British Mathematical Olympiad Past Paper

1993 – 2019 Collection

Last updated: August 19, 2020

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. Rough work *should* be handed in, but should be clearly marked.
4. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
5. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
6. Staple all the pages neatly together in the top *left* hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.
7. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 25 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 25 January (as defined locally).
8. In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (2–7 April 2019). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Bath, United Kingdom 11–22 July 2019) will then be chosen.
9. **Do not turn over until told to do so.**

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1. A list of five two-digit positive integers is written in increasing order on a blackboard. Each of the five integers is a multiple of 3, and each digit 0,1,2,3,4,5,6,7,8,9 appears exactly once on the blackboard. In how many ways can this be done? *Note that a two-digit number cannot begin with the digit 0.*

2. For each positive integer $n \geq 3$, we define an n-ring to be a circular arrangement of n (not necessarily different) positive integers such that the product of every three neighbouring integers is n. Determine the number of integers n in the range $3 \leq n \leq 2018$ for which it is possible to form an n-ring.

3. Ares multiplies two integers which differ by 9. Grace multiplies two integers which differ by 6. They obtain the same product T. Determine all possible values of T.

4. Let Γ be a semicircle with diameter AB. The point C lies on the diameter AB and points E and D lie on the arc BA, with E between B and D. Let the tangents to Γ at D and E meet at F. Suppose that $\angle ACD = \angle ECB$.

Prove that $\angle EFD = \angle ACD + \angle ECB$.

5. Two solid cylinders are mathematically similar. The sum of their heights is 1. The sum of their surface areas is 8π. The sum of their volumes is 2π. Find all possibilities for the dimensions of each cylinder.

6. Ada the ant starts at a point O on a plane. At the start of each minute she chooses North, South, East or West, and marches 1 metre in that direction. At the end of 2018 minutes she finds herself back at O. Let n be the number of possible journeys which she could have made. What is the highest power of 10 which divides n?

1. Let ABC be a triangle. Let L be the line through B perpendicular to AB. The perpendicular from A to BC meets L at the point D. The perpendicular bisector of BC meets L at the point P. Let E be the foot of the perpendicular from D to AC.

Prove that triangle BPE is isosceles.

2. For some integer n, a set of n^2 magical chess pieces arrange themselves on a square $n^2 \times n^2$ chessboard composed of n^4 unit squares. At a signal, the chess pieces all teleport to another square of the chessboard such that the distance between the centres of their old and new squares is n. The chess pieces win if, both before and after the signal, there are no two chess pieces in the same row or column. For which values of n can the chess pieces win?

3. Let p be an odd prime. How many non-empty subsets of

$$\{1, 2, 3, \ldots, p-2, p-1\}$$

have a sum which is divisible by p?

4. Find all functions f from the positive real numbers to the positive real numbers for which $f(x) \leq f(y)$ whenever $x \leq y$ and

$$f(x^4) + f(x^2) + f(x) + f(1) = x^4 + x^2 + x + 1$$

for all $x > 0$.

1. Helen divides 365 by each of $1, 2, 3, \ldots, 365$ in turn, writing down a list of the 365 remainders. Then Phil divides 366 by each of $1, 2, 3, \ldots, 366$ in turn, writing down a list of the 366 remainders. Whose list of remainders has the greater sum and by how much?

2. In a 100-day period, each of six friends goes swimming on exactly 75 days. There are n days on which at least five of the friends swim. What are the largest and smallest possible values of n?

3. The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

4. Consider sequences $a_1, a_2, a_3, \ldots$ of positive real numbers with $a_1 = 1$ and such that
$$a_{n+1} + a_n = (a_{n+1} - a_n)^2$$
for each positive integer n. How many possible values can a_{2017} take?

5. If we take a 2×100 (or 100×2) grid of unit squares, and remove alternate squares from a long side, the remaining 150 squares form a 100-*comb*. Henry takes a 200×200 grid of unit squares, and chooses k of these squares and colours them so that James is unable to choose 150 uncoloured squares which form a 100-comb. What is the smallest possible value of k?

6. Matthew has a deck of 300 cards numbered 1 to 300. He takes cards out of the deck one at a time, and places the selected cards in a row, with each new card added at the right end of the row. Matthew must arrange that, at all times, the mean of the numbers on the cards in the row is an integer. If, at some point, there is no card remaining in the deck which allows Matthew to continue, then he stops.

   When Matthew has stopped, what is the smallest possible number of cards that he could have placed in the row? Give an example of such a row.

1. Consider triangle ABC. The midpoint of AC is M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.

2. There are n places set for tea around a circular table, and every place has a small cake on a plate. Alice arrives first, sits at the table, and eats her cake (but it isn't very nice). Next the Mad Hatter arrives, and tells Alice that she will have a lonely tea party, and that she must keep on changing her seat, and each time she must eat the cake in front of her (if it has not yet been eaten). In fact the Mad Hatter is very bossy, and tells Alice that, for $i = 1, 2, \ldots, n-1$, when she moves for the i-th time, she must move a_i places and he hands Alice the list of instructions $a_1, a_2, \ldots, a_{n-1}$. Alice does not like the cakes, and she is free to choose, at every stage, whether to move clockwise or anticlockwise. For which values of n can the Mad Hatter force Alice to eat all the cakes?

3. It is well known that, for each positive integer n,
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
and so is a square. Determine whether or not there is a positive integer m such that
$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3$$
is a square.

4. Let f be a function defined on the real numbers and taking real values. We say that f is *absorbing* if $f(x) \le f(y)$ whenever $x \le y$ and $f^{2018}(z)$ is an integer for all real numbers z.

   a) Does there exist an absorbing function f such that $f(x)$ is an integer for only finitely many values of x?

   b) Does there exist an absorbing function f and an increasing sequence of real numbers $a_1 < a_2 < a_3 < \ldots$ such that $f(x)$ is an integer only if $x = a_i$ for some i?

Note that if k is a positive integer and f is a function, then f^k denotes the composition of k copies of f. For example $f^3(t) = f(f(f(t)))$ for all real numbers t.

1. The integers $1, 2, 3, \ldots, 2016$ are written down in base 10, each appearing exactly once. Each of the digits from 0 to 9 appears many times in the list. How many of the digits in the list are odd? *For example, 8 odd digits appear in the list* $1, 2, 3, \ldots, 11$.

