include this content, but for geometry in particular, here is a list of selected items that you might find it useful to revise.

TMUA Specification (April 2025, Section 1 §M5) – selected items

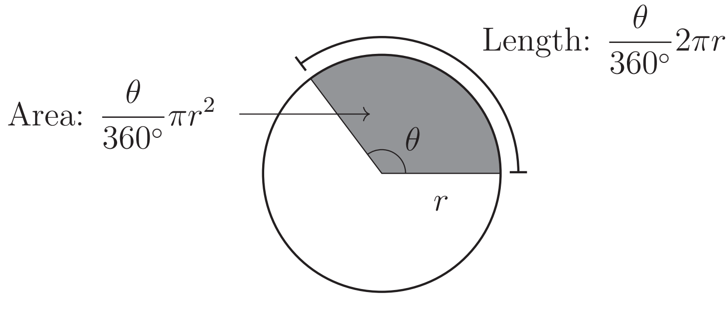
- Recall and use the properties of angles at a point, angles on a straight line, perpendicular lines and opposite angles at a vertex.
- Understand and use the angle properties of parallel lines, intersecting lines, triangles and quadrilaterals.
- Derive and apply the properties and definitions of special types of quadrilaterals, including square, rectangle, parallelogram, trapezium, kite and rhombus.

- Identify and use conventional circle terms: centre, radius, chord, diameter, circumference, tangent, arc, sector and segment (including the use of the terms minor and major for arcs, sectors and segments).
- Calculate arc lengths, angles and areas of sectors of circles.
- Describe translations as 2-dimensional vectors.
- Apply addition and subtraction of vectors, multiplication of vectors by a scalar, and diagrammatic and column representations of vectors.

Revision

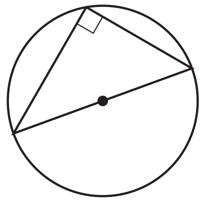
- Points in the plane can be described with two coordinates (x, y). The x-axis is the line $y = 0$, and the y-axis is the line $x = 0$.
- A straight line has equation $y = mx + c$, where m is the gradient and c is the y-intercept. Other ways to write the equation of a line are $ax + by + c = 0$ (where that's a different c to the one in the previous expression) or $y - y_1 = m(x - x_1)$. The last expression is useful because that line goes through the point (x_1, y_1) and has gradient m, which might be information that we've been given.
- Two lines are parallel if and only if they have the same gradient. Two lines are perpendicular if and only if their gradients multiply to give -1.
- The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.
- If we expand the brackets in that equation, then we would get $x^2 + y^2 - 2ax - 2bx + a^2 + b^2 - r^2$. So if we see an equation of the form $x^2 + y^2 + cx + dy + e = 0$ then that would represent a circle with centre $\left(-\frac{c}{2}, -\frac{d}{2}\right)$ and radius $\sqrt{\frac{c^2}{4} + \frac{d^2}{4} - e}$, if the term inside the square root is positive. If that term is zero, this equation describes a single point, and if it's negative then the equation does not describe any points in the (x, y)-plane.
- A circle with radius r has area πr^2 and circumference $2\pi r$.
- For a circle with radius r, suppose that two radii make an angle θ. The arc subtended by θ has length $\frac{\theta}{360^\circ} 2\pi r$ if you measure θ in degrees, or θr if you measure θ in radians.

  The area of the sector enclosed by that arc and the radii is $\frac{\theta}{360^\circ}\pi r^2$ if you measure θ in degrees, or $\frac{1}{2}\theta r^2$ if you measure θ in radians.

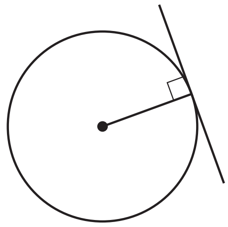


- Circle Theorems.

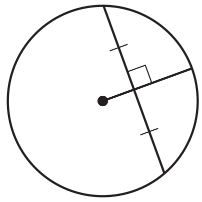
  The angle in a semicircle is a right angle; if AB is the diameter of a circle, and C is on the circle, then $\angle ACB = 90°$. Conversely, if triangle ABC is right-angled at C, then the centre of the circle passing through A, B, and C is the midpoint of the hypotenuse AB.

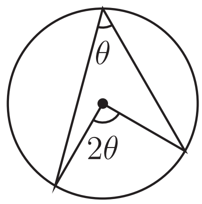
  The tangent is at right angles to the radius at any point on a circle's circumference. Conversely, if a line passes through a point on the circle's circumference and is at right angles to the radius, then it is a tangent.

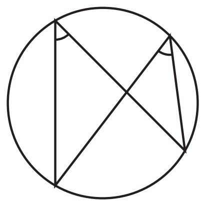
  The perpendicular from the centre to a chord bisects the chord. You can check that this is true by drawing radii that connect the centre to each point on the circumference, and then using Pythagoras to find the lengths.

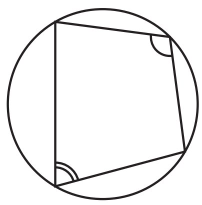
  The angle subtended by an arc at the centre of a circle is twice the angle subtended by the arc at any point on the circumference.
  You can check this by drawing in one more radius, and then naming the angles in the isosceles triangles that you've formed.

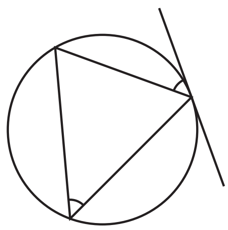
  Angles in the same segment are equal. This follows from the previous fact, because these angles subtend the same arc, so the angle at the centre is equal.

  The opposite angles in a cyclic quadrilateral add to 180°. You can check that this is true by drawing radii that connect the centre to each point, and then naming the angles in the isosceles triangles that you've formed.

  If you draw a tangent to a circle at point A, and draw a triangle ABC with points on the circle, then the angle between AB and the tangent is equal to the angle BCA. You can check this by drawing in the radii that connect the centre to A and B and chasing angles around, using the above facts.

- The angles at a point sum to 360°. Angles on a straight line sum to 180°. Opposite angles where two lines cross are equal. If a line crosses two parallel lines then the corresponding angles it makes with each of the parallel lines are equal.

- A parallelogram is a quadrilateral where both pairs of opposite sides are parallel. Each pair of opposite sides is therefore equal in length, but the four sides might not all be the same length as each other. The diagonals of a parallelogram bisect one another.

- A trapezium is a quadrilateral with one pair of opposite sides that are parallel.

- A kite is a quadrilateral with reflectional symmetry along one of its diagonals. Therefore it has two pairs of sides that are equal length. The diagonals of a kite cross at right angles.

- A rhombus is a parallelogram with all four sides the same length. Equivalently, a rhombus is a kite with all four sides the same length. The diagonals of a rhombus bisect one another at right angles.

- Triangles are *similar* if they have the same angles (they might be different sizes, and they might be translated / rotated / reflected / all of the above when compared with one another).

- More generally, two shapes are similar if one can be obtained from the other by some sequence of translations, rotations, and reflections. Note that these transformations all preserve angles and all preserve ratios of lengths (but not necessarily actual lengths).

- Triangles are *congruent* if they have the same side lengths and the same angles (they might be translated / rotated / reflected / all of the above). On the trigonometry worksheet, we will see several sufficient conditions that tell us that two triangles are congruent.

- A vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can store the same information as a pair of coordinates. Used in that sense, the vector is called a position vector.

- A vector can also describe the displacement from one point to another, so that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ could represent the displacement from $(1, 1)$ to $(3, 2)$ for example.

- Vectors can be added by adding the components separately. To show that in a diagram, we might interpret the first vector as a position vector (drawing an arrow starting from the origin) and then interpret the second as a displacement (drawing an arrow starting from the end of the first vector).

- The magnitude of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.

- The distance from A to B is the magnitude of the vector displacement from A to B. The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

- A vector can be multiplied by a number by multiplying each component by that number. The result is a vector in the same direction but with scaled magnitude.

Revision Questions

1. Find the equation of the line through the points $(1, 5)$ and $(3, -1)$.
2. Find the equation of the line through the point $(3, 5)$ with gradient 2.
3. A circle has centre $(-1, 2)$ and radius 3. Write down an equation for the circle. What's the area of this circle? Where does this circle cross the axes?
4. A circle is given by $x^2 + 9x + y^2 - 3y = 10$. Find the centre and radius of the circle.
5. Points A and B lie on a circle with centre O and radius 2. The angle $\angle AOB$ is $120°$. Find the length of the arc between A and B. Find the area enclosed by that arc and the radii OA and OB.
6. Two circles are given by $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Find the area of the region that's inside both circles.
7. A circle has centre $(c, 0)$ and radius 1. The area in the region $x > 0$ which is inside the circle depends on c, and we'll call it $A(c)$. Sketch a graph of $A(c)$ against c.
8. The points $(0, 0)$ and $(1, a)$ and $(0, a + a^{-1})$ all lie on the same circle. Find the centre of the circle in terms of a.
9. Twelve equally-spaced points are marked on the circumference of a circle. Three of them are selected and labelled A and B and C. What are the possible values for angle ABC?
10. Show that the points $(0, 5)$, $(1, 3)$, $(2, 6)$, and $(3, 4)$ lie on the corners of a square.
11. Show that the points $(0, 0)$, $(0, 4)$, $(3, 2)$, $(3, 1)$ form a trapezium. Find its area.
12. Use similar triangles to prove that the diagonals of a kite cross at right angles.
13. Find equations of three lines such that the finite region bounded by the three lines is an equilateral triangle.
14. Draw a diagram to show the three separate position vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
15. Add the vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Show the resulting vector on your diagram from the previous question.

TMUA Questions

TMUA 2020 Paper 1 Question 16

The circle C_1 has equation $(x+2)^2 + (y-1)^2 = 3$
The circle C_2 has equation $(x-4)^2 + (y-1)^2 = 3$
The straight line l is a tangent to both C_1 and C_2 and has positive gradient.
The acute angle between l and the x-axis is θ
Find the value of $\tan\theta$.

A $\frac{1}{2}$ **B** 2 **C** $\frac{\sqrt{2}}{2}$ **D** $\sqrt{2}$ **E** $\frac{\sqrt{6}}{2}$ **F** $\frac{\sqrt{6}}{3}$ **G** $\frac{\sqrt{3}}{3}$ **H** $\sqrt{3}$

[See the next page for hints]

TMUA 2021 Paper 1 Question 20

Find the length of the curve with equation

$$2\log_{10}(x-y) = \log_{10}(2-2x) + \log_{10}(y+5)$$

A 5 **B** 10 **C** 15 **D** 3π **E** 9π **F** 12π

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 16

- Draw a diagram! This will force you to think about things like the locations of the centres of the circles, and whether they overlap or intersect anywhere.
- What's the radius of each circle? What do you notice?
- Add a line to your diagram that's tangent to both circles and has positive gradient. This line is uniquely determined, which is maybe a bit of a surprise. If you draw in other tangents, convince yourself that they don't have positive gradient.
- Symmetry is powerful. If you can guess a point that lies on the line "by symmetry", that can save you work. You might like to think carefully about whether the point definitely lies on the line.
- If you spot a right-angled triangle, draw a separate diagram of that triangle, labelling what you know. You can probably deduce something, using all the many things you know about right-angled triangles.
- Try to avoid solving for the equation of the line, unless you really want to. There's an angle that you want. Mark it in on your diagram(s).
- Although the question says "the acute angle between l and the x-axis", we can replace "the x-axis" in that phrase with any line that's parallel to the x-axis. Check your understanding; why can we do this?

