STEP

SIXTH TERM EXAM PAPER MATHEMATICS

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Dear Mathematicians,

This is free material compiled from internet resources. I hope you like this free copy.

I am currently LATEX ing the solutions, so that we have a neater and nicer copy of answers too. Some drafts are already available on TSR under replies of the post.

The work is/will be free too. If you wish to join me voluntarily, please contact me at `DrYuFromShanghai@QQ` I need someone who are good at LATEX, drawing vector graphics, compiling and sorting answers. Or any other help you can think of. I'd appreciate any help with this long term project.

Best wishes and good luck everyone.
Fight with COVID19!

Section A: Pure Mathematics

1 The line $y = a^2x$ and the curve $y = x(b-x)^2$, where $0 < a < b$, intersect at the origin O and at points P and Q. The x-coordinate of P is less than the x-coordinate of Q.

(i) Find the coordinates of P and Q, and sketch the line and the curve on the same axes.

(ii) Show that the equation of the tangent to the curve at P is

$$y = a(3a-2b)x + 2a(b-a)^2.$$

(iii) This tangent meets the y-axis at R. The area of the region between the curve and the line segment OP is denoted by S. Show that

$$S = \frac{1}{12}(b-a)^3(3a+b)\,.$$

(iv) The area of triangle OPR is denoted by T. Show that $S > \frac{1}{3}T$.

2 If $x = \log_b(c)$, express c in terms of b and x and prove that $\dfrac{\log_a(c)}{\log_a(b)} = \log_b(c)$.

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2(\pi)} + \frac{1}{\log_5(\pi)} > 2\,.$$

(ii) Given that $\log_2\left(\dfrac{\pi}{\mathrm{e}}\right) > \dfrac{1}{5}$ and that $\mathrm{e}^2 < 8$, prove that $\ln\pi > \frac{17}{15}$.

(iii) Given that $\mathrm{e}^3 > 20$, $\pi^2 < 10$ and $\log_{10}(\pi) > \frac{3}{10}$, prove that $\ln(\pi) < \frac{15}{13}$.

3 The points R and S have coordinates $(-a\,,\,0)$ and $(2a\,,\,0)$, respectively, where $a > 0$. The point P has coordinates $(x\,,\,y)$ where $y > 0$ and $x < 2a$. Let $\angle PRS = \alpha$ and $\angle PSR = \beta$.

(i) Show that, if $\beta = 2\alpha$, then P lies on the curve $y^2 = 3(x^2 - a^2)$.

(ii) Find the possible relationships between α and β when $0 < \alpha < \pi$ and P lies on the curve $y^2 = 3(x^2 - a^2)$.

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{x\ln(x)}\left(1-(\ln(x))^2\right)^2 \qquad (x>0,\quad x\neq 1)\,.$$

Show that, when $(\ln x)^2 = 1$, both $\mathrm{f}(x) = 0$ and $\mathrm{f}'(x) = 0$.

The function F is defined by

$$\mathrm{F}(x) = \begin{cases} \displaystyle\int_{1/\mathrm{e}}^{x} \mathrm{f}(t)\,\mathrm{d}t & \text{for } 0 < x < 1\,, \\[2ex] \displaystyle\int_{\mathrm{e}}^{x} \mathrm{f}(t)\,\mathrm{d}t & \text{for } x > 1\,. \end{cases}$$

(i) Find $\mathrm{F}(x)$ explicitly and hence show that $\mathrm{F}(x^{-1}) = \mathrm{F}(x)$.

(ii) Sketch the curve with equation $y = \mathrm{F}(x)$.

5 **(i)** Write down the most general polynomial of degree 4 that leaves a remainder of 1 when divided by any of $x-1$, $x-2$, $x-3$ or $x-4$.

(ii) The polynomial $\mathrm{P}(x)$ has degree N, where $N \geqslant 1$, and satisfies

$$\mathrm{P}(1) = \mathrm{P}(2) = \cdots = \mathrm{P}(N) = 1\,.$$

Show that $\mathrm{P}(N+1) \neq 1$.

Given that $\mathrm{P}(N+1) = 2$, find $\mathrm{P}(N+r)$ where r is a positive integer. Find a positive integer r, independent of N, such that $\mathrm{P}(N+r) = N+r$.

(iii) The polynomial $\mathrm{S}(x)$ has degree 4. It has integer coefficients and the coefficient of x^4 is 1. It satisfies

$$\mathrm{S}(a) = \mathrm{S}(b) = \mathrm{S}(c) = \mathrm{S}(d) = 2001\,,$$

where a, b, c and d are distinct (not necessarily positive) integers.

(a) Show that there is no integer e such that $\mathrm{S}(e) = 2018$.

(b) Find the number of ways the (distinct) integers a, b, c and d can be chosen such that $\mathrm{S}(0) = 2017$ and $a < b < c < d$.

6 Use the identity

$$2\sin P\,\sin Q = \cos(Q-P) - \cos(Q+P)$$

to show that

$$2\sin\theta\,\big(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta\big) = 1 - \cos(2n\theta)\ .$$

(i) Let A_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using n rectangular strips each of width $\frac{\pi}{n}$, such that the midpoint of the top of each strip lies on the curve. Show that

$$A_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\,.$$

(ii) Let B_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using the trapezium rule with n strips each of width $\frac{\pi}{n}$. Show that

$$B_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\cos\left(\frac{\pi}{2n}\right).$$

(iii) Show that

$$\frac{1}{2}(A_n + B_n) = B_{2n}\,,$$

and that

$$A_n B_{2n} = A_{2n}^2\,.$$

7 **(i)** In the cubic equation $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, use the substitution

$$x = \frac{pz+q}{z+1}$$

to show that the equation reduces to $az^3 + b = 0$, where a and b are to be expressed in terms of p and q.

(ii) Show further that the equation $x^3 - 3cx + d = 0$, where c and d are non-zero real numbers, can be written in the form $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, provided $d^2 > 4c^3$.

(iii) Find the real root of the cubic equation $x^3 + 6x - 2 = 0$.

(iv) Find the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$, and hence show how the equation $x^3 - 3cx + d = 0$ can be solved in the case $d^2 = 4c^3$.

8 The functions s and c satisfy $\mathrm{s}(0) = 0$, $\mathrm{c}(0) = 1$ and

$$\mathrm{s}'(x) = \mathrm{c}(x)^2,$$

$$\mathrm{c}'(x) = -\mathrm{s}(x)^2.$$

You may assume that s and c are uniquely defined by these conditions.

(i) Show that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3$ is constant, and deduce that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3 = 1$.

(ii) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{s}(x)\mathrm{c}(x)\Big) = 2\mathrm{c}(x)^3 - 1$$

and find (and simplify) an expression in terms of $\mathrm{c}(x)$ for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{s}(x)}{\mathrm{c}(x)}\right)$.

(iii) Find the integrals

$$\int \mathrm{s}(x)^2 \,\mathrm{d}x \quad \text{and} \quad \int \mathrm{s}(x)^5 \,\mathrm{d}x\,.$$

(iv) Given that s has an inverse function, s^{-1}, use the substitution $u = \mathrm{s}(x)$ to show that

$$\int \frac{1}{(1-u^3)^{\frac{2}{3}}}\,\mathrm{d}u = \mathrm{s}^{-1}(u) + \text{constant.}$$

(v) Find, in terms of u, the integrals

$$\int \frac{1}{(1-u^3)^{\frac{4}{3}}}\,\mathrm{d}u \qquad \text{and} \qquad \int (1-u^3)^{\frac{1}{3}}\,\mathrm{d}u\,.$$

Section B: Mechanics

9 A straight road leading to my house consists of two sections. The first section is inclined downwards at a constant angle α to the horizontal and ends in traffic lights; the second section is inclined upwards at an angle β to the horizontal and ends at my house. The distance between the traffic lights and my house is d.

I have a go-kart which I start from rest, pointing downhill, a distance x from the traffic lights on the downward-sloping section. The go-kart is not powered in any way, all resistance forces are negligible, and there is no sudden change of speed as I pass the traffic lights.

(i) Given that I reach my house, show that $x \sin\alpha \geqslant d \sin\beta$.

(ii) Let T be the total time taken to reach my house. Show that

$$\left(\frac{g\sin\alpha}{2}\right)^{\frac{1}{2}} T = (1+k)\sqrt{x} - \sqrt{k^2x - kd}\ ,$$

where $k = \dfrac{\sin\alpha}{\sin\beta}$.

(iii) Hence determine, in terms of d and k, the value of x which minimises T.

[You need not justify the fact that the stationary value is a minimum.]

10 A train is made up of two engines, each of mass M, and n carriages, each of mass m. One of the engines is at the front of the train, and the other is coupled between the kth and $(k+1)$th carriages. When the train is accelerating along a straight, horizontal track, the resistance to the motion of each carriage is R and the driving force on each engine is D, where $2D > nR$. The tension in the coupling between the engine at the front and the first carriage is T.

(i) Show that

$$T = \frac{n(mD + MR)}{nm + 2M}\,.$$

(ii) Show that T is greater than the tension in any other coupling provided that $k > \frac{1}{2}n$.

(iii) Show also that, if $k > \frac{1}{2}n$, then at least one of the couplings is in compression (that is, there is a negative tension in the coupling).