2. For each positive real number x, we define $\{x\}$ to be the greater of x and $1/x$, with $\{1\} = 1$. Find, with proof, all positive real numbers y such that
$$5y\{8y\}\{25y\} = 1.$$

3. Determine all pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$.

4. Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.

5. Let ABC be a triangle with $\angle A < \angle B < 90°$ and let Γ be the circle through A, B and C. The tangents to Γ at A and C meet at P. The line segments AB and PC produced meet at Q. It is given that
$$[ACP] = [ABC] = [BQC].$$
Prove that $\angle BCA = 90°$. *Here* $[XYZ]$ *denotes the area of triangle* XYZ.

6. Consecutive positive integers $m, m+1, m+2$ and $m+3$ are divisible by consecutive odd positive integers $n, n+2, n+4$ and $n+6$ respectively. Determine the smallest possible m in terms of n.

1. This problem concerns triangles which have vertices with integer coordinates in the usual x, y-coordinate plane. For how many positive integers $n < 2017$ is it possible to draw a right-angled isosceles triangle such that exactly n points on its perimeter, including all three of its vertices, have integer coordinates?

2. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to the real number x. Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor\right)$$

for integers $n \geq 1$. Prove that $a_{n+1} > a_n$ for infinitely many n, and determine whether $a_{n+1} < a_n$ for infinitely many n.

[Here are some examples of the use of $\lfloor x \rfloor$: $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ and $\lfloor \frac{2017}{1000} \rfloor = 2$.]

3. Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P, and the rays AD and BC meet at Q. The internal angle bisector of angle $\angle BQA$ meets AC at R and the internal angle bisector of angle $\angle APD$ meets AD at S. Prove that RS is parallel to CD.

4. Bobby's booby-trapped safe requires a 3-digit code to unlock it. Alex has a probe which can test combinations without typing them on the safe. The probe responds *Fail* if no individual digit is correct. Otherwise it responds *Close*, including when all digits are correct. For example, if the correct code is 014, then the responses to 099 and 014 are both Close, but the response to 140 is Fail. If Alex is following an optimal strategy, what is the smallest number of attempts needed to guarantee that he knows the correct code, whatever it is?

1. On Thursday 1st January 2015, Anna buys one book and one shelf. For the next two years, she buys one book every day and one shelf on alternate Thursdays, so she next buys a shelf on 15th January 2015. On how many days in the period Thursday 1st January 2015 until (and including) Saturday 31st December 2016 is it possible for Anna to put all her books on all her shelves, so that there is an equal number of books on each shelf?

2. Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

3. Suppose that a sequence $t_0, t_1, t_2, \ldots$ is defined by a formula $t_n = An^2 + Bn + C$ for all integers $n \geq 0$. Here A, B and C are real constants with $A \neq 0$. Determine values of A, B and C which give the greatest possible number of successive terms of the sequence which are also successive terms of the Fibonacci sequence. *The Fibonacci sequence is defined by* $F_0 = 0$, $F_1 = 1$ *and* $F_m = F_{m-1} + F_{m-2}$ *for* $m \geq 2$.

4. James has a red jar, a blue jar and a pile of 100 pebbles. Initially both jars are empty. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the *state* of the game. The following conditions must be satisfied:

   a) The red jar may never contain fewer pebbles than the blue jar;
   b) The game may never be returned to a previous state.

   What is the maximum number of moves that James can make?

5. Let ABC be a triangle, and let D, E and F be the feet of the perpendiculars from A, B and C to BC, CA and AB respectively. Let P, Q, R and S be the feet of the perpendiculars from D to BA, BE, CF and CA respectively. Prove that P, Q, R and S are collinear.

6. A positive integer is called *charming* if it is equal to 2 or is of the form $3^i 5^j$ where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

1. Circles of radius r_1, r_2 and r_3 touch each other externally, and they touch a common tangent at points A, B and C respectively, where B lies between A and C. Prove that $16(r_1+r_2+r_3) \geq 9(AB+BC+CA)$.

2. Alison has compiled a list of 20 hockey teams, ordered by how good she thinks they are, but refuses to share it. Benjamin may mention three teams to her, and she will then choose either to tell him which she thinks is the weakest team of the three, or which she thinks is the strongest team of the three. Benjamin may do this as many times as he likes. Determine the largest N such that Benjamin can guarantee to be able to find a sequence $T_1, T_2, \ldots, T_N$ of teams with the property that he knows that Alison thinks that T_i is better than T_{i+1} for each $1 \leq i < N$.

3. Let $ABCD$ be a cyclic quadrilateral. The diagonals AC and BD meet at P, and DA and CB produced meet at Q. The midpoint of AB is E. Prove that if PQ is perpendicular to AC, then PE is perpendicular to BC.

4. Suppose that p is a prime number and that there are different positive integers u and v such that p^2 is the mean of u^2 and v^2. Prove that $2p-u-v$ is a square or twice a square.

1. Place the following numbers in increasing order of size, and justify your reasoning:
$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3} \text{ and } 4^{3^4}.$$
Note that a^{b^c} means $a^{(b^c)}$.

2. Positive integers p, a and b satisfy the equation $p^2 + a^2 = b^2$. Prove that if p is a prime greater than 3, then a is a multiple of 12 and $2(p + a + 1)$ is a perfect square.

3. A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?

4. Let x be a real number such that $t = x + x^{-1}$ is an integer greater than 2. Prove that $t_n = x^n + x^{-n}$ is an integer for all positive integers n. Determine the values of n for which t divides t_n.

5. Let $ABCD$ be a cyclic quadrilateral. Let F be the midpoint of the arc AB of its circumcircle which does not contain C or D. Let the lines DF and AC meet at P and the lines CF and BD meet at Q. Prove that the lines PQ and AB are parallel.

6. Determine all functions $f(n)$ from the positive integers to the positive integers which satisfy the following condition: whenever a, b and c are positive integers such that $1/a + 1/b = 1/c$, then
$$1/f(a) + 1/f(b) = 1/f(c).$$

1. The first term x_1 of a sequence is 2014. Each subsequent term of the sequence is defined in terms of the previous term. The iterative formula is
$$x_{n+1} = \frac{(\sqrt{2}+1)x_n - 1}{(\sqrt{2}+1) + x_n}.$$
Find the 2015th term x_{2015}.