TMUA 2021 Paper 1 Question 20

- If you haven't met logarithms yet, come back to this question after you have.
- The left-hand side is only defined if $x - y > 0$. Similarly, the terms on the right-hand side might not be defined for all values of x and y.
- I don't know the lengths of many curves. Something to do with circles, perhaps? I can see factors of π in half the options for this question.
- We'll need to manipulate the logarithms to get an equation we can work with, but we should remember where we started.
- Draw a diagram!

TMUA 2021 Paper 2 Question 7

A circle has equation $(x-9)^2 + (y+2)^2 = 4$

A square has vertices at $(1, 0)$, $(1, 2)$, $(-1, 2)$ and $(-1, 0)$.

A straight line bisects both the area of the circle and the area of the square.

What is the x-coordinate of the point where this straight line meets the x-axis?

A 2

B 3

C 4

D 4.5

E 5

F 6

G The straight line is not uniquely determined by the information given, so there is more than one possible point of intersection.

H There is no straight line that bisects both the area of the circle and the area of the square.

[See the next page for hints]

TMUA 2022 Paper 1 Question 4

These sectors of circles are similar.

The arc length of the smaller sector is 6.

The difference between the areas of the sectors is 21.

Find the positive difference between the perimeters of the sectors.

A 4.5 **B** 7 **C** 8 **D** 9 **E** 10.5 **F** 14 **G** 15

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 7

- Draw a diagram!
- What can you say about “a straight line that bisects the area of the circle”? Why?
- What can you say about “a straight line that bisects the area of the square”? Why?
- You’ll probably want to solve for the equation of the line for this question, before you find the x-intercept.
- Some of the options suggest that there might be no such line, or too many such lines. Focus on trying to solve for the line, and only consider one of these options if you’re really having trouble finding the equation of the line (is there some reason why you cannot find the equation of the line?)

TMUA 2022 Paper 1 Question 4

- We could write down the arc length for the larger sector, using the fact that these shapes are similar (ratios between lengths are the same in each diagram). But do we want to do this? Not yet, perhaps.
- The question asks about the “difference between areas”, so we should be careful to take the larger area and subtract the smaller area. Luckily, it’s pretty clear which sector is larger.
- All the formulas that I know for the area or the arc length of a sector involve an angle. I should give that angle a name.
- The perimeter of a sector is not something I’ve particularly thought about before, but I assume that it means that I should add all the side lengths together, like I would for a square or rectangle or triangle.

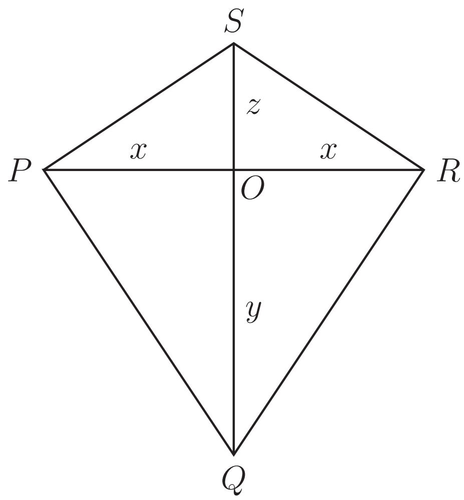
TMUA 2022 Paper 1 Question 14

A circle has centre O and radius 6.
P, Q and R are points on the circumference with angle $POQ \geq \frac{\pi}{2}$
The area of the triangle POQ is $9\sqrt{3}$
What is the greatest possible area of triangle PRQ?

A $18 + 9\sqrt{3}$ **B** $18\sqrt{3}$ **C** $27 + 9\sqrt{3}$ **D** $27\sqrt{3}$ **E** $36 + 9\sqrt{3}$ **F** $36\sqrt{3}$

[See the next page for hints]

TMUA 2022 Paper 2 Question 11



The diagram shows a kite $PQRS$ whose diagonals meet at O, with $OP = x$, $OQ = y$, $OR = x$, and $OS = z$.
Which of the following is **necessary and sufficient** for angle SPQ to be a right angle?

A $x = y = z$ **B** $2x = y + z$ **C** $x^2 = yz$ **D** $y = z$ **E** $y^2 = x^2 + z^2$

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 14

- Draw a diagram!
- You know several things about triangle POQ; the area is directly mentioned in the question, you're told a fact about an angle, and it's clearly related to the circle somehow. It's probably related to the triangle PRQ too, but that's less clear so let's ignore that for now. Can you deduce any facts about the side lengths or angles of triangle POQ?
- Don't worry about the precise locations of P and Q and R. There's definitely not enough information for us to actually work out where these points are, because the circle can be rotated! That rotation doesn't change the areas in the question, so it just doesn't matter.
- Because it's only the *relative* positions of P and Q and R that matter, you can draw your diagram so that PQ has a particularly convenient orientation, such as being horizontal or vertical. When you use the convenience of making an arbitrary choice like this, you might say that you are making your choice "without loss of generality".
- With your choice, and your knowledge of triangle POQ, you might have PQ (relatively) fixed. The only thing about triangle PRQ left to set is the position of R. How can you position this point to maximise the area of the triangle?

TMUA 2022 Paper 2 Question 11

- Recall that the diagonals of a kite meet at right angles.
- Once you've remembered that fact about kites, you can ignore the point R entirely.
- There is an approach for this question that uses right-angled triangles.
- There is an approach for this question that uses similar triangles.
- There is an approach for this question that uses the angle in a semi-circle.

TMUA Specification (April 2025, Section 1 §MM4)

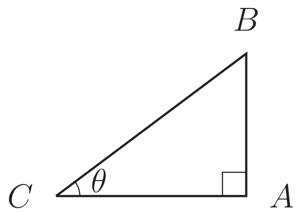
- The sine and cosine rules, and the area of a triangle in the form $\frac{1}{2}ab\sin C$.
- The sine rule includes an understanding of the "ambiguous" case (angle–side–side).
- Problems might be set in 2 or 3 dimensions.
- Radian measure, including use for arc length and area of sector and segment.
- The values of sine, cosine and tangent for the angles: 0°, 30°, 45°, 60°, 90°.
- The sine, cosine and tangent functions; their graphs, symmetries, and periodicity.
- Knowledge and use of the equations:
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
  - $\sin^2\theta + \cos^2\theta = 1$
- Solution of simple trigonometric equations in a given interval (this may involve the use of the identities above).

There is also a part of the TMUA Specification listing content that test-takers are expected to recall from GCSE or equivalent. For most of these worksheets, we haven't felt the need to include this content, but for trigonometry in particular, we want to highlight one item.

TMUA Specification (April 2025, Section 1 §M5) – selected item

- Understand and use the basic congruence criteria for triangles (SSS, SAS, ASA, RHS)

Revision



- If triangle ABC is right-angled at A and $\angle BCA = \theta$, then we define

$$\sin\theta = \frac{|AB|}{|BC|}, \qquad \cos\theta = \frac{|AC|}{|BC|}, \qquad \tan\theta = \frac{|AB|}{|AC|}.$$

- So $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

- Pythagoras' Theorem states that $|AB|^2 + |AC|^2 = |BC|^2$ so $\sin^2\theta + \cos^2\theta = 1$.

  It follows that $-1 \leqslant \sin\theta \leqslant 1$ and $-1 \leqslant \cos\theta \leqslant 1$.

- Since the angles in a triangle add up to $180°$, the angle at B is $90° - \theta$.

  Looking at the triangle that way around, we can deduce that $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$.

- The sine, cosine, and tangent functions are all periodic;

$$\sin(x + 360°) = \sin x, \qquad \cos(x + 360°) = \cos x, \qquad \tan(x + 180°) = \tan x.$$

- Also, we have $\sin(-x) = -\sin x$, and $\cos(-x) = \cos(x)$.

- Radians are another way to measure angles. An arc of a unit circle with arc length 1 subtends an angle of one radian at the origin. This means that there are 2π radians at a point, π radians on a line, $\frac{\pi}{2}$ radians in a right-angle, and so on. To convert, I remember that $180°$ is equal to π radians and work from there (for example, $60°$ is $\frac{\pi}{3}$ radians). Angles measured in degrees have the $°$ symbol and angles measured in radians do not.

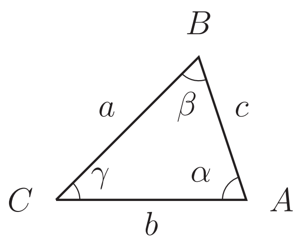
- Here are some values of sin and cos and tan. The left table uses degrees and the right table uses radians.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$*$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$*$

  where $*$ is my way of indicating that $\tan\theta$ is undefined for $\theta = 90°$.

Now consider a triangle that is not necessarily right-angled.



Let's write α, β, and γ for the angles at A, B, and C respectively, and let's call the side-lengths $a = |BC|$, $b = |AC|$, and $c = |AB|$.

- The area of this triangle is $\frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta$.

- (The cosine rule) We have $a^2 = b^2 + c^2 - 2bc\cos\alpha$.
- (The sine rule) We have $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.
- Triangles are *congruent* if they have the same side lengths and the same angles (they might be translated / rotated / reflected / all of the above).
- Here are several sufficient conditions that tell us that two triangles are congruent.
  - (SSS) if two triangles have all three side lengths in common, then they are congruent. You can check this with the cosine rule. Suppose the three side lengths are a, b, and c. The cosine rule determines $\cos\alpha$, and since cos is a one-to-one function for angles between $0°$ and $180°$, the angle α is uniquely determined. Similarly for the other angles.
  - (SAS): if two triangles share two side lengths and the angle *between* those sides, then they are congruent. You can check that the cosine rule lets you determine the other side length, and then we're in the SSS case.
  - (ASA): if two triangles share two angles and the side *between* those angles, then they are congruent. Knowing two angles determines the third, and then the sine rule determines the remaining sides.
  - (RHS): if two right-angled triangles share their hypotenuse length and one other side length, then they are congruent. This follows from Pythagoras, which determines the third side, giving SSS.

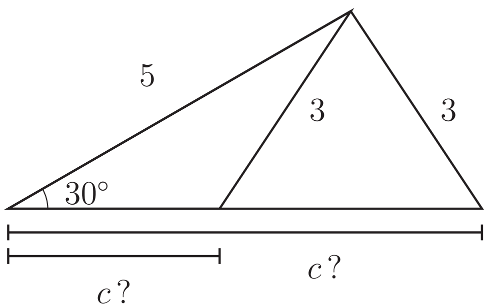
  Note that SSA (two sides and an angle that's not between those sides) is not on this list, because it's not a sufficient condition. This is called the ambiguous case of the sine rule.
- (Ambiguous case) Suppose that we're given a, b, and α, and we would like to determine c. Then the cosine rule tells us that $b^2 + c^2 - 2bc\cos\alpha = a^2$. This is a quadratic equation for c

  $$c^2 - (2b\cos\alpha)\,c + b^2 - a^2 = 0$$

  We're interested in positive solutions for the length c, and sometimes there are two such solutions. By calculating the discriminant and thinking about the signs of the coefficients, we see that this happens if $b\sin\alpha < a < b$ and α is an acute angle ($0 < \alpha < 90°$).

  Example; let's set $a = 3$, $b = 5$, and $\alpha = 30°$.

  The angle α is acute, and $b\sin\alpha < a < b$, so we expect two positive solutions for c, corresponding to a pair of non-congruent triangles with matching a, b, and α. Here's a diagram showing both of them.