11 The point O lies on a rough plane that is inclined at an angle α to the horizontal, where $\alpha < 45°$. The point A lies on the plane a distance d from O up the line L of greatest slope through O. The point B, which is not on the rough plane, lies in the same vertical plane as O and A, and AB is horizontal. The distance from O to B is d.

A particle P of mass m rests on L between O and A. One end of a light inelastic string is attached to P. The string passes over a smooth light pulley fixed at B and its other end is attached to a freely hanging particle of mass λm.

(i) Show that the acute angle, θ, between the string and the line L satisfies $\alpha \leqslant \theta \leqslant 2\alpha$.

(ii) Given that P can rest in equilibrium at every point on L between O and A, show that $2\lambda \sin\alpha \leqslant 1$.

(iii) The coefficient of friction between P and the plane is μ, and the acute angle β is given by $\mu = \tan\beta$. Show that if $\beta \geqslant 2\alpha$, then a necessary condition for equilibrium to be possible for every position of P on L between O and A is

$$\lambda \leqslant \frac{\sin(\beta - \alpha)}{\cos(\beta - 2\alpha)}.$$

Obtain the corresponding result if $\alpha \leqslant \beta \leqslant 2\alpha$.

Section C: Probability and Statistics

12 A bag contains three coins. The probabilities of their showing heads when tossed are p_1, p_2 and p_3.

(i) A coin is taken at random from the bag and tossed. What is the probability that it shows a head?

(ii) A coin is taken at random from the bag (containing three coins) and tossed; the coin is returned to the bag and again a coin is taken at random from the bag and tossed. Let N_1 be the random variable whose value is the number of heads shown on the two tosses. Find the expectation of N_1 in terms of p, where $p = \frac{1}{3}(p_1 + p_2 + p_3)$, and show that $\text{Var}(N_1) = 2p(1-p)$.

(iii) Two of the coins are taken at random from the bag (containing three coins) and tossed. Let N_2 be the random variable whose value is the number of heads showing on the two coins. Find $\text{E}(N_2)$ and $\text{Var}(N_2)$.

(iv) Show that $\text{Var}(N_2) \leqslant \text{Var}(N_1)$, with equality if and only if $p_1 = p_2 = p_3$.

13 A multiple-choice test consists of five questions. For each question, n answers are given ($n \geqslant 2$) only one of which is correct and candidates either attempt the question by choosing one of the n given answers or do not attempt it.

For each question attempted, candidates receive two marks for the correct answer and lose one mark for an incorrect answer. No marks are gained or lost for questions that are not attempted. The pass mark is five.

Candidates A, B and C don't understand any of the questions so, for any question which they attempt, they each choose one of the n given answers at random, independently of their choices for any other question.

(i) Candidate A chooses in advance to attempt exactly k of the five questions, where $k = 0$, 1, 2, 3, 4 or 5. Show that, in order to have the greatest probability of passing the test, she should choose $k = 4$.

(ii) Candidate B chooses at random the number of questions he will attempt, the six possibilities being equally likely. Given that Candidate B passed the test find, in terms of n, the probability that he attempted exactly four questions.

(iii) For each of the five questions Candidate C decides whether to attempt the question by tossing a biased coin. The coin has a probability of $\frac{n}{n+1}$ of showing a head, and she attempts the question if it shows a head. Find the probability, in terms of n, that Candidate C passes the test.

Section A: Pure Mathematics

1 **(i)** Use the substitution $u = x\sin x + \cos x$ to find

$$\int \frac{x}{x\tan x + 1}\,\mathrm{d}x\,.$$

Find by means of a similar substitution, or otherwise,

$$\int \frac{x}{x\cot x - 1}\,\mathrm{d}x\,.$$

(ii) Use a substitution to find

$$\int \frac{x\sec^2 x\ \tan x}{x\sec^2 x - \tan x}\,\mathrm{d}x$$

and

$$\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2}\,\mathrm{d}x\,.$$

2 **(i)** The inequality $\dfrac{1}{t} \leqslant 1$ holds for $t \geqslant 1$. By integrating both sides of this inequality over the interval $1 \leqslant t \leqslant x$, show that

$$\ln x \leqslant x - 1 \qquad (*)$$

for $x \geqslant 1$. Show similarly that $(*)$ also holds for $0 < x \leqslant 1$.

(ii) Starting from the inequality $\dfrac{1}{t^2} \leqslant \dfrac{1}{t}$ for $t \geqslant 1$, show that

$$\ln x \geqslant 1 - \frac{1}{x} \qquad (**)$$

for $x > 0$.

(iii) Show, by integrating $(*)$ and $(**)$, that

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y}$$

for $y > 0$ and $y \neq 1$.

3 The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p > 0$ and $q < 0$, lie on the curve C with equation

$$y^2 = 4ax\,,$$

where $a > 0$.

(i) Show that the equation of the tangent to C at P is

$$y = \frac{1}{p}x + ap\,.$$

(ii) The tangents to the curve at P and at Q meet at R. These tangents meet the y-axis at S and T respectively, and O is the origin. Prove that the area of triangle OPQ is twice the area of triangle RST.

4 **(i)** Let r be a real number with $|r| < 1$ and let

$$S = \sum_{n=0}^{\infty} r^n\,.$$

You may assume without proof that $S = \dfrac{1}{1-r}$.

Let $p = 1 + r + r^2$. Sketch the graph of the function $1 + r + r^2$ and deduce that $\frac{3}{4} \leqslant p < 3$.

Show that, if $1 < p < 3$, then the value of p determines r, and hence S, uniquely.

Show also that, if $\frac{3}{4} < p < 1$, then there are two possible values of S and these values satisfy the equation $(3 - p)S^2 - 3S + 1 = 0$.

(ii) Let r be a real number with $|r| < 1$ and let

$$T = \sum_{n=1}^{\infty} nr^{n-1}\,.$$

You may assume without proof that $T = \dfrac{1}{(1-r)^2}$.

Let $q = 1 + 2r + 3r^2$. Find the set of values of q that determine T uniquely.

Find the set of values of q for which T has two possible values. Find also a quadratic equation, with coefficients depending on q, that is satisfied by these two values.

5 A circle of radius a is centred at the origin O. A rectangle $PQRS$ lies in the minor sector OMN of this circle where M is $(a, 0)$ and N is $(a\cos\beta, a\sin\beta)$, and β is a constant with $0 < \beta < \frac{\pi}{2}$. Vertex P lies on the positive x-axis at $(x, 0)$; vertex Q lies on ON; vertex R lies on the arc of the circle between M and N; and vertex S lies on the positive x-axis at $(s, 0)$.

(i) Show that the area A of the rectangle can be written in the form

$$A = x(s - x)\tan\beta\,.$$

(ii) Obtain an expression for s in terms of a, x and β, and use it to show that

$$\frac{\mathrm{d}A}{\mathrm{d}x} = (s - 2x)\tan\beta - \frac{x^2}{s}\tan^3\beta\,.$$

(iii) Deduce that the greatest possible area of rectangle $PQRS$ occurs when $s = x(1 + \sec\beta)$ and show that this greatest area is $\frac{1}{2}a^2\tan\frac{1}{2}\beta$.

(iv) Show also that this greatest area occurs when $\angle ROS = \frac{1}{2}\beta$.

6 In this question, you may assume that, if a continuous function takes both positive and negative values in an interval, then it takes the value 0 at some point in that interval.

(i) The function f is continuous and $\mathrm{f}(x)$ is non-zero for some value of x in the interval $0 \leqslant x \leqslant 1$. Prove by contradiction, or otherwise, that if

$$\int_0^1 \mathrm{f}(x)\mathrm{d}x = 0\,,$$

then $\mathrm{f}(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$.

(ii) The function g is continuous and

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x = 1\,, \quad \int_0^1 x\mathrm{g}(x)\,\mathrm{d}x = \alpha\,, \quad \int_0^1 x^2\mathrm{g}(x)\,\mathrm{d}x = \alpha^2\,. \qquad (*)$$

Show, by considering

$$\int_0^1 (x-\alpha)^2\mathrm{g}(x)\,\mathrm{d}x\,,$$

that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

Find a function of the form $\mathrm{g}(x) = a + bx$ that satisfies the conditions $(*)$ and verify that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

(iii) The function h has a continuous derivative h' and

$$\mathrm{h}(0) = 0\,, \quad \mathrm{h}(1) = 1\,, \quad \int_0^1 \mathrm{h}(x)\,\mathrm{d}x = \beta\,, \quad \int_0^1 x\mathrm{h}(x)\,\mathrm{d}x = \tfrac{1}{2}\beta(2-\beta)\,.$$

Use the result in part **(ii)** to show that $\mathrm{h}'(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

7 The triangle ABC has side lengths $|BC| = a$, $|CA| = b$ and $|AB| = c$. Equilateral triangles BXC, CYA and AZB are erected on the sides of the triangle ABC, with X on the other side of BC from A, and similarly for Y and Z. Points L, M and N are the centres of rotational symmetry of triangles BXC, CYA and AZB respectively.

(i) Show that $|CM| = \dfrac{b}{\sqrt{3}}$ and write down the corresponding expression for $|CL|$.

(ii) Use the cosine rule to show that

$$6\,|LM|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\,\Delta\,,$$

where Δ is the area of triangle ABC. Deduce that LMN is an equilateral triangle.