2. In Oddesdon Primary School there are an odd number of classes. Each class contains an odd number of pupils. One pupil from each class will be chosen to form the school council. Prove that the following two statements are logically equivalent.

   a) There are more ways to form a school council which includes an odd number of boys than ways to form a school council which includes an odd number of girls.
   b) There are an odd number of classes which contain more boys than girls.

3. Two circles touch one another internally at A. A variable chord PQ of the outer circle touches the inner circle. Prove that the locus of the incentre of triangle AQP is another circle touching the given circles at A. *The* incentre *of a triangle is the centre of the unique circle which is inside the triangle and touches all three sides. A* locus *is the collection of all points which satisfy a given condition.*

4. Given two points P and Q with integer coordinates, we say that P *sees* Q if the line segment PQ contains no other points with integer coordinates. An *n-loop* is a sequence of n points $P_1, P_2, \ldots, P_n$, each with integer coordinates, such that the following conditions hold:
   a) P_i sees P_{i+1} for $1 \le i \le n-1$, and P_n sees P_1;
   b) No P_i sees any P_j apart from those mentioned in (a);
   c) No three of the points lie on the same straight line.

   Does there exist a 100-loop?

1. Calculate the value of
$$\frac{2014^4 + 4 \times 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \times 2013^4}{2013^2 + 4025^2}.$$

2. In the acute-angled triangle ABC, the foot of the perpendicular from B to CA is E. Let l be the tangent to the circle ABC at B. The foot of the perpendicular from C to l is F. Prove that EF is parallel to AB.

3. A number written in base 10 is a string of 3^{2013} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

4. Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?

5. Let ABC be an equilateral triangle, and P be a point inside this triangle. Let D, E and F be the feet of the perpendiculars from P to the sides BC, CA and AB respectively. Prove that

   a) $AF + BD + CE = AE + BF + CD$ and
   b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

   The area of triangle XYZ is denoted $[XYZ]$.

6. The angles A, B and C of a triangle are measured in degrees, and the lengths of the opposite sides are a, b and c respectively. Prove that
$$60 \leq \frac{aA + bB + cC}{a + b + c} < 90.$$

1. Every diagonal of a regular polygon with 2014 sides is coloured in one of n colours. Whenever two diagonals cross in the interior, they are of different colours. What is the minimum value of n for which this is possible?

2. Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. *The perimeter of a cuboid is the sum of the lengths of all its twelve edges.*

3. Let $a_0 = 4$ and define a sequence of terms using the formula $a_n = a_{n-1}^2 - a_{n-1}$ for each positive integer n.

   a) Prove that there are infinitely many prime numbers which are factors of at least one term in the sequence;

   b) Are there infinitely many prime numbers which are factors of no term in the sequence?

4. Let ABC be a triangle and P be a point in its interior. Let AP meet the circumcircle of ABC again at A'. The points B' and C' are similarly defined. Let O_A be the circumcentre of BCP. The circumcentres O_B and O_C are similarly defined. Let ${O_A}'$ be the circumcentre of $B'C'P$. The circumcentres ${O_B}'$ and ${O_C}'$ are similarly defined. Prove that the lines $O_A{O_A}', O_B{O_B}'$ and $O_C{O_C}'$ are concurrent.

1. Isaac places some counters onto the squares of an 8 by 8 chessboard so that there is at most one counter in each of the 64 squares. Determine, with justification, the maximum number that he can place without having five or more counters in the same row, or in the same column, or on either of the two long diagonals.

2. Two circles S and T touch at X. They have a common tangent which meets S at A and T at B. The points A and B are different. Let AP be a diameter of S. Prove that B, X and P lie on a straight line.

3. Find all real numbers x, y and z which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.

4. Find all positive integers n such that $12n - 119$ and $75n - 539$ are both perfect squares.

5. A triangle has sides of length at most 2, 3 and 4 respectively. Determine, with proof, the maximum possible area of the triangle.

6. Let ABC be a triangle. Let S be the circle through B tangent to CA at A and let T be the circle through C tangent to AB at A. The circles S and T intersect at A and D. Let E be the point where the line AD meets the circle ABC. Prove that D is the midpoint of AE.

1. Are there infinitely many pairs of positive integers (m, n) such that both m divides $n^2 + 1$ and n divides $m^2 + 1$?

2. The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.

3. Consider the set of positive integers which, when written in binary, have exactly 2013 digits and more 0s than 1s. Let n be the number of such integers and let s be their sum. Prove that, when written in binary, $n + s$ has more 0s than 1s.

4. Suppose that $ABCD$ is a square and that P is a point which is on the circle inscribed in the square. Determine whether or not it is possible that PA, PB, PC, PD and AB are all integers.

1. Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

2. Consider the numbers $1, 2, \ldots, n$. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

4. Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   a) Remove an equal number of balls from each bag;
   b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations?

   Operation b) is now replaced with

   b′) Triple the number of balls in one bag.

   Is it now always possible to empty both bags after a finite sequence of operations?

5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

6. Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

   The altitude from A is the line through A which is perpendicular to BC. The foot of this altitude is the point D where it meets BC. The other altitudes are similarly defined.

1. The diagonals AC and BD of a cyclic quadrilateral meet at E. The midpoints of the sides AB, BC, CD and DA are P, Q, R and S respectively. Prove that the circles EPS and EQR have the same radius.

2. A function f is defined on the positive integers by $f(1) = 1$ and, for $n > 1$,
$$f(n) = f\left(\left\lfloor \frac{2n-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{2n}{3} \right\rfloor\right)$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Is it true that $f(n) - f(n-1) \leq n$ for all $n > 1$?

   [Here are some examples of the use of $\lfloor x \rfloor$: $\lfloor \pi \rfloor = 3$, $\lfloor 1729 \rfloor = 1729$ *and* $\lfloor \frac{2012}{1000} \rfloor = 2$.*]*

3. The set of real numbers is split into two subsets which do not intersect. Prove that for each pair (m, n) of positive integers, there are real numbers $x < y < z$ all in the same subset such that $m(z-y) = n(y-x)$.

4. Show that there is a positive integer k with the following property: if a, b, c, d, e and f are integers and m is a divisor of
$$a^n + b^n + c^n - d^n - e^n - f^n$$
for all integers n in the range $1 \leq n \leq k$, then m is a divisor of $a^n + b^n + c^n - d^n - e^n - f^n$ for all positive integers n.