Revision Questions

1. Find the area of an equilateral triangle with all three sides of length a.
2. Find all the solutions to $\sin x = \frac{1}{2}$ with $0 \leqslant x < 360°$.
3. Find all the solutions to $\tan x = 1$ with $0 \leqslant x < 360°$.
4. Find all the solutions to $\tan(45x) = 1$ with $0 \leqslant x < 360°$.
5. A triangle has angles $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, and has side lengths 1, $\sqrt{3}$, 2 opposite those angles respectively. Write down three expressions for the area using the three formulas above, and check that they all give the same value.
6. Write $(\cos x + \sin x)^2$ in terms of the variable $u = \cos x \sin x$.
7. For $0 \leqslant x < \frac{\pi}{2}$, write $1 - \sin^2 x + \sin^4 x - \sin^6 x + \dots$ as a single expression (not an infinite sum) in terms of $\cos x$. Why have I excluded $\frac{\pi}{2}$ from the range here?
8. Write $\cos^4 x + \cos^2 x$ in terms of $\sin x$.
9. Simplify $\cos(450° - x)$.
10. Simplify $\cos(90° - x)\sin(180° - x) - \sin(90° - x)\cos(180° - x)$.

    (If you know a fact about $\sin(A - B)$, you may only use it here if you prove it!)
11. A triangle ABC has side lengths $|AB| = 8$ and $|BC| = 7$, and the angle $\angle ABC = \frac{\pi}{3}$. Find the remaining side length $|AC|$, the area of the triangle, and an expression for $\sin \angle BCA$.
12. A triangle ABC has side lengths $|AB| = 8$ and $|AC| = 7$, and the angle $\angle ABC = \frac{\pi}{3}$.

    Find all possibilities for the remaining side length $|BC|$.
13. This question explores the ambiguous case (SSA), this time using the sine rule rather than the cosine rule. Given a and b and α, use the sine rule to write down an expression for $\sin \beta$.

    Explain why there are no solutions if $a < b \sin \alpha$, and exactly one solution if $a = b \sin \alpha$.

    For $a > b \sin \alpha$ there are two solutions for β in the range $0 < \beta < \pi$. By considering the sum of the angles in the triangle, explain why, if $a \geqslant b$ or if $\frac{\pi}{2} \leqslant \alpha < \pi$, then the larger of these solutions for β does not correspond to a real triangle.
14. From the cosine rule, and the fact that $-1 \leqslant \cos \alpha \leqslant 1$, deduce that for any triangle with side lengths $a > b > c > 0$, we must have $a < b + c$ (this is the rule that "the longest side is shorter than the sum of the other two sides" or said differently "the shortest route from B to C is a straight line" also known as "the triangle inequality").
15. A triangle has side lengths 8, 13, and 15. You are told that one of the angles of this triangle has a value that is a whole number when measured in degrees. Find the exact value of this angle.

TMUA Questions

TMUA 2020 Paper 1 Question 12

How many real solutions are there to the equation

$$3\cos x = \sqrt{x}$$

where x is in radians?

A 0
B 1
C 2
D 3
E 4
F 5
G infinitely many

[See the next page for hints]

TMUA 2021 Paper 1 Question 6

The function f is given by

$$\mathrm{f}(x) = \frac{\cos x + 3}{7 + 5\cos x - \sin^2 x}$$

Find the positive difference between the maximum and the minimum values of $\mathrm{f}(x)$.

A 0
B $\frac{1}{3}$
C $\frac{1}{2}$
D $\frac{2}{3}$
E 1
F 2

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 12

- Don't try to solve for the actual values of these solutions! I don't even really know what that would involve.
- We're not given a range for x, so when we start the question x could be any real number. However, you know that only certain values are permitted for one of the functions in the question, so that immediately restricts the range of potential solutions x. What else can you say about the functions in the question?
- Draw a diagram or diagrams!
- This is secretly a question about inequalities. Given two functions (without any "jumps" in value), if $f(a) < g(a)$ and $f(b) > g(b)$, then somewhere between a and b there's a point where $f(x) = g(x)$.

TMUA 2021 Paper 1 Question 6

- Depending on the maths that you've learned, you might be able to differentiate this function. Even if you can technically do this, I wouldn't encourage you to do so!
- It's a shame about that $\sin^2 x$. If that were a $\cos^2 x$ then all the trigonometric functions in this question would be $\cos x$.
- Try to simplify the fraction. It's just about possible to reason about the fraction if you don't simplify it, but this question is much easier if you can simplify first.
- If in doubt, try some simple values of $\cos x$.

TMUA 2021 Paper 1 Question 19
The equation

$$\sin^2\left(4^{\cos\theta} \times 60^\circ\right) = \frac{3}{4}$$

has exactly three solutions in the range $0^\circ \leq \theta \leq x^\circ$
What is the range of all possible values of x?

A $90 \leq x < 120$
B $90 \leq x < 270$
C $120 \leq x < 240$
D $270 \leq x < 300$
E $300 \leq x < 360$
F $450 \leq x < 630$

[See the next page for hints]

TMUA 2021 Paper 2 Question 18
A student chooses two distinct real numbers x and y with $0 < x < y < 1$.
The student then attempts to draw a triangle ABC with:

$$AB = 1$$
$$\sin A = x$$
$$\sin B = y$$

Which of the following statements is/are correct?
I For some choice of x and y, there is exactly **one** triangle the student could draw.
II For some choice of x and y, there are exactly **two** different triangles the student could draw.
III For some choice of x and y, there are exactly **three** different triangles the student could draw.
(Note that congruent triangles are considered to be the same.)

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

[See the next page for hints]

Hints

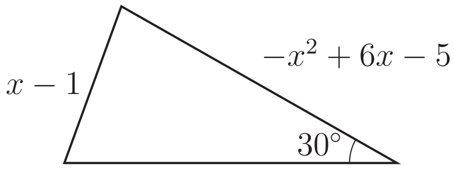
TMUA 2021 Paper 1 Question 19

- There's something slightly odd about being asked for solutions in the range $0^\circ \leqslant \theta \leqslant x^\circ$ for various values of x, rather than just solving the equation for θ. Let's try to solve the equation for θ and see how far we get.
- $4^{\cos\theta}$ doesn't vary as much as you might think. What are the minimum and maximum values of that function?
- Try to work one step at a time. Keep track of various possibilities, but be proactive about eliminating the ones that aren't going to lead anywhere.

TMUA 2021 Paper 2 Question 18

- This looks like the ASA case, I think? We're looking at the angle at A, the angle at B, and the side length in between. What could go wrong?
- Note that we're told x and y, the values of $\sin A$ and $\sin B$. That's slightly different from being told the values of the angles. Does that matter?
- Draw diagrams!
- For your diagrams, you could choose to have the line AB to be horizontal, because rotations don't matter for congruent triangles.

TMUA 2022 Paper 1 Question 17



Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A $1 < x < 3$
B $1 < x < 4$
C $1 < x < 5$
D $3 < x < 4$
E $3 < x < 5$
F $4 < x < 5$

[See the next page for hints]

TMUA 2022 Paper 2 Question 20

The functions f_1 to f_5 are defined on the real numbers by

$$\begin{aligned}
\mathrm{f}_1(x) &= \cos x\\
\mathrm{f}_2(x) &= \sin(\cos x)\\
\mathrm{f}_3(x) &= \cos(\sin(\cos x))\\
\mathrm{f}_4(x) &= \sin(\cos(\sin(\cos x)))\\
\mathrm{f}_5(x) &= \cos(\sin(\cos(\sin(\cos x))))
\end{aligned}$$

where all numbers are taken to be in radians.
These functions have maximum values m_1, m_2, m_3, m_4 and m_5, respectively.
Which one of the following statements is true?

A m_1, m_2, m_3, m_4 and m_5 are all equal to 1
B $0 < m_5 < m_4 < m_3 < m_2 < m_1 = 1$
C $m_1 = m_3 = m_5 = 1$ and $0 < m_2 = m_4 < 1$
D $m_1 = m_3 = m_5 = 1$ and $0 < m_4 < m_2 < 1$
E $m_1 = m_3 = 1$ and $0 < m_2 = m_4 < 1$ and $0 < m_5 < 1$
F $m_1 = m_3 = 1$ and $0 < m_4 < m_2 < 1$ and $0 < m_5 < 1$

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 17

- This is the ambiguous SSA case.
- I sketched a little right-angled triangle for this.
- It might be a good idea to factorise that quadratic, if we can.
- Why can we say that $x > 1$? (I'm not looking for "because all the options have $x > 1$"!)

TMUA 2022 Paper 2 Question 20

- The question asks about the maximum value, but you should also track the minimum value for f_1, f_2, and so on.
- $y = \sin(\cos x)$ is a bit of a classic for curve-sketching. It's relevant that $1 < \dfrac{\pi}{2}$.
- Note that when you apply cosine to a range of values, the maximum output might come from the minimum input, the maximum input, or some other value.
- The options look really complicated, but you can eliminate options as you go, working through the functions one at a time.

TMUA Specification (April 2025, Section 1 §MM5)

- $y = a^x$ and its graph, for simple positive values of a.
- Laws of logarithms:
  - $a^b = c \Leftrightarrow b = \log_a c$
  - $\log_a x + \log_a y = \log_a(xy)$
  - $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$
  - $k \log_a x = \log_a(x^k)$ including the special cases:
  - $\log_a \left(\dfrac{1}{x}\right) = -\log_a x$
  - $\log_a a = 1$.
- Questions requiring knowledge of the change of base formula will not be set.
- The solution of equations of the form $a^x = b$, and equations which can be reduced to this form, including those that need prior algebraic manipulation.

Revision

- The solution x to the equation $a^x = b$ where a and b are positive numbers (with $a \neq 1$) is called $\log_a(b)$. In this expression, the number a is called the base of the logarithm.
- $\log_a(x)$ is a function of x which is defined when $x > 0$. Like with $\sin x$, sometimes the brackets are omitted if it's clear what the function is being applied to, so we might write $\log_a x$.
- $\log_a x$ is not defined for $x = 0$. If x is positive and close to zero, then $|\log_a x|$ is large.
- $\log_a x$ is a one-to-one function on $x > 0$; if $\log_a x = \log_a y$ then $x = y$. If $a > 1$ then $\log_a x$ is an increasing function.
- $\log_a(a^k) = k$ for any value of k. To see why, remember that the value of $\log_a(a^k)$ is defined to be the solution x to the equation $a^x = a^k$. That solution is just k. This includes the special cases $\log_a 1 = 0$ and $\log_a a = 1$.
- With fixed $a \neq 1$, you can think of $\log_a x$ as the inverse function for $f(x) = a^x$.
- It's also true that $a^{\log_a x} = x$. To see why, let $y = \log_a x$. That would mean that $a^y = x$. Now replace the y in that equation with the expression $\log_a x$.
- (Product rule) If $a > 0$ and $a \neq 1$ and $x > 0$ and $y > 0$, then $\log_a(xy) = \log_a(x) + \log_a(y)$.
- (Power rule) If $a > 0$ and $a \neq 1$ and $x > 0$ and k is real, then $\log_a(x^k) = k \log_a x$.
- You are not required to know that $\log_a b = \dfrac{\log_c b}{\log_c a}$ for positive a, b, c with $a \neq 1$ and $c \neq 1$.