Show further that the areas of triangles LMN and ABC are equal if and only if

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta\,.$$

(iii) Show that the conditions

$$(a-b)^2 = -2ab\big(1 - \cos(C - 60^\circ)\big)$$

and

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta$$

are equivalent.

Deduce that the areas of triangles LMN and ABC are equal if and only if ABC is equilateral.

8 Two sequences are defined by $a_1 = 1$ and $b_1 = 2$ and, for $n \geqslant 1$,

$$a_{n+1} = a_n + 2b_n\,,$$
$$b_{n+1} = 2a_n + 5b_n\,.$$

(i) Prove by induction that, for all $n \geqslant 1$,

$$a_n^2 + 2a_nb_n - b_n^2 = 1\,. \qquad (*)$$

(ii) Let $c_n = \dfrac{a_n}{b_n}$. Show that $b_n \geqslant 2 \times 5^{n-1}$ and use $(*)$ to show that

$$c_n \to \sqrt{2} - 1 \text{ as } n \to \infty\,.$$

(iii) Show also that $c_n > \sqrt{2} - 1$ and hence that $\dfrac{2}{c_n + 1} < \sqrt{2} < c_n + 1\,.$

Deduce that $\dfrac{140}{99} < \sqrt{2} < \dfrac{99}{70}\,.$

Section B: Mechanics

9 A particle is projected at speed u from a point O on a horizontal plane. It passes through a fixed point P which is at a horizontal distance d from O and at a height $d\tan\beta$ above the plane, where $d > 0$ and β is an acute angle. The angle of projection α is chosen so that u is as small as possible.

(i) Show that $u^2 = gd\tan\alpha$ and $2\alpha = \beta + 90°$.

(ii) At what angle to the horizontal is the particle travelling when it passes through P? Express your answer in terms of α in its simplest form.

10 Particles P_1, P_2, ... are at rest on the x-axis, and the x-coordinate of P_n is n. The mass of P_n is $\lambda^n m$. Particle P, of mass m, is projected from the origin at speed u towards P_1. A series of collisions takes place, and the coefficient of restitution at each collision is e, where $0 < e < 1$. The speed of P_n immediately after its first collision is u_n and the speed of P_n immediately after its second collision is v_n. No external forces act on the particles.

(i) Show that $u_1 = \dfrac{1+e}{1+\lambda}\,u$ and find expressions for u_n and v_n in terms of e, λ, u and n.

(ii) Show that, if $e > \lambda$, then each particle (except P) is involved in exactly two collisions.

(iii) Describe what happens if $e = \lambda$ and show that, in this case, the fraction of the initial kinetic energy lost approaches e as the number of collisions increases.

(iv) Describe what happens if $\lambda e = 1$. What fraction of the initial kinetic energy is eventually lost in this case?

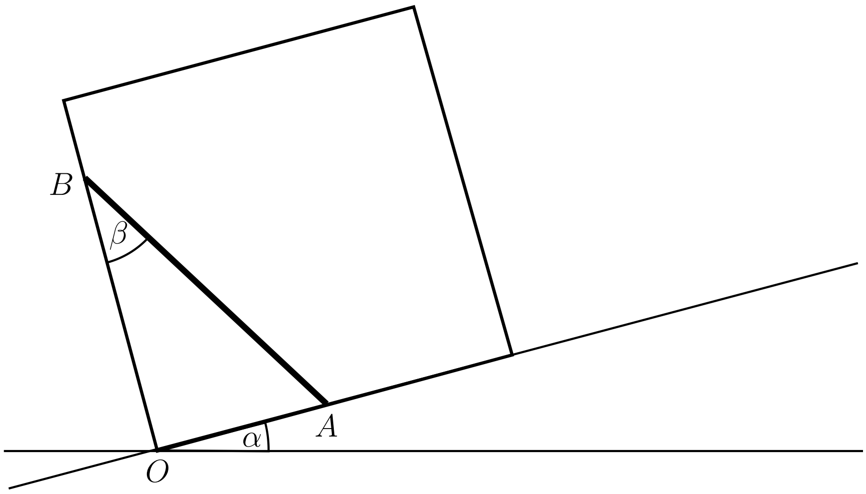
11 A plane makes an acute angle α with the horizontal. A box in the shape of a cube is fixed onto the plane in such a way that four of its edges are horizontal and two of its sides are vertical.

A uniform rod of length $2L$ and weight W rests with its lower end at A on the bottom of the box and its upper end at B on a side of the box, as shown in the diagram below. The vertical plane containing the rod is parallel to the vertical sides of the box and cuts the lowest edge of the box at O. The rod makes an acute angle β with the side of the box at B.

The coefficients of friction between the rod and the box at the two points of contact are both $\tan\gamma$, where $0 < \gamma < \frac{1}{2}\pi$.

The rod is in limiting equilibrium, with the end at A on the point of slipping in the direction away from O and the end at B on the point of slipping towards O. Given that $\alpha < \beta$, show that $\beta = \alpha + 2\gamma$.

[**Hint**: You may find it helpful to take moments about the midpoint of the rod.]



Section C: Probability and Statistics

12 In a lottery, each of the N participants pays $£c$ to the organiser and picks a number from 1 to N. The organiser picks at random the winning number from 1 to N and all those participants who picked this number receive an equal share of the prize, $£J$.

(i) The participants pick their numbers independently and with equal probability. Obtain an expression for the probability that no participant picks the winning number, and hence determine the organiser's expected profit.

Use the approximation

$$\left(1-\frac{a}{N}\right)^N \approx \mathrm{e}^{-a} \qquad (*)$$

to show that if $2Nc = J$ then the organiser will expect to make a loss.

Note: $\mathrm{e} > 2$.

(ii) Instead of the numbers being equally popular, a fraction γ of the numbers are popular and the rest are unpopular. For each participant, the probability of picking any given popular number is $\dfrac{a}{N}$ and the probability of picking any given unpopular number is $\dfrac{b}{N}$.

Find a relationship between a, b and γ.

Show that, using the approximation $(*)$, the organiser's expected profit can be expressed in the form

$$A\mathrm{e}^{-a} + B\mathrm{e}^{-b} + C\,,$$

where A, B and C can be written in terms of J, c, N and γ.

In the case $\gamma = \frac{1}{8}$ and $a = 9b$, find a and b. Show that, if $2Nc = J$, then the organiser will expect to make a profit.

Note: $\mathrm{e} < 3$.

13 I have a sliced loaf which initially contains n slices of bread. Each time I finish setting a STEP question, I make myself a snack: either toast, using one slice of bread; or a sandwich, using two slices of bread. I make toast with probability p and I make a sandwich with probability q, where $p+q=1$, unless there is only one slice left in which case I must, of course, make toast.

(i) Let s_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is the second of two slices used to make a sandwich and let t_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is used to make toast. What is the value of s_1?

(ii) Explain why the following equations hold:

$$t_r = (s_{r-1} + t_{r-1})\, p \qquad (2 \leqslant r \leqslant n-1)\,;$$
$$s_r = 1 - (s_{r-1} + t_{r-1}) \qquad (2 \leqslant r \leqslant n)\,.$$

(iii) Hence, or otherwise, show that $s_r = q(1 - s_{r-1})$ for $2 \leqslant r \leqslant n-1$.

(iv) Show further that

$$s_r = \frac{q + (-q)^r}{1+q} \qquad (1 \leqslant r \leqslant n-1)\,,$$

and find the corresponding expression for t_r.

Find also expressions for s_n and t_n in terms of q.

Section A: Pure Mathematics

1 **(i)** For $n = 1$, 2, 3 and 4, the functions p_n and q_n are defined by

$$\mathrm{p}_n(x) = (x+1)^{2n} - (2n+1)x(x^2+x+1)^{n-1}$$

and

$$\mathrm{q}_n(x) = \frac{x^{2n+1}+1}{x+1} \qquad (x \neq -1)\,.$$

Show that $\mathrm{p}_n(x) \equiv \mathrm{q}_n(x)$ (for $x \neq -1$) in the cases $n = 1$, $n = 2$ and $n = 3$.

Show also that this does not hold in the case $n = 4$.

(ii) Using results from part (i):

(a) express $\dfrac{300^3+1}{301}$ as the product of two factors (neither of which is 1);

(b) express $\dfrac{7^{49}+1}{7^7+1}$ as the product of two factors (neither of which is 1), each written in terms of various powers of 7 which you should not attempt to calculate explicitly.

2 Differentiate, with respect to x,

$$(ax^2+bx+c)\,\ln\big(x+\sqrt{1+x^2}\big) + \big(dx+e\big)\sqrt{1+x^2}\,,$$

where a, b, c, d and e are constants. You should simplify your answer as far as possible.

Hence integrate:

(i) $\ln\big(x+\sqrt{1+x^2}\,\big)$;

(ii) $\sqrt{1+x^2}$;

(iii) $x\ln\big(x+\sqrt{1+x^2}\,\big)$.

3 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that (for example) $\lfloor 2.9 \rfloor = 2$, $\lfloor 2 \rfloor = 2$ and $\lfloor -1.5 \rfloor = -2$.

On separate diagrams draw the graphs, for $-\pi \leqslant x \leqslant \pi$, of:

(i) $y = \lfloor x \rfloor$; (ii) $y = \sin\lfloor x \rfloor$; (iii) $y = \lfloor \sin x \rfloor$; (iv) $y = \lfloor 2\sin x \rfloor$.