1. One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

2. Let s be an integer greater than 6. A solid cube of side s has a square hole of side $x < 6$ drilled directly through from one face to the opposite face (so the drill removes a cuboid). The volume of the remaining solid is numerically equal to the total surface area of the remaining solid. Determine all possible integer values of x.

3. Let ABC be a triangle with $\angle CAB$ a right-angle. The point L lies on the side BC between B and C. The circle ABL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M and N lie on a straight line.

4. Isaac has a large supply of counters, and places one in each of the 1×1 squares of an 8×8 chessboard. Each counter is either red, white or blue. A particular pattern of coloured counters is called an *arrangement*. Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters. *Note that* 0 *is an even number.*

5. Circles S_1 and S_2 meet at L and M. Let P be a point on S_2. Let PL and PM meet S_1 again at Q and R respectively. The lines QM and RL meet at K. Show that, as P varies on S_2, K lies on a fixed circle.

6. Let a, b and c be the lengths of the sides of a triangle. Suppose that $ab + bc + ca = 1$. Show that $(a + 1)(b + 1)(c + 1) < 4$.

1. Let ABC be a triangle and X be a point inside the triangle. The lines AX, BX and CX meet the circle ABC again at P, Q and R respectively. Choose a point U on XP which is between X and P. Suppose that the lines through U which are parallel to AB and CA meet XQ and XR at points V and W respectively. Prove that the points R, W, V and Q lie on a circle.

2. Find all positive integers x and y such that $x + y + 1$ divides $2xy$ and $x + y - 1$ divides $x^2 + y^2 - 1$.

3. The function f is defined on the positive integers as follows;

$$
\begin{aligned}
f(1) &= 1; \\
f(2n) &= f(n) \text{ if } n \text{ is even}; \\
f(2n) &= 2f(n) \text{ if } n \text{ is odd}; \\
f(2n+1) &= 2f(n) + 1 \text{ if } n \text{ is even}; \\
f(2n+1) &= f(n) \text{ if } n \text{ is odd}.
\end{aligned}
$$

Find the number of positive integers n which are less than 2011 and have the property that $f(n) = f(2011)$.

4. Let G be the set of points (x, y) in the plane such that x and y are integers in the range $1 \leq x, y \leq 2011$. A subset S of G is said to be *parallelogram-free* if there is no proper parallelogram with all its vertices in S. Determine the largest possible size of a parallelogram-free subset of G. *Note that a proper parallelogram is one where its vertices do not all lie on the same line*

1. Find all integers x, y and z such that
$$x^2 + y^2 + z^2 = 2(yz + 1) \text{ and } x + y + z = 4018.$$

2. Points A, B, C, D and E lie, in that order, on a circle and the lines AB and ED are parallel. Prove that $\angle ABC = 90°$ if, and only if, $AC^2 = BD^2 + CE^2$.

3. Isaac attempts all six questions on an Olympiad paper in order. Each question is marked on a scale from 0 to 10. He never scores more in a later question than in any earlier question. How many different possible sequences of six marks can he achieve?

4. Two circles, of different radius, with centres at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.

5. Find all functions f, defined on the real numbers and taking real values, which satisfy the equation $f(x)f(y) = f(x + y) + xy$ for all real numbers x and y.

6. Long John Silverman has captured a treasure map from Adam McBones. Adam has buried the treasure at the point (x, y) with integer co-ordinates (not necessarily positive). He has indicated on the map the values of $x^2 + y$ and $x + y^2$, and these numbers are distinct. Prove that Long John has to dig only in one place to find the treasure.

1. There are 2010^{2010} children at a mathematics camp. Each has at most three friends at the camp, and if A is friends with B, then B is friends with A. The camp leader would like to line the children up so that there are at most 2010 children between any pair of friends. Is it always possible to do this?

2. In triangle ABC the centroid is G and D is the midpoint of CA. The line through G parallel to BC meets AB at E. Prove that $\angle AEC = \angle DGC$ if, and only if, $\angle ACB = 90°$. *The centroid of a triangle is the intersection of the three medians, the lines which join each vertex to the midpoint of the opposite side.*

3. The integer x is at least 3 and $n = x^6 - 1$. Let p be a prime and k be a positive integer such that p^k is a factor of n. Show that $p^{3k} < 8n$.

4. Prove that, for all positive real numbers x, y and z,

$$4(x + y + z)^3 > 27(x^2y + y^2z + z^2x).$$

1. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

2. Find all real values of x, y and z such that

$$(x+1)yz = 12,\ (y+1)zx = 4 \text{ and } (z+1)xy = 4.$$

3. Let $ABPC$ be a parallelogram such that ABC is an acute-angled triangle. The circumcircle of triangle ABC meets the line CP again at Q. Prove that $PQ = AC$ if, and only if, $\angle BAC = 60°$. *The circumcircle of a triangle is the circle which passes through its vertices.*

4. Find all positive integers n such that both $n + 2008$ divides $n^2 + 2008$ and $n + 2009$ divides $n^2 + 2009$.

5. Determine the sequences $a_0, a_1, a_2, \ldots$ which satisfy all of the following conditions:
a) $a_{n+1} = 2a_n^2 - 1$ for every integer $n \geq 0$,
b) a_0 is a rational number and
c) $a_i = a_j$ for some i, j with $i \neq j$.

6. The obtuse-angled triangle ABC has sides of length a, b and c opposite the angles $\angle A, \angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc.$$

1. Find all solutions in non-negative integers a, b to $\sqrt{a} + \sqrt{b} = \sqrt{2009}$.

2. Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcentre be O and the orthocentre be H. Prove that the centre of the circle BOH lies on the line AB. *The circumcentre of a triangle is the centre of its circumcircle. The orthocentre of a triangle is the point where its three altitudes meet.*

3. Find all functions f from the real numbers to the real numbers which satisfy
$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy))$$
for all real numbers x and y.

4. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.