Revision Questions

1. Simplify $(2^3)^4$ and $(2^4)^3$ and $2^4 2^3$ and $2^3 2^4$.
2. Solve $x^{-2} + 4x^{-1} + 3 = 0$.
3. Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
4. Write $\log_3(x^2 + 3x + 2)$ as the sum of two terms, each involving a logarithm.
5. Solve $\log_x(x^2) = x^3$.
6. Solve $\log_x(2x) = 3$ for $x > 0$.
7. Solve $\log_{x+5}(6x + 22) = 2$.
8. Let $a = \log_3 2$ and $b = \log_3 5$. Write the following in terms of a and b.
$$\log_3 1024, \quad \log_3 40, \quad \log_3 \sqrt{\frac{2}{5}}, \quad \log_3 \frac{1}{10}, \quad \log_3 1.024.$$
9. Expand $(2^x + 2^{-x})(2^y - 2^{-y}) + (2^x - 2^{-x})(2^y + 2^{-y})$.
Expand $(2^x + 2^{-x})(2^y + 2^{-y}) + (2^x - 2^{-x})(2^y - 2^{-y})$.
10. Use logarithms to solve $2^x = 3$, then $0.5^x = 3$, then $4^x = 3$.
11. Suppose that a and b are positive numbers and $a \neq 1$. Using the definition of $\log_a b$ as the number x that satisfies $a^x = b$, explain why $\log_a b = 0$ if and only if $b = 1$.
12. Given $\log_{10}(\log_{10} x) = 6$, how many zeros are there at the end of the number x?
13. Use the product rule and the power rule to prove the "quotient rule", which is; if $a > 0$ and $a \neq 1$ and $x > 0$ and $y > 0$, then $\log_a \left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.
14. Solve $2^x + 2^{-x} = 4$.
How many solutions are there to $2^x + 2^{-x} = c$? Identify different cases in terms of c.
15. Prove that $\log_a(N + \sqrt{N^2 - 1}) = -\log_a(N - \sqrt{N^2 - 1})$ for any number $N \geqslant 1$ and any base $a > 1$.
16. Consider the equation $x^y = y^x$ with $x, y > 0$. Use logarithms to turn this into an equation of the form $f(x) = f(y)$. [Harder] Sketch $f(x)$.
17. Simplify $a^{k \log_a b}$ for positive numbers a, b, k with $a \neq 1$ and $b \neq 1$.
Consider the number $x = (\log_a b)(\log_b c)$. By simplifying a^x, show that $x = \log_a c$.
Deduce that $\log_b c = \dfrac{\log_a c}{\log_a b}$ for positive numbers a, b, c with $a \neq 1$ and $b \neq 1$.
18. Let $a = \log_3 2$ and $b = \log_3 5$. Use the result in the previous question to write $\log_6(45)$ in terms of a and b.
19. The revision notes say "if $a > 1$ then $\log_a x$ is an increasing function". What can you say if $a < 1$?

TMUA Questions

TMUA 2020 Paper 1 Question 7

Given that

$$2^{3x} = 8^{(y+3)}$$

and

$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$

what is the value of $x + y$?

A -23 **B** -22 **C** -15 **D** -14 **E** -11 **F** -10

[See the next page for hints]

TMUA 2021 Paper 1 Question 10

Use the trapezium rule with 3 strips to estimate

$$\int_{\frac{1}{2}}^{2} 2\log_{10} x \, \mathrm{d}x$$

A $\log_{10}\frac{\sqrt{6}}{2}$ **B** $\log_{10}\frac{3}{2}$ **C** $\log_{10}\frac{9}{4}$ **D** $\log_{10} 3$ **E** $\log_{10}\frac{81}{16}$ **F** $\log_{10}\frac{\sqrt{23}}{2}$

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 7

- The question just asks for $x + y$, but we should probably try to solve for both x and y.
- The variable x only appears in exponents. The variable y only appears in exponents.
- How are the numbers 2, 4, 8, and 16 related?
- Try not to evaluate 8^3 as 512, unless you have to.

TMUA 2021 Paper 1 Question 10

- If you haven't met the trapezium rule yet, come back to this question after you have.
- Don't forget the factor of $\frac{1}{2}$, or the factor of 2, or the other factor of $\frac{1}{2}$.
- During your work, you can choose whether you prefer to write things like $4\log_{10} 3$ or $\log_{10} 81$. I personally prefer the first one.

TMUA 2022 Paper 1 Question 11

Evaluate

$$\sum_{n=1}^{100} \log_{10}\left(3^{1-n}\right)$$

A $-4950 \log_{10} 3$

B $4950 \log_{10} 3$

C $-5050 \log_{10} 3$

D $5050 \log_{10} 3$

E $1 - 4950 \log_{10} 3$

F $1 + 4950 \log_{10} 3$

G $1 - 5050 \log_{10} 3$

H $1 + 5050 \log_{10} 3$

[See the next page for hints]

TMUA 2021 Paper 2 Question 14

Consider the following simultaneous equations, where p is a real number:

$$p2^x + \log_2 y = 2$$
$$2^x + \log_2 y = 1$$

What is the complete range of p for which these simultaneous equations have a real solution (x, y)?

A $p < 1$

B $p \neq 1$

C $p > 1$

D $p < 1$ or $p > 2$

E $p \neq 1$ and $p < 2$

F $p > 1$ and $p < 2$

G $p > 2$

H All real values of p

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 11

- Write out the first couple of terms of the sum, and the last term.
- Careful with the last term!
- Use laws of logarithms on each term.
- Remember that you can recognise certain sorts of sequences (in particular, arithmetic progressions and geometric progressions).

TMUA 2021 Paper 2 Question 14

- Try to solve these equations for x and y, and see what happens.
- The variable x only appears in terms like 2^x, and the variable y only appears in terms like $\log_2 y$. If you write $a = 2^x$ and $b = \log_2 y$, can you solve the equations for a and b?
- By making that substitution, we've obscured the fact that one of the variables appears in an exponential and one appears in a logarithm. Don't forget that! Given that we actually want to solve for x and y, not a and b, what might go wrong?

TMUA 2021 Paper 2 Question 17
Consider the following functions defined for $x > 1$:

$$f(x) = \log_2\left(\log_2 \sqrt{x}\right)$$

$$g(x) = \log_2\left(\sqrt{\log_2 x}\right)$$

Which one of the following is true for **all** values of $x > 1$?

A $0 \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le 0$
B $0 \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le 0$
C $\frac{1}{2} \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le \frac{1}{2}$
D $\frac{1}{2} \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le \frac{1}{2}$
E $1 \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le 1$
F $1 \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le 1$

[See the next page for hints]

TMUA 2022 Paper 2 Question 15
The real numbers x, y and z are all greater than 1, and satisfy the equations

$$\log_x y = z \quad \text{and} \quad \log_y z = x$$

Which one of the following equations for $\log_z x$ **must** be true?

A $\log_z x = y$
B $\log_z x = \dfrac{1}{y}$
C $\log_z x = xy$
D $\log_z x = \dfrac{1}{xy}$
E $\log_z x = xz$
F $\log_z x = \dfrac{1}{xz}$
G $\log_z x = yz$
H $\log_z x = \dfrac{1}{yz}$

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 17

- Don't try to understand the options! Compare $\mathrm{f}(x)$ and $\mathrm{g}(x)$. For which values of x is $\mathrm{f}(x) \leqslant \mathrm{g}(x)$?
- You know that $\log_2 x$ is an increasing function of x, so you can just compare the arguments of that function if you want to know which will give the larger output.
- If it helps, write u for $\log_2 x$ to make things look simpler.
- The question wants bounds on $\mathrm{f}(x)$ and $\mathrm{g}(x)$ in various cases. If you work out values of x for which $\mathrm{f}(x) \leqslant \mathrm{g}(x)$, then that will give you a starting point to use when you try to find inequalities for the values of $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

TMUA 2022 Paper 2 Question 15

- Write out what the given equations mean in terms of exponentials.
- Don't think too much about what it means for x to appear both as a base in the first equation, and then also as a value in the second statement. Aim for equations that you can work with!
- If you can eliminate y from your equations then you've shown that there is some relationship between x and z. That might let you rule out six of the eight options.
- You're aiming for a fact about $\log_z x$, so you're looking for an expression with a power of z that involves x.

TMUA Specification (April 2025, Section 1 §MM6)

- The derivative of $f(x)$ as the gradient of the tangent to the graph $y = f(x)$ at a point.
  - Interpretation of a derivative as a rate of change.
  - Second-order derivatives.
  - Knowledge of notation: $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $f'(x)$, and $f''(x)$.
- Differentiation from first principles is excluded.
- Differentiation of x^n for rational n, and related sums and differences. This might require some simplification before differentiating.
- Applications of differentiation to gradients, tangents, normals, stationary points (maxima and minima only), strictly increasing functions [if $f'(x) > 0$] and strictly decreasing functions [if $f'(x) < 0$]. Points of inflexion will not be examined, although a qualitative understanding of points of inflexion in the curves of simple polynomial functions is expected.

Revision

- The tangent to a graph at a particular point is a line which has the same value and gradient as the graph at that point. The gradient of the graph means something like the instantaneous rate of change of the graph at that specific point (differentiation from first principles is excluded, so you're not expected to know how it's possible to work out the gradient at a single point on a graph!)
- The derivative of a function is a new function that tells you the gradient. If your function is $f(x)$ then it might be written as $f'(x)$. It's also common to see a graph or curve in the (x, y)-plane, and in that case the derivative might be written as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
- For powers of x, we have some rules that let us work out the derivative.
- The derivative of x^a is ax^{a-1}, including for fractional exponents like $a = \frac{1}{2}$.
- If c is a constant then the derivative of $c \times f(x)$ is $c \times f'(x)$.
- The derivative of $y_1 + y_2$ is (the derivative of y_1)+(the derivative of y_2). Perhaps this looks too obvious to need stating, but remember that, for example, the square of $y_1 + y_2$ is not equal to (the square of y_1)+(the square of y_2).
- Example: if we want the tangent to the graph $y = x^3 + 2x$ at $x = 1$, then we need the value of y (which is 3), and the value of the derivative, which is the value of $3x^2 + 2$ when $x = 1$, which is 5. To find that derivative, I've used the rule for adding together powers of x, the rule for a function that's multiplied by 2, and I've remembered that $x = x^1$. The derivative of a line is its gradient, so we can write $y = 5x + c$ and solve for c using the value at $x = 1$ to get $y = 5x - 2$.

- The normal to a graph at a point is a line through that point which is perpendicular to the tangent. Two lines are perpendicular if their gradients multiply to -1.

- If the derivative is positive, that means that the function is increasing. If it's negative, that's a decreasing function. In general a function might increase in some regions and decrease in other regions.

- If the derivative changes sign at a point, that's a turning point. You'll have zero derivative at the turning point, but that's not actually sufficient for the derivative to *change* sign (e.g. x^3 has zero derivative at $x = 0$, but that's not a turning point because the derivative is positive on both sides). A point with zero derivative is called a stationary point.

- The derivative of the derivative is called the second derivative. You can work out the derivatives one at a time. So the second derivative of x^a would be the derivative of ax^{a-1}, which is $a(a-1)x^{a-2}$. The second derivative is the rate of change of the derivative. For the second derivative of a function $f(x)$, we might use the notation $f''(x)$. In cases where we wrote the first derivative as $\frac{\mathrm{d}y}{\mathrm{d}x}$, then we might use the notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the second derivative.

- A turning point is a local maximum if the second derivative is negative at that point. It's a local minimum if the second derivative is positive. "Maxima" is the plural of "maximum". "Minima" is the plural of "minimum". Overall, the function might have several local maxima, or none, and it might increase without bound (like $y = x$ for example).