In each case, you should indicate clearly the value of y at points where the graph is discontinuous.

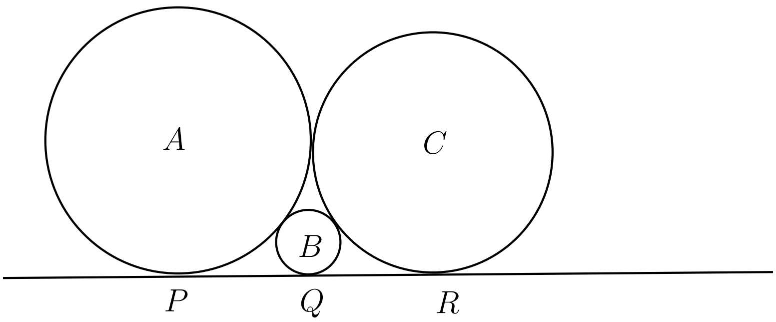
4 **(i)** Differentiate $\dfrac{z}{(1+z^2)^{\frac{1}{2}}}$ with respect to z.

(ii) The *signed curvature* κ of the curve $y = \mathrm{f}(x)$ is defined by

$$\kappa = \frac{\mathrm{f}''(x)}{\big(1 + (\mathrm{f}'(x))^2\big)^{\frac{3}{2}}}.$$

Use this definition to determine all curves for which the signed curvature is a non-zero constant. For these curves, what is the geometrical significance of κ?

5 **(i)**



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

6 The sides OA and CB of the quadrilateral $OABC$ are parallel. The point X lies on OA, between O and A. The position vectors of A, B, C and X relative to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively.

(i) Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \quad \text{and} \quad \mathbf{x} = m\mathbf{a},$$

where k and m are scalars, and state the range of values that each of k and m can take.

(ii) The lines OB and AC intersect at D, the lines XD and BC intersect at Y and the lines OY and AB intersect at Z. Show that the position vector of Z relative to O can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk + 1}.$$

(iii) The lines DZ and OA intersect at T. Show that

$$OT \times OA = OX \times TA \quad \text{and} \quad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA},$$

where, for example, OT denotes the length of the line joining O and T.

7 The set S consists of all the positive integers that leave a remainder of 1 upon division by 4. The set T consists of all the positive integers that leave a remainder of 3 upon division by 4.

(i) Describe in words the sets $S \cup T$ and $S \cap T$.

(ii) Prove that the product of any two integers in S is also in S. Determine whether the product of any two integers in T is also in T.

(iii) Given an integer in T that is not a prime number, prove that at least one of its prime factors is in T.

(iv) For any set X of positive integers, an integer in X (other than 1) is said to be X-*prime* if it cannot be expressed as the product of two or more integers *all in* X (and all different from 1).

(a) Show that every integer in T is either T-prime or is the product of an odd number of T-prime integers.

(b) Find an example of an integer in S that can be expressed as the product of S-prime integers in two distinct ways. [Note: $s_1 s_2$ and $s_2 s_1$ are not counted as distinct ways of expressing the product of s_1 and s_2.]

8 Given an infinite sequence of numbers u_0, u_1, u_2, $\ldots$, we define the *generating function*, f, for the sequence by

$$\mathrm{f}(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots .$$

Issues of convergence can be ignored in this question.

(i) Using the binomial series, show that the sequence given by $u_n = n$ has generating function $x(1-x)^{-2}$, and find the sequence that has generating function $x(1-x)^{-3}$.

Hence, or otherwise, find the generating function for the sequence $u_n = n^2$. You should simplify your answer.

(ii) **(a)** The sequence u_0, u_1, u_2, $\ldots$ is determined by $u_n = k u_{n-1}$ $(n \geqslant 1)$, where k is independent of n, and $u_0 = a$. By summing the identity $u_n x^n \equiv k u_{n-1} x^n$, or otherwise, show that the generating function, f, satisfies

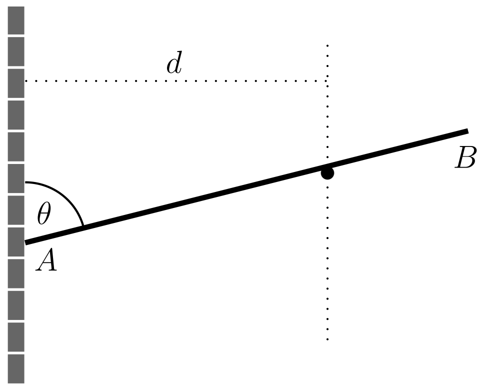
$$\mathrm{f}(x) = a + kx\mathrm{f}(x) .$$

Write down an expression for $\mathrm{f}(x)$.

(b) The sequence u_0, u_1, u_2, $\ldots$ is determined by $u_n = u_{n-1} + u_{n-2}$ $(n \geqslant 2)$ and $u_0 = 0$, $u_1 = 1$. Obtain the generating function.

Section B: Mechanics

9 A horizontal rail is fixed parallel to a vertical wall and at a distance d from the wall. A uniform rod AB of length $2a$ rests in equilibrium on the rail with the end A in contact with the wall. The rod lies in a vertical plane perpendicular to the wall. It is inclined at an angle θ to the vertical (where $0 < \theta < \frac{1}{2}\pi$) and $a \sin\theta < d$, as shown in the diagram.



The coefficient of friction between the rod and the wall is μ, and the coefficient of friction between the rod and the rail is λ.

(i) Show that in limiting equilibrium, with the rod on the point of slipping at both the wall and the rail, the angle θ satisfies

$$d \operatorname{cosec}^2 \theta = a\big((\lambda + \mu)\cos\theta + (1 - \lambda\mu)\sin\theta\big)\,.$$

(ii) Derive the corresponding result if, instead, $a \sin\theta > d$.

10 Four particles A, B, C and D are initially at rest on a smooth horizontal table. They lie equally spaced a small distance apart, in the order $ABCD$, in a straight line. Their masses are λm, m, m and m, respectively, where $\lambda > 1$.

Particles A and D are simultaneously projected, both at speed u, so that they collide with B and C (respectively). In the following collision between B and C, particle B is brought to rest. The coefficient of restitution in each collision is e.

(i) Show that $e = \dfrac{\lambda - 1}{3\lambda + 1}$ and deduce that $e < \frac{1}{3}$.

(ii) Given also that C and D move towards each other with the same speed, find the value of λ and of e.

11 The point O is at the top of a vertical tower of height h which stands in the middle of a large horizontal plain. A projectile P is fired from O at a fixed speed u and at an angle α above the horizontal.

(i) Show that the distance x from the base of the tower when P hits the plain satisfies

$$\frac{gx^2}{u^2} = h(1 + \cos 2\alpha) + x \sin 2\alpha\,.$$

(ii) Show that the greatest value of x as α varies occurs when $x = h \tan 2\alpha$ and find the corresponding value of $\cos 2\alpha$ in terms of g, h and u.

(iii) Show further that the greatest achievable distance between O and the landing point is $\dfrac{u^2}{g} + h$.

Section C: Probability and Statistics

12 **(i)** Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.

(ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.

(iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n+1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2, and hence obtain a generalisation of the results of parts (i) and (ii).

13 An internet tester sends n e-mails simultaneously at time $t = 0$. Their arrival times at their destinations are independent random variables each having probability density function $\lambda e^{-\lambda t}$ ($0 \leqslant t < \infty$, $\lambda > 0$).

(i) The random variable T is the time of arrival of the e-mail that arrives first at its destination. Show that the probability density function of T is

$$n\lambda e^{-n\lambda t},$$

and find the expected value of T.

(ii) Write down the probability that the second e-mail to arrive at its destination arrives later than time t and hence derive the density function for the time of arrival of the second e-mail. Show that the expected time of arrival of the second e-mail is

$$\frac{1}{\lambda}\left(\frac{1}{n-1}+\frac{1}{n}\right).$$

Section A: Pure Mathematics

1 **(i)** Sketch the curve $y = \mathrm{e}^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $\mathrm{e}^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n\mathrm{e}^x \to 0$ as $x \to -\infty$ for any integer n.

(ii) Sketch the curve $y = \mathrm{e}^{x^2}(2x^4 - 5x^2 + 2)$.

2 **(i)** Show that $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and find a similar expression for $\sin 15^\circ$.

(ii) Show that $\cos\alpha$ is a root of the equation

$$4x^3 - 3x - \cos 3\alpha = 0,$$

and find the other two roots in terms of $\cos\alpha$ and $\sin\alpha$.

(iii) Use parts (i) and (ii) to solve the equation $y^3 - 3y - \sqrt{2} = 0$, giving your answers in surd form.

3 A prison consists of a square courtyard of side b bounded by a perimeter wall and a square building of side a placed centrally within the courtyard. The sides of the building are parallel to the perimeter walls.

Guards can stand either at the middle of a perimeter wall or in a corner of the courtyard. If the guards wish to see as great a length of the perimeter wall as possible, determine which of these positions is preferable. You should consider separately the cases $b < 3a$ and $b > 3a$.

4 The midpoint of a rod of length $2b$ slides on the curve $y = \frac{1}{4}x^2$, $x \geqslant 0$, in such a way that the rod is always tangent, at its midpoint, to the curve.