1. Find the value of
$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}.$$

2. Find all solutions in positive integers x, y, z to the simultaneous equations
$$x + y - z = 12$$
$$x^2 + y^2 - z^2 = 12.$$

3. Let ABC be a triangle, with an obtuse angle at A. Let Q be a point (other than A, B or C) on the circumcircle of the triangle, on the same side of chord BC as A, and let P be the other end of the diameter through Q. Let V and W be the feet of the perpendiculars from Q onto CA and AB respectively. Prove that the triangles PBC and AWV are similar. [*Note: the circumcircle of the triangle ABC is the circle which passes through the vertices A, B and C.*]

4. Let S be a subset of the set of numbers $\{1, 2, 3, ..., 2008\}$ which consists of 756 distinct numbers. Show that there are two distinct elements a, b of S such that $a + b$ is divisible by 8.

5. Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N. Prove that
$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \leq \frac{1}{8}$$
and locate the position of P in triangle ABC when equality holds.

6. The function f is defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n)$, and $nf(2n+1) = (2n+1)(f(n) + n)$ for all $n \geq 1$.
   i) Prove that $f(n)$ is always an integer.
   ii) For how many positive integers less than 2007 is $f(n) = 2n$?

1. Find the minimum value of $x^2 + y^2 + z^2$ where x, y, z are real numbers such that $x^3 + y^3 + z^3 - 3xyz = 1$.

2. Let triangle ABC have incentre I and circumcentre O. Suppose that $\angle AIO = 90^\circ$ and $\angle CIO = 45^\circ$. Find the ratio $AB : BC : CA$.

3. Adrian has drawn a circle in the xy-plane whose radius is a positive integer at most 2008. The origin lies somewhere inside the circle. You are allowed to ask him questions of the form "Is the point (x, y) inside your circle?" After each question he will answer truthfully "yes" or "no". Show that it is always possible to deduce the radius of the circle after at most sixty questions. [*Note: Any point which lies exactly on the circle may be considered to lie inside the circle.*]

4. Prove that there are infinitely many pairs of distinct positive integers x, y such that $x^2 + y^3$ is divisible by $x^3 + y^2$.

1. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

2. In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that

$$\text{Area of } AMCP = \text{Area of } MNPQ = \frac{1}{3} \text{ Area of } ABCD.$$

3. The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

4. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

5. For positive real numbers a, b, c, prove that

$$(a^2 + b^2)^2 \geq (a + b + c)(a + b - c)(b + c - a)(c + a - b).$$

6. Let n be an integer. Show that, if $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

1. Triangle ABC has integer-length sides, and $AC = 2007$. The internal bisector of $\angle BAC$ meets BC at D. Given that $AB = CD$, determine AB and BC.

2. Show that there are infinitely many pairs of positive integers (m, n) such that
$$\frac{m+1}{n} + \frac{n+1}{m}$$
is a positive integer.

3. Let ABC be an acute-angled triangle with $AB > AC$ and $\angle BAC = 60^o$. Denote the circumcentre by O and the orthocentre by H and let OH meet AB at P and AC at Q. Prove that $PO = HQ$.

Note: The circumcentre of triangle ABC is the centre of the circle which passes through the vertices A, B and C. The orthocentre is the point of intersection of the perpendiculars from each vertex to the opposite side.

4. In the land of Hexagonia, the six cities are connected by a rail network such that there is a direct rail line connecting each pair of cities. On Sundays, some lines may be closed for repair. The passengers' rail charter stipulates that any city must be accessible by rail from any other (not necessarily directly) at all times. In how many different ways can some of the lines be closed subject to this condition?

1. Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

2. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

   i) In how many ways can he split them into two teams of six?

   ii) In how many ways can he split them into three teams of four?

3. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

4. The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length 1.

   A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find, with proof, the greatest number of cells which can be visited.

5. Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.

6. Let T be a set of 2005 coplanar points with no three collinear. Show that, for any of the 2005 points, the number of triangles it lies strictly within, whose vertices are points in T, is even.

1. Find the minimum possible value of $x^2 + y^2$ given that x and y are real numbers satisfying
$$xy(x^2 - y^2) = x^2 + y^2 \text{ and } x \neq 0.$$

2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy
$$x^2 - y^2 = 2^k$$
for some non-negative integer k.

3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that
$$\angle BPC + \angle BAC = 180^o.$$

4. An exam consisting of six questions is sat by 2006 children. Each question is marked either right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (i.e. the total number of questions solved by child 1 + the total solved by child 2 + $\cdots$ + the total solved by child 2006). Find the least possible value of N.

1. Each of Paul and Jenny has a whole number of pounds.
   He says to her: "If you give me £3, I will have n times as much as you".
   She says to him: "If you give me £n, I will have 3 times as much as you".
   Given that all these statements are true and that n is a positive integer, what are the possible values for n?

2. Let ABC be an acute-angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC, CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC, and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $CP = CQ$.

3. Determine the least natural number n for which the following result holds:
   No matter how the elements of the set $\{1, 2, \ldots, n\}$ are coloured red or blue, there are integers x, y, z, w in the set (not necessarily distinct) of the same colour such that $x + y + z = w$.

4. Determine the least possible value of the largest term in an arithmetic progression of seven distinct primes.

5. Let S be a set of rational numbers with the following properties:
   i) $\frac{1}{2} \in S$;
   ii) If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.
   Prove that S contains all rational numbers in the interval $0 < x < 1$.

1. The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$
Prove that N is a perfect square.

2. In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides in D, E and F respectively.
Prove that the circle on diameter EF passes through D.

3. Let a, b, c be positive real numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

4. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \ldots, 36\}$ such that
i) A_i and A_j have non-empty intersection for every i, j.
ii) The intersection of all the elements of X is the empty set.
Show that $n \leq 100$. How many such sets X are there when $n = 100$?

1. Solve the simultaneous equations

$$ab + c + d = 3, \quad bc + d + a = 5, \quad cd + a + b = 2, \quad da + b + c = 6,$$

where a, b, c, d are real numbers.

2. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD.
Prove that the triangle BQC is isosceles.

3. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.
Prove that Alice can always obtain a score at least as great as Barbara's.

4. A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set.
Find the number of wicked subsets of the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

5. Let p, q and r be prime numbers. It is given that p divides $qr - 1$, q divides $rp - 1$, and r divides $pq - 1$.
Determine all possible values of pqr.

1. Let ABC be an equilateral triangle and D an internal point of the side BC. A circle, tangent to BC at D, cuts AB internally at M and N, and AC internally at P and Q.

   Show that $BD + AM + AN = CD + AP + AQ$.