- If you're looking for the overall maximum value of a function, then you might need to look beyond the turning points. For example; if $\mathrm{f}(x)$ is defined as $2x^4 - x^2$ when $-1 \leqslant x \leqslant 1$, then the maximum value comes at the endpoints $x = \pm 1$, not at the local maximum at $x = 0$.

- You're not required to know about points of inflexion, but you should be aware that the second derivative of the curve can change sign; for example, $y = x^3 - x$ has such a point when $x = 0$ because the second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$. You should also be aware that the second derivative might be zero at a stationary point. In such a case you might need to sketch the graph to work out if it's a turning point (and if so, whether it's a maximum or a minimum).

- Remember that you know how to find the turning point of a quadratic without differentiating. You could complete the square instead.

- Remember that in the context of circle geometry, you know facts about tangents that do not require you to differentiate anything.

Revision Questions

1. Differentiate $x^{17} - x^{-17}$ with respect to x.
2. Differentiate $2\sqrt{x} + 3\sqrt[3]{x}$ with respect to x.
3. Differentiate $\dfrac{x+3}{x^2}$ with respect to x.
4. Differentiate $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!}$. As a reminder, $n!$ means $n \times (n-1) \times \cdots \times 2 \times 1$.
5. Find the tangent to the curve $y = x^3 - x$ at $x = 1$.
6. Find the tangent to the curve $x^2 + y^2 + 2x - 6y - 15 = 0$ at the point $(3, 0)$.
7. Find the normal to the parabola $y = x^2$ at $x = 3$.
8. Find the normal to the curve $y = \sqrt{1 - x^2}$ at $x = p$ in terms of p, given $0 < p < 1$.
9. Find the minimum value of the function $\mathrm{f}(x) = x^4 - 4x^2 + 3$.
10. Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima. For which values of x is $\mathrm{f}(x) = x^4 - 2x^3 + x^2$ an increasing function? For which values of x is it decreasing? Sketch the curve.
11. Consider the function $\mathrm{f}(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Check that there is a stationary point at $x = 1$, and decide whether this is a local minimum or a local maximum.

TMUA Questions

TMUA 2021 Paper 1 Question 11

The function f is given by

$$\mathrm{f}(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$

Find the fraction of the interval $0 < x < 1$ for which $\mathrm{f}(x)$ is decreasing.

A $\frac{2}{15}$ **B** $\frac{1}{5}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$ **E** $\frac{2}{3}$ **F** $\frac{4}{5}$ **G** $\frac{13}{15}$

[See the next page for hints]

TMUA 2021 Paper 1 Question 12

The minimum value of the function $x^4 - p^2x^2$ is -9
p is a real number.
Find the minimum value of the function $x^2 - px + 6$

A -3 **B** $6 - \frac{3\sqrt{2}}{2}$ **C** $\frac{3}{2}$ **D** 3 **E** $\frac{9}{2}$ **F** $6 + \frac{3\sqrt{2}}{2}$

[See the next page for hints]

Hints

TMUA 2021 Paper 1 Question 11

- The question asks for the fraction of the interval where $f(x)$ is decreasing. We should try to find all values of x where $f(x)$ is decreasing, and then work out the fraction.
- We could multiply out $f(x)$. It's not going to be fun, but it will give us something we can work with.
- Don't bother sketching this function.
- "Decreasing" means something about the derivative $f'(x)$.
- Once you've worked out $f'(x)$, it will be messy. Try to simplify or factorise a bit, before you move to inequalities or turning points.
- Taking out a factor of $x^{1/7}$ from $f'(x)$ (the opposite of the advice in the first bullet point!) might help you work with the derivative.

TMUA 2021 Paper 1 Question 12

- You could differentiate both of these functions.
- Find expressions for both minimum values in terms of p.
- It's time for an equation; the given information tells you something about p, and you want to know the value of the other minimum. Hopefully this is a matter of solving one equation and substituting the value into your other expression.
- Don't panic if your equation for p does not have a unique solution. Check both, or convince yourself that it doesn't matter which solution you choose.

TMUA 2021 Paper 2 Question 6

Consider the following two statements about the polynomial $\mathrm{f}(x)$:

P: $\mathrm{f}(x) = 0$ for exactly three real values of x

Q: $\mathrm{f}'(x) = 0$ for exactly two real values of x

Which one of the following is correct?

A P is **necessary** but **not sufficient** for Q.

B P is **sufficient** but **not necessary** for Q.

C P is **necessary and sufficient** for Q.

D P is **not necessary** and **not sufficient** for Q.

[See the next page for hints]

TMUA 2021 Paper 2 Question 8

Consider the following statement about the polynomial $\mathrm{p}(x)$, where a and b are real numbers with $a < b$:

$(*)$ There exists a number c with $a < c < b$ such that $\mathrm{p}'(c) = 0$.

Which one of the following is true?

A The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary and sufficient** for $(*)$

B The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary** but **not sufficient** for $(*)$

C The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **sufficient** but **not necessary** for $(*)$

D The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **not necessary** and **not sufficient** for $(*)$

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 6

- Sketch a few polynomials with $f(x) = 0$ for exactly three real values of x. Do all of your sketches have exactly two points where $f'(x) = 0$? If yes, then perhaps P is sufficient for Q. Keep sketching!
- Sketch a few polynomials with exactly two points where $f'(x) = 0$. Do all of your sketches have exactly three real values of x for which $f(x) = 0$? If yes, then perhaps P is necessary for Q. Keep sketching!
- Returning to the first set of sketches, note that the points where $f'(x) = 0$ do not have to be "between" the roots, but there must be exactly two of them in total.
- This question might sound like it's about cubic polynomials. It is not about cubic polynomials.

TMUA 2021 Paper 2 Question 8

- This question looks quite similar to the previous one!
- Remember that "P is sufficient for Q" means "if P then Q".
- Remember that "P is necessary for Q" means "if Q then P".
- Sketch a few polynomials with $p(a) = p(b)$. Do all of your sketches have a point in between with zero derivative?
- Sketch a few polynomials with a point where the derivative is zero. Do all of your sketches have $p(a) = p(b)$? Really? I haven't even told you what a and b are yet!

TMUA 2022 Paper 1 Question 15
A rectangle is drawn in the region enclosed by the curves p and q, where

$$p(x) = 8 - 2x^2, \qquad q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x- and y-axes.
What is the maximum possible area of the rectangle?

A $\frac{26}{9}$ **B** $\frac{52}{9}$ **C** $\frac{4\sqrt{6}}{3}$ **D** $\frac{8\sqrt{6}}{3}$ **E** $4\sqrt{2}$ **F** $8\sqrt{2}$ **G** $\frac{20\sqrt{10}}{9}$ **H** $\frac{40\sqrt{10}}{9}$

[See the next page for hints]

TMUA 2022 Paper 2 Question 17
A student answered the following question:
"a and b are non-zero real numbers. Prove that the equation $x^3 + ax^2 + b = 0$ has three distinct real roots if $27b\left(b + \frac{4a^3}{27}\right) < 0$."
Here is the student's solution:

I We differentiate $y = x^3 + ax^2 + b$ to get $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax = x(3x + 2a)$.
Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ shows that the stationary points are at $(0, b)$ and $\left(-\frac{2a}{3}, b + \frac{4a^3}{27}\right)$

II If $27b\left(b + \frac{4a^3}{27}\right) < 0$, then b and $b + \frac{4a^3}{27}$ must have opposite signs, and so one of the stationary points is above the x-axis and one is below.

III If the cubic has three distinct real roots, then one of the stationary points is above the x-axis and one is below.

IV Hence if $27b\left(b + \frac{4a^3}{27}\right) < 0$, then the equation has three distinct real roots.

Which one of the following options best describes the student's solution?

A It is a completely correct solution.
B The student has instead proved the converse of the statement in the question.
C The solution is wrong, because the student should have stated step II after step III.
D The solution is wrong, because the student should have shown the converse of the result in step II.
E The solution is wrong, because the student should have shown the converse of the result in step III.

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 15

- Draw a diagram!
- Work out what the question means by "the region enclosed by the curves p and q". Does your diagram have a single finite region bounded by the curves?
- Sketch a little rectangle in the region with the sides parallel to the axes. You can make this rectangle larger by making it wider or taller. Whichever you choose, how far can you push this?
- Time for algebra; write down the height of the rectangle in terms of its width. Write down the area.

TMUA 2022 Paper 2 Question 17

- Read the student's solution line by line, but regularly remind yourself of what they're supposed to be proving.
- It's unlikely that the student has made any small algebraic mistakes (e.g. finding the stationary points incorrectly) because that wouldn't be a very interesting question! Indeed, none of the answers refer to the differentiation in step I.
- For this question, you need to know that the converse of "if [this] then [that]" is "if [that] then [this]". For example, the converse of "if it rains then I wear a coat" is "if I wear a coat then it rains". The converse of a true statement might be false (often, but not always).
- The statement that they're trying to prove is phrased as "[this] if [that]", which is a little bit awkward to read. We could re-write it as "Prove that if $27b\left(b + \frac{4a^3}{27}\right) < 0$ then the equation $x^3 + ax^2 + b = 0$ has three distinct real roots". That looks like their last line, which is a good sign for them I guess!
- I'm always looking out for lines where the student has previously shown [this], and now the student is proving "if [that] then [this]". That sets off an alarm bell for me, because the student is probably going to conclude "therefore [that]", which doesn't follow logically from the work they've done.
- Example; imagine that a student writes "2 is even. All multiples of 4 are even. Therefore 2 is a multiple of 4." The sentence in the middle isn't false, but it's not going in the right direction to match up nicely with the previous statement.

TMUA Specification (April 2025, Section 1 §MM7)

- Definite integration as related to the "area between a curve and an axis". The difference between finding a definite integral and finding the area between a curve and an axis is expected to be understood.
- Finding definite and indefinite integrals of x^n for n rational, $n \neq -1$, and related sums and differences, including expressions which require simplification prior to integrating.
- An understanding of the Fundamental Theorem of Calculus and its significance to integration. Simple examples of its use may be required in the forms:
  - $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ where $F'(x) = f(x)$.
  - $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$
- Combining integrals with either equal or contiguous ranges.
- Approximation of the area under a curve using the trapezium rule; determination of whether this constitutes an overestimate or an underestimate.
- Solving differential equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.

Revision

- A definite integral is written as $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ where a and b are real numbers sometimes called the two end-points of the integral, and where $f(x)$ is defined for $a \leqslant x \leqslant b$.
- The definite integral is a number (not, for instance, a function of x), and the value of that number depends on a and b and the function f. We sometimes say "dummy variable" to describe x inside the integral, and it's a sort of placeholder. The letter x is often used, but for the same function, same a, and same b, the integrals $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ or $\displaystyle\int_a^b f(q)\,\mathrm{d}q$ would mean the same thing and have the same value. The "$\mathrm{d}x$" at the end indicates which letter is the dummy variable.
- To calculate definite integrals, we first introduce the idea of an indefinite integral, written $\displaystyle\int f(x)\,\mathrm{d}x$ without limits. In contrast to the definite integral, the indefinite integral is a function of x. It is only defined up to a constant.
- The indefinite integral of x^n is $\dfrac{x^{n+1}}{n+1} + c$, provided that $n \neq -1$, where c is an arbitrary constant.

- Note that differentiation is the reverse of indefinite integration in the sense that if you differentiate $\dfrac{x^{n+1}}{n+1} + c$ then you get back to x^n. So we might say that we've identified a function $F(x)$ with derivative $F'(x) = f(x)$.