(i) Show that the curve traced out by one end of the rod can be written in the form

$$x = 2\tan\theta - b\cos\theta$$
$$y = \tan^2\theta - b\sin\theta$$

for some suitably chosen angle θ which satisfies $0 \leqslant \theta < \frac{1}{2}\pi$.

(ii) When one end of the rod is at a point A on the y-axis, the midpoint is at point P and $\theta = \alpha$. Let R be the region bounded by the following:

the curve $y = \frac{1}{4}x^2$ between the origin and P;

the y-axis between A and the origin;

the half-rod AP.

Show that the area of R is $\frac{2}{3}\tan^3\alpha$.

5 **(i)** The function f is defined, for $x > 0$, by

$$\mathrm{f}(x) = \int_1^3 (t-1)^{x-1}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{f}(x)$.

(ii) The function g is defined, for $-\infty < x < \infty$, by

$$\mathrm{g}(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-2xt+x^2}}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{g}(x)$.

6 The vertices of a plane quadrilateral are labelled A, B, A' and B', in clockwise order. A point O lies in the same plane and within the quadrilateral. The angles AOB and $A'OB'$ are right angles, and $OA = OB$ and $OA' = OB'$.

(i) Use position vectors relative to O to show that the midpoints of AB, BA', $A'B'$ and $B'A$ are the vertices of a square.

(ii) Given that the lengths of OA and OA' are fixed (and the conditions of the first paragraph still hold), find the value of angle BOA' for which the area of the square is greatest.

7 Let

$$\mathrm{f}(x) = 3ax^2 - 6x^3$$

and, for each real number a, let $\mathrm{M}(a)$ be the greatest value of $\mathrm{f}(x)$ in the interval $-\frac{1}{3} \leqslant x \leqslant 1$.

Determine $\mathrm{M}(a)$ for $a \geqslant 0$. [The formula for $\mathrm{M}(a)$ is different in different ranges of a; you will need to identify three ranges.]

8 Show that:

(i) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$;

(ii) if N is a positive integer, m is a non-negative integer and k is a positive odd integer, then $(N-m)^k + m^k$ is divisible by N.

Let $S = 1^k + 2^k + 3^k + \cdots + n^k$, where k is a positive odd integer. Show that if n is odd then S is divisible by n and that if n is even then S is divisible by $\frac{1}{2}n$.

Show further that S is divisible by $1 + 2 + 3 + \cdots + n$.

Section B: Mechanics

9 A short-barrelled machine gun stands on horizontal ground. The gun fires bullets, from ground level, at speed u continuously from $t = 0$ to $t = \dfrac{\pi}{6\lambda}$, where λ is a positive constant, but does not fire outside this time period. During this time period, the angle of elevation α of the barrel decreases from $\frac{1}{3}\pi$ to $\frac{1}{6}\pi$ and is given at time t by

$$\alpha = \tfrac{1}{3}\pi - \lambda t\,.$$

(i) Let $k = \dfrac{g}{2\lambda u}$. Show that, in the case $\frac{1}{2} \leqslant k \leqslant \frac{1}{2}\sqrt{3}$, the last bullet to hit the ground does so at a distance

$$\frac{2ku^2\sqrt{1-k^2}}{g}$$

from the gun.

(ii) What is the corresponding result if $k < \frac{1}{2}$?

10 A bus has the shape of a cuboid of length a and height h. It is travelling northwards on a journey of fixed distance at constant speed u (chosen by the driver). The maximum speed of the bus is w. Rain is falling from the southerly direction at speed v in straight lines inclined to the horizontal at angle θ, where $0 < \theta < \frac{1}{2}\pi$.

(i) By considering first the case $u = 0$, show that for $u > 0$ the total amount of rain that hits the roof and the back or front of the bus in unit time is proportional to

$$h\big|v\cos\theta - u\big| + av\sin\theta\,.$$

(ii) Show that, in order to encounter as little rain as possible on the journey, the driver should choose $u = w$ if either $w < v\cos\theta$ or $a\sin\theta > h\cos\theta$. How should the speed be chosen if $w > v\cos\theta$ and $a\sin\theta < h\cos\theta$? Comment on the case $a\sin\theta = h\cos\theta$.

(iii) How should the driver choose u on the return journey?

11 Two long circular cylinders of equal radius lie in equilibrium on an inclined plane, in contact with one another and with their axes horizontal. The weights of the upper and lower cylinders are W_1 and W_2, respectively, where $W_1 > W_2$. The coefficients of friction between the inclined plane and the upper and lower cylinders are μ_1 and μ_2, respectively, and the coefficient of friction between the two cylinders is μ. The angle of inclination of the plane is α (which is positive).

(i) Let F be the magnitude of the frictional force between the two cylinders, and let F_1 and F_2 be the magnitudes of the frictional forces between the upper cylinder and the plane, and the lower cylinder and the plane, respectively. Show that $F = F_1 = F_2$.

(ii) Show that

$$\mu \geqslant \frac{W_1 + W_2}{W_1 - W_2},$$

and that

$$\tan\alpha \leqslant \frac{2\mu_1 W_1}{(1+\mu_1)(W_1 + W_2)}.$$

Section C: Probability and Statistics

12 The number X of casualties arriving at a hospital each day follows a Poisson distribution with mean 8; that is,

$$\mathrm{P}(X=n) = \frac{\mathrm{e}^{-8}8^n}{n!}, \qquad n = 0,\ 1,\ 2,\ \ldots\ .$$

Casualties require surgery with probability $\frac{1}{4}$. The number of casualties arriving on any given day is independent of the number arriving on any other day and the casualties require surgery independently of one another.

(i) What is the probability that, on a day when exactly n casualties arrive, exactly r of them require surgery?

(ii) Prove (algebraically) that the number requiring surgery each day also follows a Poisson distribution, and state its mean.

(iii) Given that in a particular randomly chosen week a total of 12 casualties require surgery on Monday and Tuesday, what is the probability that 8 casualties require surgery on Monday? You should give your answer as a fraction in its lowest terms.

13 A fair die with faces numbered 1, …, 6 is thrown repeatedly. The events A, B, C, D and E are defined as follows.

A: the first 6 arises on the nth throw.
B: at least one 5 arises before the first 6.
C: at least one 4 arises before the first 6.
D: exactly one 5 arises before the first 6.
E: exactly one 4 arises before the first 6.

Evaluate the following probabilities:

(i) $\mathrm{P}(A)$ **(ii)** $\mathrm{P}(B)$ **(iii)** $\mathrm{P}(B \cap C)$ **(iv)** $\mathrm{P}(D)$ **(v)** $\mathrm{P}(D \cup E)$.

For some parts of this question, you may want to make use of the binomial expansion in the form:

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2}x^2 + \cdots + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots .$$

Section A: Pure Mathematics

1 *All numbers referred to in this question are non-negative integers.*

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

2 **(i)** Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c\,,$ where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}$.

(iv) The curve B is given by $y = \big|\ln|x^2-4|\big|$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

3 The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2\,\mathrm{d}x = \left(\int_a^b x\,\mathrm{d}x\right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.

4 An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the centre of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leqslant \theta < \pi$.

Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.

In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

5 **(i)** Let $\mathrm{f}(x) = (x + 2a)^3 - 27a^2x$, where $a \geqslant 0$. By sketching $\mathrm{f}(x)$, show that $\mathrm{f}(x) \geqslant 0$ for $x \geqslant 0$.

(ii) Use part (i) to find the greatest value of xy^2 in the region of the x-y plane given by $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \leqslant 3$. For what values of x and y is this greatest value achieved?

(iii) Use part (i) to show that $(p + q + r)^3 \geqslant 27pqr$ for any non-negative numbers p, q and r. If $(p + q + r)^3 = 27pqr$, what relationship must p, q and r satisfy?

6 **(i)** The sequence of numbers $u_0, u_1, \ldots$ is given by $u_0 = u$ and, for $n \geqslant 0$,

$$u_{n+1} = 4u_n(1 - u_n)\,. \qquad (*)$$

In the case $u = \sin^2\theta$ for some given angle θ, write down and simplify expressions for u_1 and u_2 in terms of θ. Conjecture an expression for u_n and prove your conjecture.

(ii) The sequence of numbers $v_0, v_1, \ldots$ is given by $v_0 = v$ and, for $n \geqslant 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r\,,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2\,.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geqslant 0$, $v_{n+1} = -v_n^2 + 2v_n + 2\,.$

7 In the triangle OAB, the point D divides the side BO in the ratio $r : 1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s : 1$ (so that $OE = sEA$), where r and s are both positive.

(i) The lines AD and BE intersect at G. Show that

$$\mathbf{g} = \frac{rs}{1+r+rs}\,\mathbf{a} + \frac{1}{1+r+rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t : 1$, find an expression for t in terms of r and s.

8 Let L_a denote the line joining the points $(a, 0)$ and $(0, 1-a)$, where $0 < a < 1$. The line L_b is defined similarly.

(i) Determine the point of intersection of L_a and L_b, where $a \neq b$.

(ii) Show that this point of intersection, in the limit as $b \to a$, lies on the curve C given by

$$y = (1 - \sqrt{x})^2 \qquad (0 < x < 1)\,.$$

(iii) Show that every tangent to C is of the form L_a for some a.