2. Show that there is an integer n with the following properties:

   (i) the binary expansion of n has precisely 2004 0s and 2004 1s;

   (ii) 2004 divides n.

3. (a) Given real numbers a, b, c, with $a + b + c = 0$, prove that
$$a^3 + b^3 + c^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 > 0.$$

   (b) Given real numbers a, b, c, d, with $a + b + c + d = 0$, prove that
$$a^3 + b^3 + c^3 + d^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 + d^5 > 0.$$

4. The real number x between 0 and 1 has decimal representation
$$0{\cdot}a_1a_2a_3a_4\ldots$$
   with the following property: the number of *distinct* blocks of the form
$$a_ka_{k+1}a_{k+2}\ldots a_{k+2003},$$
   as k ranges through all positive integers, is less than or equal to 2004. Prove that x is rational.

1. Given that

$$34! = 295\,232\,799\,cd9\,604\,140\,847\,618\,609\,643\,5ab\,000\,000,$$

determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.
Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$.
Prove that

$$x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation

$$a!\,b! = a! + b! + c!$$

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples x, y, z of distinct positive integers satisfying
   (i) x, y, z are in arithmetic progression, and
   (ii) $p(xyz) \leq 3$.

2. Let ABC be a triangle and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.
   (i) Show that there is an arithmetic progression of positive integers $a, a+d, a+2d$, where $d > 0$, such that
   $$f(a) < f(a+d) < f(a+2d).$$
   (ii) Must there be an arithmetic progression $a, a+d, \ldots, a+2003d$, where $d > 0$, such that
   $$f(a) < f(a+d) < \ldots < f(a+2003d)?$$
   [*A permutation of* $\mathbb{N}$ *is a one-to-one function whose image is the whole of* $\mathbb{N}$; *that is, a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists a unique* $n \in \mathbb{N}$ *such that* $f(n) = m$.]

4. Let f be a function from the set of non-negative integers into itself such that for all $n \geq 0$
   (i) $\big(f(2n+1)\big)^2 - \big(f(2n)\big)^2 = 6f(n) + 1$, and
   (ii) $f(2n) \geq f(n)$.
   How many numbers less than 2003 are there in the image of f?

1. Find all positive integers m, n, where n is odd, that satisfy

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.
Given that $CD = CP = DQ$, prove that $\angle CAD = 60°$.

3. Find all positive real solutions to the equation

$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$

where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

5. f is a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:-
a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,
b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.
Find all possible values of $f(2001)$.

1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

2. A conference hall has a round table wth n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter the $(k+1)$ th delegate sits k places to the right of the k th delegate, for $1 \leq k \leq n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

   Find the set of values n for which this is possible.

3. Prove that the sequence defined by

$$y_0 = 1, \qquad y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right), \quad (n \geq 0)$$

   consists only of integers.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \leq i \leq N)$ is denoted by t_i. Prove the product of the quantities t_i is not more than the product of the distances PC_i.

1. Find all two-digit integers N for which the sum of the digits of $10^N - N$ is divisible by 170.

2. Circle S lies inside circle T and touches it at A. From a point P (distinct from A) on T, chords PQ and PR of T are drawn touching S at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

3. A *tetromino* is a figure made up of four unit squares connected by common edges.
   (i) If we do not distinguish between the possible rotations of a tetromino within its plane, prove that there are seven distinct tetrominoes.
   (ii) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

4. Define the sequence (a_n) by
$$a_n = n + \{\sqrt{n}\},$$
where n is a positive integer and $\{x\}$ denotes the nearest integer to x, where halves are rounded up if necessary. Determine the smallest integer k for which the terms $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers.

5. A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

1. Ahmed and Beth have respectively p and q marbles, with $p > q$.

   Starting with Ahmed, each in turn gives to the other as many marbles as the other already possesses. It is found that after $2n$ such transfers, Ahmed has q marbles and Beth has p marbles.

   Find $\frac{p}{q}$ in terms of n.

2. Find all pairs of integers (x, y) satisfying

$$1 + x^2 y = x^2 + 2xy + 2x + y.$$

3. A triangle ABC has $\angle ACB > \angle ABC$.

   The internal bisector of $\angle BAC$ meets BC at D.

   The point E on AB is such that $\angle EDB = 90°$.

   The point F on AC is such that $\angle BED = \angle DEF$.

   Show that $\angle BAD = \angle FDC$.

4. N dwarfs of heights $1, 2, 3, \ldots, N$ are arranged in a circle. For each pair of neighbouring dwarfs the positive difference between the heights is calculated; the sum of these N differences is called the "total variance" V of the arrangement.

   Find (with proof) the maximum and minimum possible values of V.

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

2. Show that, for every positive integer n,
$$121^n - 25^n + 1900^n - (-4)^n$$
is divisible by 2000.

3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that
$$AP + BP + CP$$
is minimized.

4. For each positive integer $k > 1$, define the sequence $\{a_n\}$ by
$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1} \qquad \text{for each} \quad n \geq 1.$$
Determine all values of k for which 2000 is a term of the sequence.

5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy & Grumpy, and one of Bashful & Sneezy. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow-White agreed to take part as well. In how many ways could the four teams then be formed?

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

2. Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of
$$x^2 + 4xy + 4y^2 + 2z^2.$$

3. Find positive integers a and b such that
$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

4. (a) Find a set A of ten positive integers such that no six distinct elements of A have a sum which is divisible by 6.
   (b) Is it possible to find such a set if "ten" is replaced by "eleven"?

1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children.
Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

2. A circle has diameter AB and X is a fixed point of AB lying between A and B. A point P, distinct from A and B, lies on the circumference of the circle. Prove that, for all possible positions of P,
$$\frac{\tan \angle APX}{\tan \angle PAX}$$
remains constant.

3. Determine a positive constant c such that the equation
$$xy^2 - y^2 - x + y = c$$
has precisely three solutions (x, y) in positive integers.

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,
$$98 = (1\times 81) + (0\times 27) + (1\times 9) + (2\times 3) + (2\times 1) = (10122)_3.$$
Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance
$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$
Let n be any fixed positive integer. Define the sequence (u_r) by
$$u_1 = n \qquad \text{and} \qquad u_r = c(u_{r-1}) \quad \text{for} \quad r \geq 2.$$
Show that there is a positive integer r for which $u_r = 1, 2$ or 17.