- If we want to calculate $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ and we know that the function $F(x)$ has $F'(x) = f(x)$, then the definite integral is just $F(b) - F(a)$. This is an example of something called the Fundamental Theorem of Calculus.

- Note that, if you find $F(x)$ with indefinite integration, then the arbitrary constant "$+c$" doesn't matter for your definite integral, because when you calculate $F(b) - F(a)$ that constant will cancel.

- The integral $\displaystyle\int_a^x f(t)\,\mathrm{d}t$ with constant a is worth thinking about. For any particular value of x, it's a definite integral, so we just get a number. But that makes it a function of x. The Fundamental Theorem of Calculus states that, as a function of x, it has derivative $f(x)$.

- Integration can be used to find areas, but we have to be careful.

- If $a < b$ and $f(x) > 0$ for $a < x < b$ then $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

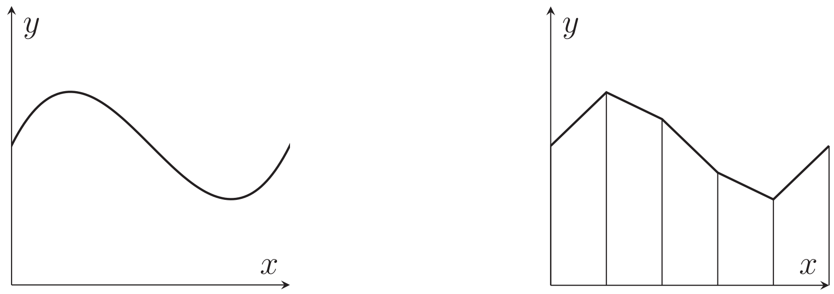
- If $a < b$ and $f(x) < 0$ for $a < x < b$ then $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is -1 times the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Areas are supposed to be positive! The integral here is sometimes called the "signed area" to reflect the fact that it's got a minus sign. This is a key difference between finding a definite integral and finding the area between a curve and the x-axis.

- If $f(x)$ is sometimes positive and sometimes negative in $a < x < b$ then we can identify separate regions where $f(x)$ is positive or negative before applying the above. The total area is the sum of the (positive!) areas of the separate parts.

- Some rules worth knowing;
  - $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$
  - $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$ for any constant k.
  - (Contiguous ranges) $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x.$
  - (Equal ranges) $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x = \int_a^b \Big(f(x) + g(x)\Big)\,\mathrm{d}x.$

- The trapezium rule gives us a way to estimate a definite integral by representing the curve with a series of straight lines ("chords"), and the region between the curve and the x-axis with a series of trapeziums. To estimate the area between $x = a$ and $x = b$ and the x-axis and the curve $y = f(x)$, we split the interval from a to b into n "strips" of equal width. This requires $n + 1$ points, and we use an arithmetic sequence with $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $\dots$, $x_n = b$, where $h = \dfrac{b-a}{n}$.
- The trapezium rule adds expressions for the signed areas of the trapeziums to get

$$\int_a^b f(x)\,\mathrm{d}x \;\approx\; \frac{1}{2} \times h \times \Big(f(x_0) + 2 \times \Big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\Big) + f(x_n)\Big)$$



- The trapezium rule gives an underestimate of the integral if the graph $y = f(x)$ curves downward (concave). This happens if the second derivative $f''(x)$ is negative throughout $a < x < b$. The chords lie below the curve.
- The trapezium rule gives an overestimate of the integral if the graph $y = f(x)$ curves upward (convex). This happens if the second derivative $f''(x)$ is positive throughout $a < x < b$. The chords lie above the curve.
- If $f''(x) = 0$ throughout $a < x < b$ then the graph $y = f(x)$ is a straight line and the trapezium rule is exactly correct. The chords lie on the curve.
- It's often the case that $f''(x)$ changes sign between a and b. In that case, the trapezium rule might give an underestimate or an overestimate, or it might even be exactly correct.
- Note that if $f(x) < 0$ then the area is -1 times the integral, so if the trapezium rule gives an underestimate for the (negative) integral, that will be an overestimate for the (positive) area.
- The solutions to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ are the indefinite integrals $y = \displaystyle\int f(x)\,\mathrm{d}x$. Remember to include the arbitrary constant "$+c$".

Revision Questions

1. Find the area enclosed between the polynomial $y = x^2 + 4x + 3$ and the x-axis.

2. Find
$$\int \frac{x+3}{x^3}\,\mathrm{d}x, \qquad \int \sqrt[3]{x}\,\mathrm{d}x, \qquad \int \left(\left(x^2\right)^3\right)^5 \mathrm{d}x, \qquad \int \left(x^2+1\right)^3 \mathrm{d}x$$

3. Suppose that $f(x) > 0$ for $-1 < x < 1$. By thinking about the area that the integral represents, explain why
$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(-x)\,\mathrm{d}x.$$

4. Let $I_1 = \displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$ and $I_2 = \displaystyle\int_{10}^{100} \frac{1}{x}\,\mathrm{d}x$. You are not expected to calculate either of these integrals. By considering a transformation of the graph $y = \dfrac{1}{x}$, and areas under that graph, prove that $I_1 = I_2$.

   Deduce that $\displaystyle\int_1^N \frac{1}{x}\,\mathrm{d}x$ with $N > 1$ can be made arbitrarily large by increasing N.

5. Suppose that $f(x) > 0$ for $-3 \leqslant x \leqslant 3$ and let
$$a = \int_0^1 f(x)\,\mathrm{d}x, \quad b = \int_1^2 f(x)\,\mathrm{d}x, \quad c = \int_2^3 f(x)\,\mathrm{d}x.$$

   Find the following in terms of a and b and c.

$$\text{(i)} \int_0^3 f(x)\,\mathrm{d}x, \qquad \text{(ii)} \int_3^1 f(x)\,\mathrm{d}x, \qquad \text{(iii)} \int_0^2 3f(x)\,\mathrm{d}x,$$
$$\text{(iv)} \int_0^2 f(x)\,\mathrm{d}x + \int_2^1 f(x)\,\mathrm{d}x, \qquad \text{(v)} \int_{-2}^0 f(-x)\,\mathrm{d}x, \qquad \text{(vi)} \int_0^1 f(2x)\,\mathrm{d}x.$$

   (For the last two, use graph transformations).

6. Let $I_3 = \displaystyle\int_1^3 \frac{1}{1+x^2}\,\mathrm{d}x$. Without calculating any of these integrals, find expressions for $\displaystyle\int_1^3 \frac{x^2}{1+x^2}\,\mathrm{d}x$ and $\displaystyle\int_1^3 \frac{x^4}{1+x^2}\,\mathrm{d}x$ in terms of I_3.

7. Use the Fundamental Theorem of Calculus to simplify $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_0^x t^2\,\mathrm{d}t\right)$. Check your answer by calculating the integral.

8. For $x > -2$, find an expression for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_x^{10} \sqrt{t^3+8}\,\mathrm{d}t\right)$.

9. Find an expression for $\displaystyle\int_x^x f(t)\,\mathrm{d}t$.

10. Use the trapezium rule to estimate $\displaystyle\int_0^1 4^x\,\mathrm{d}x$ using 4 strips.

    Is this an underestimate or an overestimate?

11. Use the trapezium rule to estimate $\displaystyle\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ using 3 strips.

    Is this an underestimate or an overestimate?

    Without using integration, find the exact value of this integral.

12. Find the solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x^2}$ that has $y \approx 3$ for large values of x.

TMUA Questions

TMUA 2020 Paper 2 Question 13

$\mathrm{f}(x)$ is a function for which

$$\int_0^3 \left(\mathrm{f}(x)\right)^2 \mathrm{d}x \;+\; \int_0^3 \mathrm{f}(x)\,\mathrm{d}x \;=\; \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$$

Which of the following claims about $\mathrm{f}(x)$ is/are **necessarily** true?

I $\quad \mathrm{f}(x) \leq 0$ for some x with $1 \leq x \leq 3$

II $\quad \displaystyle\int_0^3 \mathrm{f}(x)\,\mathrm{d}x \;\leq\; \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$

A neither of them
B I only
C II only
D I and II

[See the next page for hints]

TMUA 2020 Paper 2 Question 16

The Fundamental Theorem of Calculus (FTC) tells us that for any polynomial f:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \mathrm{f}(t)\,\mathrm{d}t\right) = \mathrm{f}(x)$$

A student calculates $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right)$ as follows:

(I) $\displaystyle\int_x^{2x} t^2\,\mathrm{d}t = \int_0^{2x} t^2\,\mathrm{d}t - \int_0^x t^2\,\mathrm{d}t$

(II) By FTC, $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x t^2\,\mathrm{d}t\right) = x^2$

(III) By FTC, $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) = (2x)^2 = 4x^2$

(IV) So $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 4x^2 - x^2$

(V) giving $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 3x^2$

Which of the following best describes the student's calculation?

A The calculation is completely correct.
B The calculation is incorrect, and the first error occurs on line (I).
C The calculation is incorrect, and the first error occurs on line (II).
D The calculation is incorrect, and the first error occurs on line (III).
E The calculation is incorrect, and the first error occurs on line (IV).
F The calculation is incorrect, and the first error occurs on line (V).

[See the next page for hints]

Hints

TMUA 2020 Paper 2 Question 13

- This question includes an integral from 0 to 3, an integral from 0 to 1, and a reference to the range $1 \leqslant x \leqslant 3$. What's the relationship between those things?
- Do not try to solve for $f(x)$.
- Note that we cannot assume that $\displaystyle\int (\mathrm{f}(x))^2 \, \mathrm{d}x = \left(\int \mathrm{f}(x) \, \mathrm{d}x \right)^2$.
- Note that $(\mathrm{f}(x))^2 \geqslant 0$. What does that tell you about the integral $\displaystyle\int_0^3 (\mathrm{f}(x))^2 \, \mathrm{d}x$?

TMUA 2020 Paper 2 Question 16

- You could always just do the integral yourself and then differentiate. This will reveal that the student's final answer is wrong. If need be, we could calculate the integrals on each line to see where things went wrong.
- Note that line (I) is fine for any function that's being integrated, using ideas from the Revision Notes.
- Line (II) is very much an application of the Fundamental Theorem of Calculus as written in the question.
- After line (III), the student's work is just substitution and tidying up, so those steps are easy to check.

TMUA 2021 Paper 1 Question 7

The function f is such that $\mathrm{f}(0) = 0$, and $x\mathrm{f}(x) > 0$ for all non-zero values of x.
It is given that

$$\int_{-2}^{2} \mathrm{f}(x)\,\mathrm{d}x = 4 \quad \text{and} \quad \int_{-2}^{2} |\mathrm{f}(x)|\,\mathrm{d}x = 8.$$

Evaluate

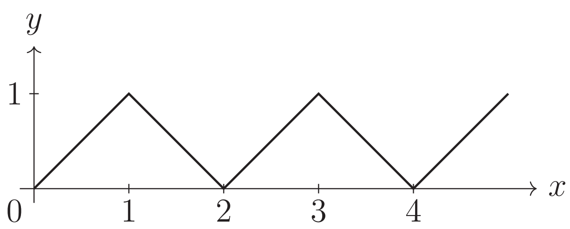
$$\int_{-2}^{0} \mathrm{f}(|x|)\,\mathrm{d}x$$

A -8 **B** -6 **C** -4 **D** -2 **E** 2 **F** 4 **G** 6 **H** 8

[See the next page for hints]

TMUA 2021 Paper 1 Question 15

The diagram shows the graph of $y = \mathrm{f}(x)$.