Section B: Mechanics

9 A particle of mass m is projected due east at speed U from a point on horizontal ground at an angle θ above the horizontal, where $0 < \theta < 90°$. In addition to the gravitational force mg, it experiences a horizontal force of magnitude mkg, where k is a positive constant, acting due west in the plane of motion of the particle.

(i) Determine expressions in terms of U, θ and g for the time, T_H, at which the particle reaches its greatest height and the time, T_L, at which it lands.

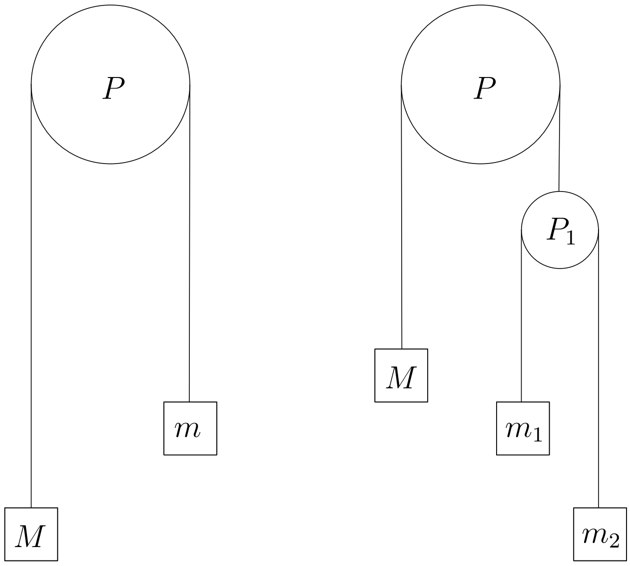
(ii) Let $T = U\cos\theta/(kg)$. By considering the relative magnitudes of T_H, T_L and T, or otherwise, sketch the trajectory of the particle in the cases $k\tan\theta < \frac{1}{2}$, $\frac{1}{2} < k\tan\theta < 1$, and $k\tan\theta > 1$. What happens when $k\tan\theta = 1$?

10 **(i)** A uniform spherical ball of mass M and radius R is released from rest with its centre a distance $H + R$ above horizontal ground. The coefficient of restitution between the ball and the ground is e. Show that, after bouncing, the centre of the ball reaches a height $R + He^2$ above the ground.

(ii) A second uniform spherical ball, of mass m and radius r, is now released from rest together with the first ball (whose centre is again a distance $H + R$ above the ground when it is released). The two balls are initially one on top of the other, with the second ball (of mass m) above the first. The two balls separate slightly during their fall, with their centres remaining in the same vertical line, so that they collide immediately after the first ball has bounced on the ground. The coefficient of restitution between the balls is also e. The centre of the second ball attains a height h above the ground.

Given that $R = 0.2$, $r = 0.05$, $H = 1.8$, $h = 4.5$ and $e = \frac{2}{3}$, determine the value of M/m.

11 The diagrams below show two separate systems of particles, strings and pulleys. In both systems, the pulleys are smooth and light, the strings are light and inextensible, the particles move vertically and the pulleys labelled with P are fixed. The masses of the particles are as indicated on the diagrams.



System I System II

(i) For system I show that the acceleration, a_1, of the particle of mass M, measured in the downwards direction, is given by

$$a_1 = \frac{M-m}{M+m}\, g\,,$$

where g is the acceleration due to gravity. Give an expression for the force on the pulley due to the tension in the string.

(ii) For system II show that the acceleration, a_2, of the particle of mass M, measured in the downwards direction, is given by

$$a_2 = \frac{M-4\mu}{M+4\mu}\, g\,,$$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2}$.

In the case $m = m_1 + m_2$, show that $a_1 = a_2$ if and only if $m_1 = m_2$.

Section C: Probability and Statistics

12 A game in a casino is played with a fair coin and an unbiased cubical die whose faces are labelled $1, 1, 1, 2, 2$ and 3. In each round of the game, the die is rolled once and the coin is tossed once. The outcome of the round is a random variable X. The value, x, of X is determined as follows. If the result of the toss is heads then $x = |ks - 1|$, and if the result of the toss is tails then $x = |k - s|$, where s is the number on the die and k is a given number.

(i) Show that $\mathrm{E}(X^2) = k + 13(k-1)^2/6$.

(ii) Given that both $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$ are positive integers, and that k is a single-digit positive integer, determine the value of k, and write down the probability distribution of X.

(iii) A gambler pays £1 to play the game, which consists of two rounds. The gambler is paid:

£w, where w is an integer, if the sum of the outcomes of the two rounds exceeds 25;
£1 if the sum of the outcomes equals 25;
nothing if the sum of the outcomes is less that 25.

Find, in terms of w, an expression for the amount the gambler expects to be paid in a game, and deduce the maximum possible value of w, given that the casino's owners choose w so that the game is in their favour.

13 A continuous random variable X has a *triangular* distribution, which means that it has a probability density function of the form

$$\mathrm{f}(x) = \begin{cases} \mathrm{g}(x) & \text{for } a < x \leqslant c \\ \mathrm{h}(x) & \text{for } c \leqslant x < b \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{g}(x)$ is an increasing linear function with $\mathrm{g}(a) = 0$, $\mathrm{h}(x)$ is a decreasing linear function with $\mathrm{h}(b) = 0$, and $\mathrm{g}(c) = \mathrm{h}(c)$.

Show that $\mathrm{g}(x) = \dfrac{2(x-a)}{(b-a)(c-a)}$ and find a similar expression for $\mathrm{h}(x)$.

(i) Show that the mean of the distribution is $\frac{1}{3}(a+b+c)$.

(ii) Find the median of the distribution in the different cases that arise.

Section A: Pure Mathematics

1 **(i)** Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2}\, - 22 = 0\,;$

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3}\, - 9 = 0\,.$

2 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $\mathrm{f}(x) = \dfrac{\lfloor x \rfloor}{x}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leqslant x \leqslant 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $\mathrm{f}(x) = \frac{7}{12}$.

Solve also the equations $\mathrm{f}(x) = \frac{17}{24}$ and $\mathrm{f}(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $\mathrm{f}(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $\mathrm{f}(x) = c$ to have exactly n roots, where $n \geqslant 1$.

3 For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points P_1, P_2, ... are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

4 **(i)** Show that, for $n > 0$,

$$\int_0^{\frac{1}{4}\pi} \tan^n x \sec^2 x \, \mathrm{d}x = \frac{1}{n+1} \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} \sec^n x \tan x \, \mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n}.$$

(ii) Evaluate the following integrals:

$$\int_0^{\frac{1}{4}\pi} x \sec^4 x \tan x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \tan x \, \mathrm{d}x\,.$$

5 The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \qquad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45^\circ$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

6 **(i)** By considering the coefficient of x^r in the series for $(1 + x)(1 + x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \qquad (1 \leqslant r \leqslant n).$$

(ii) The sequence of numbers B_0, B_1, B_2, ... is defined by

$$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \qquad \text{and} \qquad B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}.$$

Show that $B_{n+2} - B_{n+1} = B_n$ $(n = 0, 1, 2, \dots)$.

(iii) What is the relation between the sequence B_0, B_1, B_2, ... and the Fibonacci sequence F_0, F_1, F_2, ... defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$?

7 **(i)** Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

8 **(i)** The functions $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are defined by

$$\begin{aligned}
&\mathrm{a}(x) = x^2 \quad (-\infty < x < \infty),\\
&\mathrm{b}(x) = \ln x \quad (x > 0),\\
&\mathrm{c}(x) = 2x \quad (-\infty < x < \infty),\\
&\mathrm{d}(x) = \sqrt{x} \quad (x \geqslant 0)\,.
\end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

$$\mathrm{cb}, \qquad \mathrm{ab}, \qquad \mathrm{da}, \qquad \mathrm{ad}.$$

(ii) The functions f and g are defined by

$$\begin{aligned}
&\mathrm{f}(x) = \sqrt{x^2 - 1} \quad (|x| \geqslant 1),\\
&\mathrm{g}(x) = \sqrt{x^2 + 1} \quad (-\infty < x < \infty).
\end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned}
&\mathrm{h}(x) = x + \sqrt{x^2 - 1} \quad (x \geqslant 1),\\
&\mathrm{k}(x) = x - \sqrt{x^2 - 1} \quad (|x| \geqslant 1),
\end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

Section B: Mechanics

9 Two particles, A and B, are projected simultaneously towards each other from two points which are a distance d apart in a horizontal plane. Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has mass M and is projected at speed v at angle β above the horizontal. The trajectories of the two particles lie in the same vertical plane.

The particles collide directly when each is at its point of greatest height above the plane.

(i) Given that both A and B return to their starting points, and that momentum is conserved in the collision, show that

$$m \cot \alpha = M \cot \beta \,.$$

(ii) Show further that the collision occurs at a point which is a horizontal distance b from the point of projection of A where

$$b = \frac{Md}{m+M} \,,$$

and find, in terms of b and α, the height above the horizontal plane at which the collision occurs.

10 Two parallel vertical barriers are fixed a distance d apart on horizontal ice. A small ice hockey puck moves on the ice backwards and forwards between the barriers, in the direction perpendicular to the barriers, colliding with each in turn. The coefficient of friction between the puck and the ice is μ and the coefficient of restitution between the puck and each of the barriers is r.