5. Consider all functions f from the positive integers to the positive integers such that
(i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;
(ii) for each positive integer n, we have
$f(n+1)$ is **either** $4f(n) - 1$ **or** $f(n) - 1$.
Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

1. For each positive integer n, let S_n denote the set consisting of the first n natural numbers, that is
$$S_n = \{1, 2, 3, 4, \ldots, n-1, n\}.$$
   (i) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sums?
   (ii) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sums?

2. Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA respectively. Prove that PY, QZ, RX are concurrent.

3. Non-negative real numbers p, q and r satisfy $p + q + r = 1$. Prove that
$$7(pq + qr + rp) \leq 2 + 9pqr.$$

4. Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.
   (i) How small can the sum of the digits (in base 10) of such a number be?
   (ii) Can such a number have the sum of its digits (in base 10) equal to 1999?

1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.
Show that it is impossible for the score to be 20.
What is the highest possible score?

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geq 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when
$$a_1^2 + a_2^2 + \cdots + a_{1998}^2$$
is divided by 8?

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.
Prove that M is the midpoint of DA.

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:
$$a_1 = 1, \quad a_2 = 2, \quad a_4 = 12,$$
$$a_{n+1}a_{n-1} = a_n^2 \pm 1 \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

1. A booking office at a railway station sells tickets to 200 destinations. One day, tickets were issued to 3800 passengers. Show that
   (i) there are (at least) 6 destinations at which the passenger arrival numbers are the same;
   (ii) the statement in (i) becomes false if '6' is replaced by '7'.

2. A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC = \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$.

   Prove that the circumcircle of ABC touches the circumcircle of PQR.

3. Suppose x, y, z are positive integers satisfying the equation
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z},$$
   and let h be the highest common factor of x, y, z.

   Prove that $hxyz$ is a perfect square.

   Prove also that $h(y - x)$ is a perfect square.

4. Find a solution of the simultaneous equations
$$xy + yz + zx = 12$$
$$xyz = 2 + x + y + z$$
   in which all of x, y, z are positive, and prove that it is the only such solution.

   Show that a solution exists in which x, y, z are real and distinct.

1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$.
   Determine all such integers N for which $R(N) = 4N + 3$.

2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by
   $$a_1 = 1; \quad a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + \cdots + a_{n-1}), \quad n > 1.$$
   Determine the value of a_{1997}.

3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
   In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

5. Let x, y and z be positive real numbers.
   (i) If $x + y + z \geq 3$, is it necessarily true that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \leq 3$?
   (ii) If $x + y + z \leq 3$, is it necessarily true that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 3$?

1. Let M and N be two 9-digit positive integers with the property that if **any** one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

   Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

   Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

3. Find the number of polynomials of degree 5 with **distinct** coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

4. The set $S = \{1/r : r = 1, 2, 3, \ldots\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $1/20, 1/8, 1/5$ is such a progression, of length 3 (and common difference $3/40$). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and $11/40$ not being members of S).

   (i) Find a maximal progression in S of length 1996.

   (ii) Is there a maximal progression in S of length 1997?

1. Consider the pair of four-digit positive integers

$$(M, N) = (3600, 2500).$$

Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

2. A function f is defined over the set of all positive integers and satisfies

$$f(1) = 1996$$

and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad \text{for all } n > 1.$$

Calculate the exact value of $f(1996)$.

3. Let ABC be an acute-angled triangle, and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.
(*Given any triangle XYZ, its* **circumcentre** *is the centre of the circle which passes through the three vertices X, Y and Z.*)

4. For any real number x, let $[x]$ denote the greatest integer which is less than or equal to x. Define

$$q(n) = \left[\frac{n}{[\sqrt{n}]}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Determine all positive integers n for which $q(n) > q(n+1)$.

5. Let a, b and c be positive real numbers.
(i) Prove that $4(a^3 + b^3) \geq (a+b)^3$.
(ii) Prove that $9(a^3 + b^3 + c^3) \geq (a+b+c)^3$.

1. Determine all sets of non-negative integers x, y and z which satisfy the equation
$$2^x + 3^y = z^2.$$

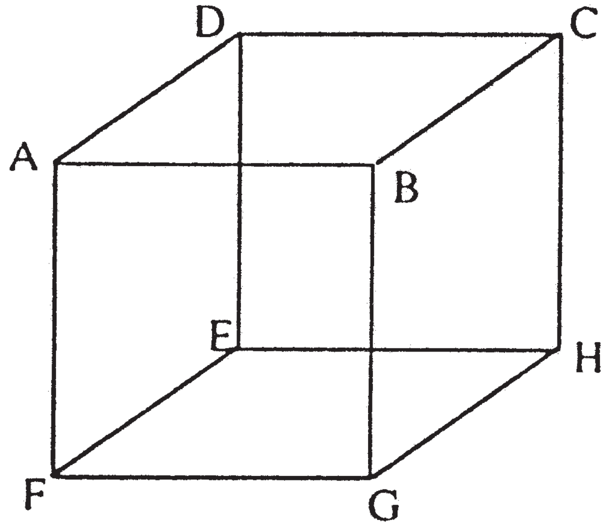
2. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations
$$u(v + w - u) = a^2,$$
$$v(w + u - v) = b^2,$$
$$w(u + v - w) = c^2.$$
Prove that triangle ABC is acute-angled and express the angles U, V, W in terms of A, B, C.

3. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2 respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

4. Let a, b, c and d be positive real numbers such that
$$a + b + c + d = 12$$
and
$$abcd = 27 + ab + ac + ad + bc + bd + cd.$$
Find all possible values of a, b, c, d satisfying these equations.

1. Find the first positive integer whose square ends in three 4's.
Find all positive integers whose squares end in three 4's.
Show that no perfect square ends with four 4's.

2. $ABCDEFGH$ is a cube of side 2.
(a) Find the area of the quadrilateral $AMHN$, where M is the midpoint of BC, and N is the midpoint of EF.
(b) Let P be the midpoint of AB, and Q the midpoint of HE. Let AM meet CP at X, and HN meet FQ at Y. Find the length of XY.