The graph consists of alternating straight-line segments of gradient 1 and -1 and continues in this way for all values of x.
The function g is defined as

$$\mathrm{g}(x) = \sum_{r=1}^{10} \mathrm{f}\left(2^{r-1}x\right)$$

Find the value of

$$\int_{0}^{1} \mathrm{g}(x)\,\mathrm{d}x$$

A $\frac{1023}{1024}$ **B** $\frac{1023}{512}$ **C** 5 **D** 10 **E** $\frac{55}{2}$ **F** 55

[See the next page for hints]

Hints

TMUA 2021 Paper 1 Question 7

- The inequality $x\mathrm{f}(x) > 0$ is an unusual condition. What does that tell you about $\mathrm{f}(x)$?
- If you know that $\mathrm{f}(x)$ is positive in some region(s) and negative in some region(s) then you can split up the integral of $|\mathrm{f}(x)|$ to think about those regions separately.
- Note that we're asked for the integral of $\mathrm{f}(|x|)$ not $\mathrm{f}(x)$, with $-2 \leqslant x \leqslant 0$. What can you say about $|x|$ for this range of x?
- Note that reflections do not change the areas of regions of the (x, y)-plane (remembering that areas are always positive).

TMUA 2021 Paper 1 Question 15

- The function $\mathrm{g}(x)$ is a sum of functions. Write out a few of the terms of the sum, including the first and last terms.
- We're going to handle these ten functions separately.
- What is $\displaystyle\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$?
- If you used integration for the previous hint, explain your answer to yourself in terms of an area.
- What does the graph of $y = \mathrm{f}(2x)$ look like for $0 \leqslant x \leqslant 1$?
- What does the graph of $y = \mathrm{f}(4x)$ look like for $0 \leqslant x \leqslant 1$?

TMUA 2021 Paper 2 Question 20

A sequence of functions $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3, \ldots$ is defined by

$$\mathrm{f}_1(x) = |x|$$

$$\mathrm{f}_{n+1}(x) = |\mathrm{f}_n(x) + x| \quad \text{for } n \geq 1$$

Find the value of

$$\int_{-1}^{1} \mathrm{f}_{99}(x)\,\mathrm{d}x$$

A 0 **B** 0.5 **C** 1 **D** 49.5 **E** 50 **F** 99 **G** 99.5 **H** 100

[See the next page for hints]

TMUA 2022 Paper 1 Question 7

Find the finite area enclosed between the line $y = 0$ and the curve $y = x^2 - 4|x| - 12$

A $\frac{128}{3}$ **B** $\frac{176}{3}$ **C** $\frac{256}{3}$ **D** 108 **E** 144 **F** 288

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 20

- 99 is a huge number. You're not expected to see what $\mathrm{f}_{99}(x)$ is just by looking at it.
- Do not "start at 99 and work down". It'll be many nested $|x|$ before you get back to anything that you can simplify.
- Instead, start small, and build up. What's $\mathrm{f}_2(x)$? Split into cases where $x \geqslant 0$ and $x < 0$ because that's how $|x|$ works.

TMUA 2022 Paper 1 Question 7

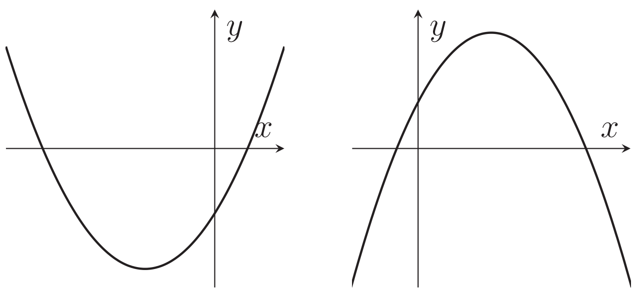
- Start by sketching the graph in the region $x \geqslant 0$, where $|x|$ is just x.
- For the region $x \leqslant 0$, simplify the expression $y = x^2 - 4|x| - 12$.
- How is the graph in $x \leqslant 0$ related to the graph in $x \geqslant 0$? Can we use that to save time with our integration?
- None of the options are negative. Why not?

TMUA Specification (April 2025, Section 1 §MM8)

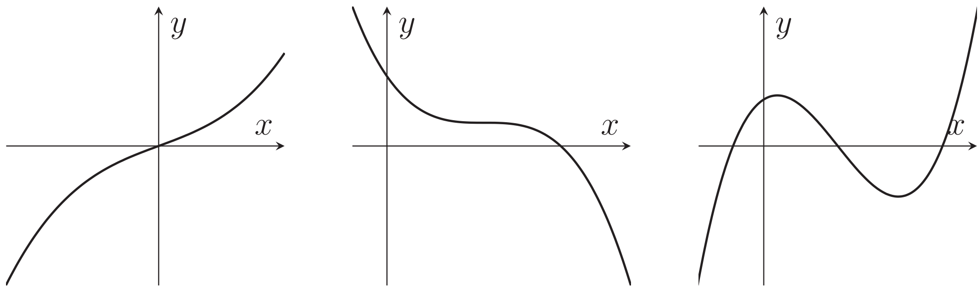
- Recognise and be able to sketch the graphs of common functions that appear in this specification: these include lines, quadratics, cubics, trigonometric functions, logarithmic functions, exponential functions, square roots, and the modulus function.
- Knowledge of the effect of simple transformations on the graph of $y = f(x)$ with positive or negative value of a as represented by:
  - $y = af(x)$
  - $y = f(x) + a$
  - $y = f(x + a)$
  - $y = f(ax)$
- Compositions of these transformations. Knowledge and use of the notation $f(g(x))$.
- Understand how altering the values of m and c affects the graph of $y = mx + c$.
- Understand how altering the values of a, b and c in $y = a(x + b)^2 + c$ affects the corresponding graph.
- Use differentiation to help determine the shape of the graph of a given function, including:
  - finding stationary points (excluding inflexions)
  - when the graph is increasing or decreasing
- Use algebraic techniques to determine where the graph of a function intersects the coordinate axes; appreciate the possible numbers of real roots that a general polynomial can possess.
- Geometric interpretation of algebraic solutions of equations; relationship between the intersections of two graphs and the solutions of the corresponding simultaneous equations.

Revision

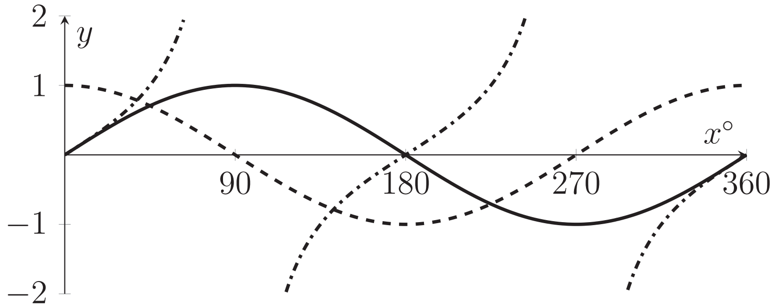
- The graph of an equation involving x and y is all the points in the (x, y) plane that satisfy the equation. For a function $f(x)$, the graph of $y = f(x)$ shows the value of f at each value of x.
- Quadratics $y = ax^2 + bx + c$ have graphs like these



- Cubics $y = ax^3 + bx^2 + cx + d$ can have 0 or 1 or 2 stationary points.

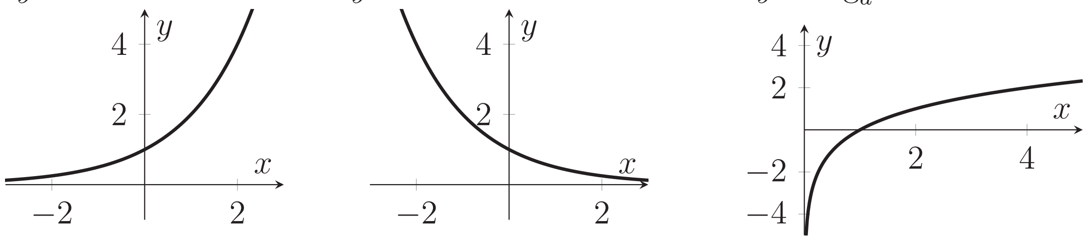


- Other polynomials have graphs that might have more stationary points; up to $(n-1)$ of them for a polynomial of degree n.
- Graphs of $y = \sin x$ (solid line) and $y = \cos x$ (dashed line) and $y = \tan x$ (dot-dashed line);



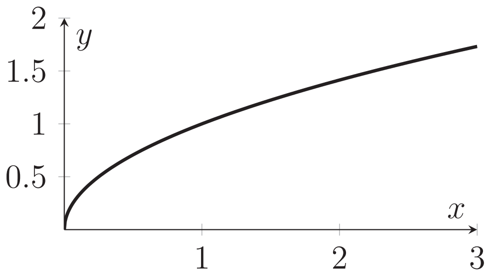
- Here are some graphs of exponential functions a^x and the logarithmic function $\log_a x$. Note that $\log_a x$ is very negative for x close to zero, if $a > 1$.

$y = a^x$ with $a > 1$ $y = a^x$ with $0 < a < 1$ $y = \log_a x$ with $a > 1$

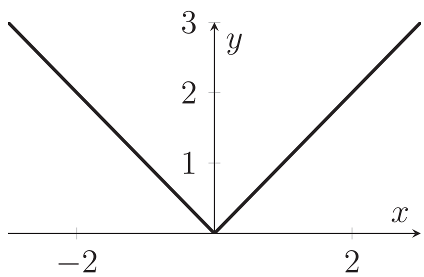


- Here are graphs of the square root function $\sqrt{x}$ and the modulus function $|x|$. For $\sqrt{x}$, note that $\sqrt{x} = x^{1/2}$ so the derivative is $\frac{1}{2}x^{-1/2}$, which gets arbitrarily large near $x = 0$.

$y = \sqrt{x}$



$y = |x|$



- The graph of $y = f(x - a)$ is the translation of the graph of $y = f(x)$ by a distance a in the positive x-direction.

- The graph of $y = f(x) + a$ is the translation of the graph of $y = f(x)$ by a distance a in the positive y-direction.

- The graph of $y = f(ax)$ is a stretch of the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ parallel to the x-axis. If $a = -1$ then the transformation is a reflection in the y-axis. If a is some other negative number then the transformation is a combination of that reflection and a stretch parallel to the x-axis.

- The graph of $y = af(x)$ is a stretch of the graph of $y = f(x)$ by a factor of a parallel to the y-axis. If $a = -1$ then the transformation is a reflection in the x-axis. If a is some other negative number then the transformation is a combination of that reflection and a stretch parallel to the y-axis.

- If you do two or more transformations then (depending on the transformations) the order in which you do them might matter.

- The notation $f(g(x))$ means that we take the value of x, apply the function g, and then apply the function f to that result. So if $f(x) = x^2$ and $g(x) = x + 1$ then $f(g(x)) = (x + 1)^2$. The brackets are supposed to help, but note that sometimes people write $fg(x)$ to mean $f(g(x))$.

- Recall that you can use differentiation to find the stationary points of a curve $y = f(x)$, and to find out whether the function is increasing or decreasing.

- If you want to find the points where the graph of a function $y = f(x)$ intersects the x-axis, then you need to find the solutions to $f(x) = 0$. If there are no solutions then there are no such points.

- If you want to find the point where the graph of a function $y = f(x)$ intersects the y-axis, then you just need to calculate $f(0)$. If this is not defined then there is no such point.