(i) The puck starts at one of the barriers, moving with speed v towards the other barrier. Show that

$$v_{i+1}^2 - r^2 v_i^2 = -2r^2 \mu g d$$

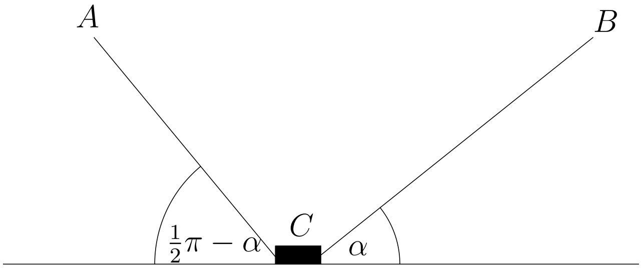
where v_i is the speed of the puck just after its ith collision.

(ii) The puck comes to rest against one of the barriers after traversing the gap between them n times. In the case $r \neq 1$, express n in terms of r and k, where $k = \dfrac{v^2}{2\mu g d}$. If $r = \mathrm{e}^{-1}$ (where e is the base of natural logarithms) show that

$$n = \tfrac{1}{2} \ln \big(1 + k(\mathrm{e}^2 - 1)\big) \,.$$

Give an expression for n in the case $r = 1$.

11



The diagram shows a small block C of weight W initially at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is μ. Two light strings, AC and BC, are attached to the block, making angles $\frac{1}{2}\pi - \alpha$ and α to the horizontal, respectively. The tensions in AC and BC are $T\sin\beta$ and $T\cos\beta$ respectively, where $0 < \alpha + \beta < \frac{1}{2}\pi$.

(i) In the case $W > T\sin(\alpha + \beta)$, show that the block will remain at rest provided

$$W\sin\lambda \geqslant T\cos(\alpha + \beta - \lambda)\,,$$

where λ is the acute angle such that $\tan\lambda = \mu$.

(ii) In the case $W = T\tan\phi$, where $2\phi = \alpha + \beta$, show that the block will start to move in a direction that makes an angle ϕ with the horizontal.

Section C: Probability and Statistics

12 Each day, I have to take k different types of medicine, one tablet of each. The tablets are identical in appearance. When I go on holiday for n days, I put n tablets of each type in a container and on each day of the holiday I select k tablets at random from the container.

(i) In the case $k = 3$, show that the probability that I will select one tablet of each type on the first day of a three-day holiday is $\frac{9}{28}$.

Write down the probability that I will be left with one tablet of each type on the last day (irrespective of the tablets I select on the first day).

(ii) In the case $k = 3$, find the probability that I will select one tablet of each type on the first day of an n-day holiday.

(iii) In the case $k = 2$, find the probability that I will select one tablet of each type on each day of an n-day holiday, and use Stirling's approximation

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n$$

to show that this probability is approximately $2^{-n}\sqrt{n\pi}$.

13 From the integers $1, 2, \ldots, 52$, I choose seven (distinct) integers at random, all choices being equally likely. From these seven, I discard any pair that sum to 53. Let X be the random variable the value of which is the number of discarded pairs. Find the probability distribution of X and show that $\mathrm{E}(X) = \frac{7}{17}$.

Note: $7 \times 17 \times 47 = 5593$.

Section A: Pure Mathematics

1 The line L has equation $y = c - mx$, with $m > 0$ and $c > 0$. It passes through the point $R(a, b)$ and cuts the axes at the points $P(p, 0)$ and $Q(0, q)$, where a, b, p and q are all positive.

(i) Find p and q in terms of a, b and m.

(ii) As L varies with R remaining fixed, show that the minimum value of the sum of the distances of P and Q from the origin is $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$, and find in a similar form the minimum distance between P and Q. (You may assume that any stationary values of these distances are minima.)

2 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

3 **(i)** Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and add to your diagram the tangent to the curve at the origin and the chord joining the origin to the point $(b, \sin b)$, where $0 < b < \frac{1}{2}\pi$.

By considering areas, show that

$$1 - \tfrac{1}{2}b^2 < \cos b < 1 - \tfrac{1}{2}b\sin b\,.$$

(ii) By considering the curve $y = a^x$, where $a > 1$, show that

$$\frac{2(a-1)}{a+1} < \ln a < -1 + \sqrt{2a-1}\,.$$

[**Hint**: You may wish to write a^x as $\mathrm{e}^{x\ln a}$.]

4 The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N.

(i) Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right).$$

(ii) In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

5 **(i)** Show that

$$\int_0^{\frac{1}{4}\pi} \sin(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{4}(\ln 2 - 1)\,,$$

(ii) and that

$$\int_0^{\frac{1}{4}\pi} \cos(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{8}(\pi - \ln 4 - 2)\,.$$

(iii) Hence evaluate

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \big(\cos(2x) + \sin(2x)\big)\,\ln\big(\cos x + \sin x\big)\,\mathrm{d}x\,.$$

6 A thin circular path with diameter AB is laid on horizontal ground. A vertical flagpole is erected with its base at a point D on the diameter AB. The angles of elevation of the top of the flagpole from A and B are α and β respectively (both are acute). The point C lies on the circular path with DC perpendicular to AB and the angle of elevation of the top of the flagpole from C is ϕ.

(i) Show that $\cot\alpha\cot\beta = \cot^2\phi$.

(ii) Show that, for any p and q,

$$\cos p\cos q\sin^2\tfrac{1}{2}(p+q) - \sin p\sin q\cos^2\tfrac{1}{2}(p+q) = \tfrac{1}{2}\cos(p+q) - \tfrac{1}{2}\cos(p+q)\cos(p-q).$$

(iii) Deduce that, if p and q are positive and $p+q \leqslant \frac{1}{2}\pi$, then

$$\cot p\cot q \geqslant \cot^2\tfrac{1}{2}(p+q)$$

and hence show that $\phi \leqslant \frac{1}{2}(\alpha+\beta)$ when $\alpha+\beta \leqslant \frac{1}{2}\pi$.

7 A sequence of numbers t_0, t_1, t_2, ... satisfies

$$t_{n+2} = pt_{n+1} + qt_n \qquad (n \geqslant 0),$$

where p and q are real. Throughout this question, x, y and z are non-zero real numbers.

(i) Show that, if $t_n = x$ for all values of n, then $p+q=1$ and x can be any (non-zero) real number.

(ii) Show that, if $t_{2n} = x$ and $t_{2n+1} = y$ for all values of n, then $q \pm p = 1$. Deduce that either $x = y$ or $x = -y$, unless p and q take certain values that you should identify.

(iii) Show that, if $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$ for all values of n, then

$$p^3 + q^3 + 3pq - 1 = 0.$$

Deduce that either $p+q=1$ or $(p-q)^2 + (p+1)^2 + (q+1)^2 = 0$. Hence show that either $x = y = z$ or $x+y+z=0$.

8 **(i)** Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad (x \neq 0).$$

Section B: Mechanics

9 A tall shot-putter projects a small shot from a point $2.5\,\text{m}$ above the ground, which is horizontal. The speed of projection is $10\,\text{m}\,\text{s}^{-1}$ and the angle of projection is θ above the horizontal.

(i) Taking the acceleration due to gravity to be $10\,\text{m}\,\text{s}^{-2}$, show that the time, in seconds, that elapses before the shot hits the ground is

$$\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right),$$

where $c = \cos 2\theta$.

(ii) Find an expression for the range in terms of c and show that it is greatest when $c = \frac{1}{5}$.

(iii) Show that the extra distance attained by projecting the shot at this angle rather than at an angle of $45°$ is $5(\sqrt{6}-\sqrt{2}-1)\,\text{m}$.

10 I stand at the top of a vertical well. The depth of the well, from the top to the surface of the water, is D. I drop a stone from the top of the well and measure the time that elapses between the release of the stone and the moment when I hear the splash of the stone entering the water.

In order to gauge the depth of the well, I climb a distance δ down into the well and drop a stone from my new position. The time until I hear the splash is t less than the previous time.

(i) Show that

$$t = \sqrt{\frac{2D}{g}} - \sqrt{\frac{2(D-\delta)}{g}} + \frac{\delta}{u},$$

where u is the (constant) speed of sound.

(ii) Hence show that

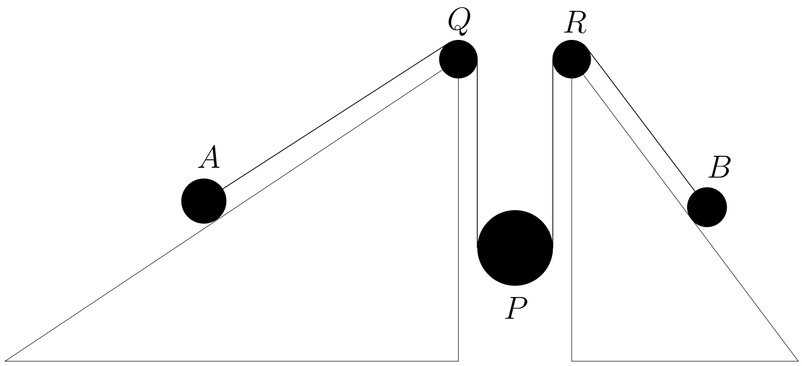
$$D = \tfrac{1}{2}gT^2,$$

where $T = \dfrac{1}{2}\beta + \dfrac{\delta}{\beta g}$ and $\beta = t - \dfrac{\delta}{u}$.