3. (a) Find the maximum value of the expression $x^2y - y^2x$ when $0 \le x \le 1$ and $0 \le y \le 1$.
(b) Find the maximum value of the expression

$$x^2y + y^2z + z^2x - x^2z - y^2x - z^2y$$

when $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

4. ABC is a triangle, right-angled at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

5. The seven dwarfs walk to work each morning in single file. As they go, they sing their famous song, "*High - low - high -low, it's off to work we go ...*". Each day they line up so that no three successive dwarfs are either increasing or decreasing in height. Thus, the line-up must go *up-down-up-down-* $\cdots$ or *down-up-down-up-* $\cdots$. If they all have different heights, for how many days they go to work like this if they insist on using a different order each day?
What if Snow White always came along too?

1. Find all triples of positive integers (a, b, c) such that
$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

2. Let ABC be a triangle, and D, E, F be the midpoints of BC, CA, AB, respectively.
Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ca = 9$.
Prove that $0 < a < 1 < b < 3 < c < 4$.

4. (a) Determine, with careful explanation, how many ways $2n$ people can be paired off to form n teams of 2.
(b) Prove that $\{(mn)!\}^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.

1. Starting with any three digit number n (such as $n = 625$) we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs, and the product of all three digits.
   (i) Find the value of $n/f(n)$ when $n = 625$. (The answer is an integer!)
   (ii) Find all three digit numbers such that the ratio $n/f(n)=1$.

2. In triangle ABC the point X lies on BC.
   (i) Suppose that $\angle BAC = 90°$, that X is the midpoint of BC, and that $\angle BAX$ is one third of $\angle BAC$. What can you say (and prove!) about triangle ACX?
   (ii) Suppose that $\angle BAC = 60°$, that X lies one third of the way from B to C, and that AX bisects $\angle BAC$. What can you say (and prove!) about triangle ACX?

3. The sequence of integers $u_0, u_1, u_2, u_3, \ldots$ satisfies $u_0 = 1$ and
$$u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1,$$
where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

4. The points Q, R lie on the circle γ, and P is a point such that PQ, PR are tangents to γ. A is a point on the extension of PQ, and γ' is the circumcircle of triangle PAR. The circle γ' cuts γ again at B, and AR cuts γ at the point C. Prove that $\angle PAR = \angle ABC$.

5. An *increasing* sequence of integers is said to be **alternating** if it *starts* with an *odd* term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence (with no term at all!) is considered to be alternating. Let $A(n)$ denote the number of alternating sequences which only involve integers from the set $\{1, 2, \ldots, n\}$. Show that $A(1) = 2$ and $A(2) = 3$. Find the value of $A(20)$, and prove that your value is correct.

1. Find the first integer $n > 1$ such that the average of
$$1^2, 2^2, 3^2, \ldots, n^2$$
is itself a perfect square.

2. How many different (i.e. pairwise non-congruent) triangles are there with integer sides and with perimeter 1994?

3. AP, AQ, AR, AS are chords of a given circle with the property that
$$\angle PAQ = \angle QAR = \angle RAS.$$
Prove that
$$AR(AP + AR) = AQ(AQ + AS).$$

4. How many perfect squares are there (mod 2^n)?

1. Find, showing your method, a six-digit integer n with the following properties: (i) n is a perfect square, (ii) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. (Thus n might look like 123124, although this is not a square.)

2. A square piece of toast $ABCD$ of side length 1 and centre O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

3. For each positive integer c, the sequence u_n of integers is defined by

   $u_1 = 1,\ u_2 = c, \quad u_n = (2n+1)u_{n-1} - (n^2-1)u_{n-2},\ (n \geq 3).$

   For which values of c does this sequence have the property that u_i divides u_j whenever $i \leq j$?
   (Note: If x and y are integers, then x *divides* y if and only if there exists an integer z such that $y = xz$. For example, $x = 4$ divides $y = -12$, since we can take $z = -3$.)

4. Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$.

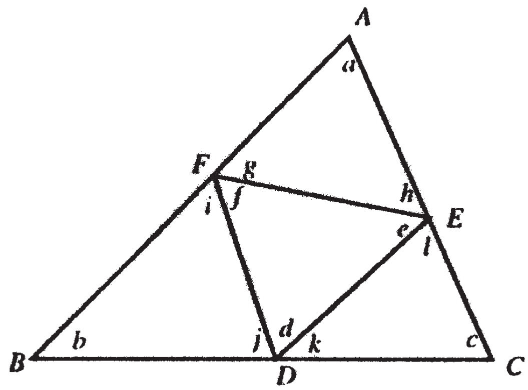
5. Let x, y, z be positive real numbers satisfying

   $$\frac{1}{3} \leq xy + yz + zx \leq 3.$$

   Determine the range of values for (i) xyz, and (ii) $x + y + z$.

1. We usually measure angles in degrees, but we can use any other unit we choose. For example, if we use 30° as a new unit, then the angles of a 30°, 60°, 90° triangle would be equal to 1, 2, 3 new units respectively.

   The diagram shows a triangle ABC with a second triangle DEF inscribed in it. All the angles in the diagram are whole number multiples of some new (unknown unit); their sizes $a, b, c, d, e, f, g, h, i, j, k, \ell$ with respect to this new angle unit are all distinct.



   Find the smallest possible value of $a+b+c$ for which such an angle unit can be chosen, and mark the corresponding values of the angles a to ℓ in the diagram.

2. Let $m = (4^p - 1)/3$, where p is a prime number exceeding 3. Prove that 2^{m-1} has remainder 1 when divided by m.

3. Let P be an internal point of triangle ABC and let α, β, γ be defined by

$$\alpha = \angle BPC - \angle BAC,\ \beta = \angle CPA - \angle CBA,\ \gamma = \angle APB - \angle ACB.$$

   Prove that

$$PA\,\frac{\sin \angle BAC}{\sin \alpha} = PB\,\frac{\sin \angle CBA}{\sin \beta} = PC\,\frac{\sin \angle ACB}{\sin \gamma}.$$

4. The set $Z(m,n)$ consists of all integers N with mn digits which have precisely n ones, n twos, n threes, …, n ms. For each integer $N \in Z(m,n)$, define $d(N)$ to be the sum of the absolute values of the differences of all pairs of consecutive digits. For example, $122313 \in Z(3,2)$ with $d(122313) = 1+0+1+2+2 = 6$. Find the average value of $d(N)$ as N ranges over all possible elements of $Z(m,n)$.

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