- A polynomial with degree n has at most n real roots.

- If you have two equations, you could graph the points that satisfy each equation and look for points that lie on both graphs. Any such point satisfies both equations simultaneously.

Revision Questions

1. Let $f(x) = x^2 + 4x + 3$. Sketch the graph of $y = f(x + 2)$.

   Sketch the graph of $y = 3f(2x)$.

   Sketch the graph of $y = 2f(3x)$. Is that the same as the previous graph?

   Give an example of a function $g(x)$ such that $y = 5g(4x)$ and $y = 4g(5x)$ have the same graph.

2. Let $f(x) = x^3 - x$. Sketch the graph of $y = 2f(x + 1)$.

   Sketch the graph of $y = 2f(x) + 1$. Is that the same as the previous graph?

   Give an example of a function $g(x)$ such that $y = 3g(x) + 2$ and $y = 3g(x + 2)$ have the same graph.

3. Sketch the graph of $y = x^n$ for various values of n; large, small, or negative.

4. Sketch the graph of $\sqrt{4x + 1}$ for $x \geqslant -\frac{1}{4}$.

5. Sketch the graph of $y = \sqrt{x^2}$.

6. Sketch the graph of $y = \sin(x^2)$.

7. Sketch the graph of $y = \log_2 x$. Sketch the graph of $y = \log_2(x^2 - 2x + 1)$.

8. Sketch the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ on the same axes. Describe the relationship between the graphs.

9. Sketch the graph of $y = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$.

10. Sketch all the points (x, y) that satisfy $y = 4 - x$.

    Sketch all the points (x, y) that satisfy $y = 4 - x^2$.

    Sketch all the points (x, y) that satisfy $y^2 = 4 - x^2$.

11. Let $f(x) = \cos x$. Sketch all the points (x, y) that satisfy $f(x) = f(y)$.

12. Let $f(x) = x^3 - x$. Sketch all the points (x, y) that satisfy $f(x) = f(y)$.

13. Sketch all the points (x, y) that satisfy $x^4 + 2x^2y^2 + y^4 - 3x^2 - 3y^2 + 2 = 0$.

14. Sketch all the points (x, y) that satisfy $x^6 + 3x^4y^2 + 3x^2y^4 + y^6 = 1$.

15. Sketch all the points (x, y) that satisfy $xy + x^2y^2 = x^3 + y^3$.

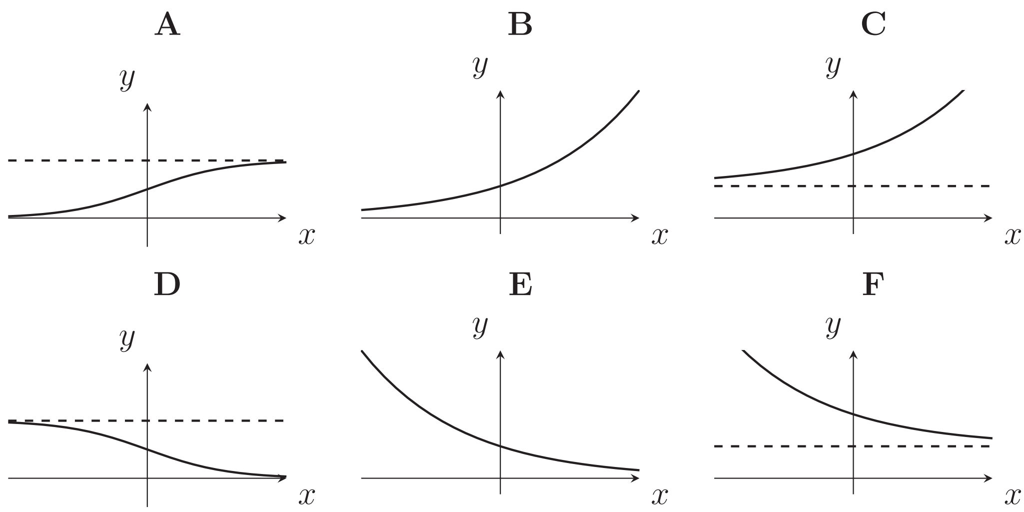
TMUA Questions

TMUA 2020 Paper 2 Question 5

Which one of the following shows the graph of

$$y = \frac{2^x}{1 + 2^x}$$

(Dotted lines indicate asymptotes.)



[See the next page for hints]

TMUA 2021 Paper 2 Question 13

A region R in the (x, y)-plane is defined by the simultaneous inequalities

$$y - x < 3$$

$$y - x^2 < 1$$

Which of the following statements is/are true for **every** point in R?

I $\quad -1 < x < 2$

II $\quad (y - x)(y - x^2) < 3$

III $\quad y < 5$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

[See the next page for hints]

Hints

TMUA 2020 Paper 2 Question 5

- Think about the approximate value of y when $|x|$ is very large, considering the cases $x > 0$ and $x < 0$ separately.
- You might find it helpful to define $f(x) = \dfrac{x}{1+x}$ and $g(x) = 2^x$, thinking of the graph in the question as $y = f(g(x))$.

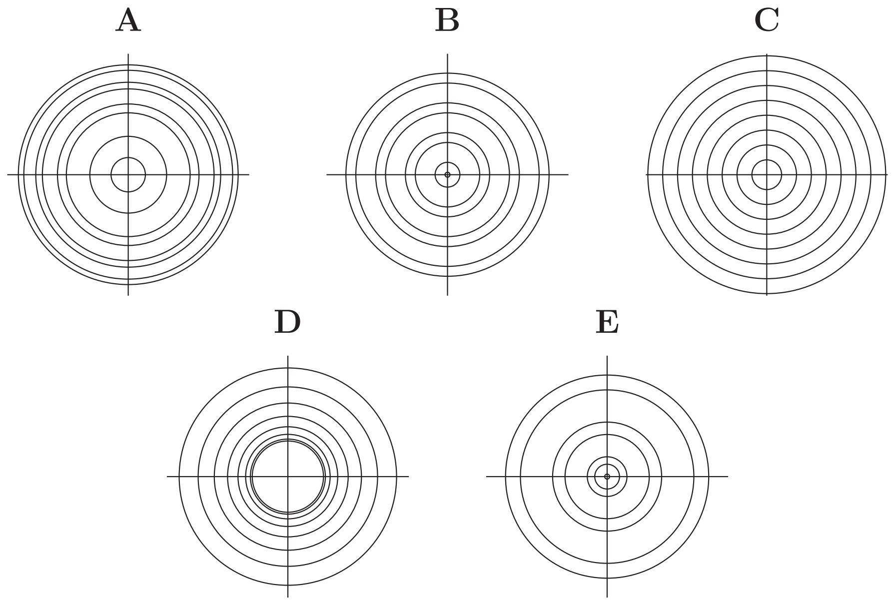
TMUA 2021 Paper 2 Question 13

- Sketch a graph of $y - x = 3$ and a graph of $y - x^2 = 1$ on the same axes.
- Find the point(s) where those graphs cross the axes or each other.
- Think carefully about which regions obey each of the inequalities and therefore which region is R.
- For I and III, we're being asked if the region is bounded by particular horizontal / vertical lines. Now that you've got a sketch, do you agree that all points in R are bounded by such lines?
- For II, remember that multiplying inequalities is "dangerous". Note that a single counterexample (a point in R that does not satisfy this inequality) would be enough to show that it is not true for **every** point in R.

TMUA 2021 Paper 1 Question 17
Which of the following sketches shows the graph of

$$\sin(x^2 + y^2) = \frac{1}{2}$$

where $x^2 + y^2 \leq 8\pi$?
[This question uses radians]



[See the next page for hints]

TMUA 2021 Paper 1 Question 18
The curve with equation

$$x = y^2 - 6y + 11$$

is rotated 90° clockwise about the point P to give the curve C.
P has x-coordinate -2 and y-coordinate 3.
What is the equation of C?

- **A** $y = -x^2 - 4x - 3$
- **B** $y = -x^2 - 4x - 5$
- **C** $y = -x^2 - 6x - 7$
- **D** $y = -x^2 - 6x - 11$
- **E** $y = x^2 - 4x + 5$
- **F** $y = x^2 + 4x + 3$
- **G** $y = x^2 - 6x + 11$
- **H** $y = x^2 + 6x + 7$

[See the next page for hints]

Hints

TMUA 2021 Paper 1 Question 17

- You know solutions for x such that $\sin x = \frac{1}{2}$. In this question, those solutions will give different parts of the graph.
- What would a graph look like if $x^2 + y^2$ were some constant?
- If the constant c increases steadily, what happens to $\sqrt{c}$?

TMUA 2021 Paper 1 Question 18

- Sketch the curve with equation $x = y^2 - 6y + 11$, and mark the point P on your sketch.
- The curve is a parabola, and after rotation we might expect a parabola.
- Sketch the graph that you expect after a $90°$ clockwise rotation about P.
- Describe the relationship between the initial graph and the point P, and then describe the relationship between the new graph and the point P.
- If we can identify the location of the turning point of the new graph, then that is (almost) good enough!

TMUA 2022 Paper 1 Question 10

A sequence of translations is applied to the graph of $y = x^3$

Which of the following graphs could be the result of this sequence of translations?

I $\quad y = x^3 - 3x^2 + 9x - 27$

II $\quad y = x^3 - 9x^2 + 27x - 3$

III $\quad y = 27x^3 - 9x^2 + x - 3$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

[See the next page for hints]

TMUA 2022 Paper 1 Question 18

It is given that

$$\mathrm{f}(x) = x^2(x-1)^2(x-2)$$

$$\mathrm{g}(x) = -p(x-q)^2(x-r)^2$$

where p, q and r are positive and $q < r$

Find the set of values of q and r that guarantees the greatest number of distinct real solutions of the equation $\mathrm{f}(x) = \mathrm{g}(x)$ for all p.

A $q < 1$ and $r < 1$

B $q < 1$ and $1 < r < 2$

C $q < 1$ and $r > 2$

D $1 < q < 2$ and $1 < r < 2$

E $1 < q < 2$ and $r > 2$

F $q > 2$ and $r > 2$

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 10

- The effect of "a sequence of translations" is not as complicated as it sounds. Do not write out a sequence like T_1, T_2, ... for the translations. Think about what the overall effect could be.
- What happens to the equation $y = x^3$ if you translate parallel to the x-axis by a units and then translate parallel to the y-axis by b units?
- For each of I, II, and III, we must decide whether it's possible for such an equation to be the result of such a transformation. We could try to solve for a and b in each case.

TMUA 2022 Paper 1 Question 18

- Both $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are polynomials. The "greatest number of distinct real solutions" of the equation $\mathrm{f}(x) = \mathrm{g}(x)$ is something to do with the degrees of these polynomials.
- Sketch $y = \mathrm{f}(x)$, which does not depend on p or q or r.
- Imagining for a moment that $p = 1$, separately sketch $y = -(x-q)^2(x-r)^2$. What is the effect of changing p to some other positive real number?
- We're being asked whether we would like the roots of $\mathrm{g}(x)$ to be between the roots of $\mathrm{f}(x)$ or not. Our aim is to make sure that, for any value of p, the number of crossings between f and g is as high as it could be, independent of the value of p.
- What happens when x is very negative? Which of the two functions has the more negative value?
- A crossing is guaranteed if there's a change of sign from $\mathrm{f}(x) < \mathrm{g}(x)$ to $\mathrm{f}(x) > \mathrm{g}(x)$ in some region.