(iii) Taking $u = 300\,\text{m}\,\text{s}^{-1}$ and $g = 10\,\text{m}\,\text{s}^{-2}$, show that if $t = \frac{1}{5}\,\text{s}$ and $\delta = 10\,\text{m}$, the well is approximately $185\,\text{m}$ deep.

11 The diagram shows two particles, A of mass $5m$ and B of mass $3m$, connected by a light inextensible string which passes over two smooth, light, fixed pulleys, Q and R, and under a smooth pulley P which has mass M and is free to move vertically.

Particles A and B lie on fixed rough planes inclined to the horizontal at angles of $\arctan\frac{7}{24}$ and $\arctan\frac{4}{3}$ respectively. The segments AQ and RB of the string are parallel to their respective planes, and segments QP and PR are vertical. The coefficient of friction between each particle and its plane is μ.



(i) Given that the system is in equilibrium, with both A and B on the point of moving up their planes, determine the value of μ and show that $M = 6m$.

(ii) In the case when $M = 9m$, determine the initial accelerations of A, B and P in terms of g.

Section C: Probability and Statistics

12 Fire extinguishers may become faulty at any time after manufacture and are tested annually on the anniversary of manufacture.

The time T years after manufacture until a fire extinguisher becomes faulty is modelled by the continuous probability density function

$$\mathrm{f}(t) = \begin{cases} \dfrac{2t}{(1+t^2)^2} & \text{for } t \geqslant 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

A faulty fire extinguisher will fail an annual test with probability p, in which case it is destroyed immediately. A non-faulty fire extinguisher will always pass the test. All of the annual tests are independent.

(i) Show that the probability that a randomly chosen fire extinguisher will be destroyed exactly three years after its manufacture is $p(5p^2 - 13p + 9)/10$.

(ii) Find the probability that a randomly chosen fire extinguisher that was destroyed exactly three years after its manufacture was faulty 18 months after its manufacture.

13 I choose at random an integer in the range 10000 to 99999, all choices being equally likely. Given that my choice does not contain the digits 0, 6, 7, 8 or 9, show that the expected number of different digits in my choice is 3.3616.

Section A: Pure Mathematics

1 **(i)** Show that the gradient of the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $b\neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p,q) lies on both the straight line $ax+by=1$ and the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $ab\neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p=\pm q$.

Show further that either $(a-b)^2=1$ or $(a+b)^2=1$.

(ii) Show that if the straight line $ax+by=1$, where $ab\neq 0$, is a normal to the curve $\dfrac{a}{x}-\dfrac{b}{y}=1$, then $a^2-b^2=\frac{1}{2}$.

2 The number E is defined by $E=\displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x=\mathrm{e}-1-E\,,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x\,.$

3 Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

(i) By differentiating $(*)$, or otherwise, show that

$$\cot\tfrac{1}{9}\pi-\cot\tfrac{2}{9}\pi+\cot\tfrac{4}{9}\pi=\sqrt{3}\,.$$

(ii) By setting $\theta=\frac{1}{6}\pi-\phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi-\theta)\cot(\tfrac{1}{3}\pi+\theta)=\cot 3\theta\,.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi-\operatorname{cosec}\tfrac{5}{9}\pi+\operatorname{cosec}\tfrac{7}{9}\pi=2\sqrt{3}\,.$$

4 The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2=4ax$. The tangents to the curve at P and Q meet at the point T.

(i) Show that T has coordinates $\big(apq, a(p+q)\big)$. You may assume that $p\neq 0$ and $q\neq 0$.

(ii) The point F has coordinates $(a,0)$ and ϕ is the angle TFP. Show that

$$\cos\phi=\frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

5 **(i)** Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

(ii) Let

$$I = \int_0^{\pi} \left| \sin x - kx \right| \mathrm{d}x \, .$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha \, ,$$

where α is the unique solution of $(*)$.

(iii) Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum.

(iv) Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}} \, .$$

6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \cdots \, .$$

(ii) Find the sum of the series

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots \, .$$

7 In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h)\,,$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

8 **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n) = (1,0)$ and $(m,n) = (1,-1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

Section B: Mechanics

9 A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}\,.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

10 A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic.

(i) Explain why the speeds of A and B immediately before the first collision are the same.

(ii) The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M-m)u^2}{(M+m)^2 g}\,.$$

11 A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A.

(i) Show that

$$3\sin\alpha = 4\sin\beta\,,$$

where α and β are the angles PAB and PBA, respectively.

(ii) Given that $\cos\beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan\frac{1}{7}$.

Section C: Probability and Statistics

12 I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

Section A: Pure Mathematics

1 **(i)** Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

(ii) Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

2 The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

3 **(i)** Show that

$$\sin(x+y) - \sin(x-y) = 2\cos x \, \sin y$$

and deduce that

$$\sin A - \sin B = 2\cos \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

(ii) Show also that

$$\cos A - \cos B = -2\sin \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

(iii) The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

4 **(i)** Use the substitution $x = \dfrac{1}{t^2-1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\,\right) + c\,.$$

[Note You may use without proof the result $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$ **]**

(ii) The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360^o about the x-axis. Show that the volume enclosed is $2\pi \ln \frac{5}{4}$.

5 By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \cdots + \dbinom{n}{n} = 2^n\,;$

(ii) $\dbinom{n}{1} + 2\dbinom{n}{2} + 3\dbinom{n}{3} + \cdots + n\dbinom{n}{n} = n2^{n-1}\,;$

(iii) $\dbinom{n}{0} + \dfrac{1}{2}\dbinom{n}{1} + \dfrac{1}{3}\dbinom{n}{2} + \cdots + \dfrac{1}{n+1}\dbinom{n}{n} = \dfrac{1}{n+1}\left(2^{n+1}-1\right)\,;$

(iv) $\dbinom{n}{1} + 2^2\dbinom{n}{2} + 3^2\dbinom{n}{3} + \cdots + n^2\dbinom{n}{n} = n\,(n+1)\,2^{n-2}\,.$

6 **(i)** Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

(ii) In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

(iii) By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

7 Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$.

(i) Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

(ii) Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha+\beta}\,,$$

lies on the lines OC and AB.

(iii) The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

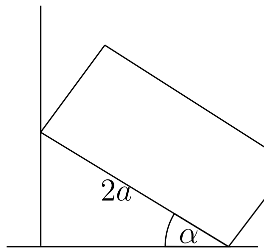
(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4\,.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

Section B: Mechanics

9



The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane.

(i) Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

(ii) The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

$$a\cos(2\lambda + \alpha) = b\sin\alpha\,,$$

where $\tan\lambda = \mu$.

(iii) Show also that if the lamina is square, then $\lambda = \frac{1}{4}\pi - \alpha$.

10 A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \big(\mathrm{e}^t\cos t\big)\,\mathbf{i} + \big(\mathrm{e}^t\sin t\big)\,\mathbf{j}\,.$$

(i) Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement.

(ii) Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

(iii) A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

11 Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions.

(i) Find the speeds of the particles after they collide for the first time and

(ii) show that they will both change direction if $2em > M - m$.

(iii) After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why

$$mv - MV = u(M - m),$$

and find v and V in terms of m, M, e, u and n.

Section C: Probability and Statistics

12 **(i)** A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ for this case, and show that

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} \mathrm{P}\left(X \geqslant n\right).$$

(ii) I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

(iii) Hence show that $\mathrm{E}(X) \geqslant 3$.

13 The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour.

(i) Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$p\mathrm{e}^{2\lambda} - \mathrm{e}^{\lambda} + 1 = 0.$$

(ii) Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

(iii) The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

(iv) Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Section A: Pure Mathematics

1 A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

2 A curve has the equation

$$y^3 = x^3 + a^3 + b^3 \,,$$

where a and b are positive constants.

(i) Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2y - a^2x = a^3 + b^3 \,.$$

(ii) In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0 \,.$$

(iii) Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3 \,.$$

3 **(i)** By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a - x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.

Find the number of distinct real solutions of the equation

$$x = \big((1+a)x - a\big)^{\frac{1}{3}}$$

in the cases that arise according to the value of a.

(ii) Find the number of distinct real solutions of the equation

$$x = (b + x)^{\frac{1}{2}}$$

in the cases that arise according to the value of b.

4 **(i)** The sides of a triangle have lengths $p - q$, p and $p + q$, where $p > q > 0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta\,.$$

(ii) In the case $\alpha = 2\beta$, show that $\cos\beta = \frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

5 A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

$$V = \tfrac{1}{3}\pi r^2 h\,, \qquad A = \pi r \ell\,.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $\ell = \sqrt{3}\,r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

6 **(i)** Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

(ii) Show by means of a substitution that

$$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

(iii) Evaluate:

(a) $\displaystyle\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x$;

(b) $\displaystyle\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x$.

7 Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx\,\mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi}-1\right).$$

(i) Expand $\cos(A+B)+\cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x\,\mathrm{d}x = \tfrac{19}{650}\left(\mathrm{e}^{2\pi}-1\right).$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x\,\mathrm{d}x$.

8 **(i)** The equation of the circle C is

$$(x-2t)^2 + (y-t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y=0$.

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y=0$, $3y=4x$ and $4y+3x=15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Section B: Mechanics

9 Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection *above* the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection *below* the horizontal is β. The particles collide at time T after projection.

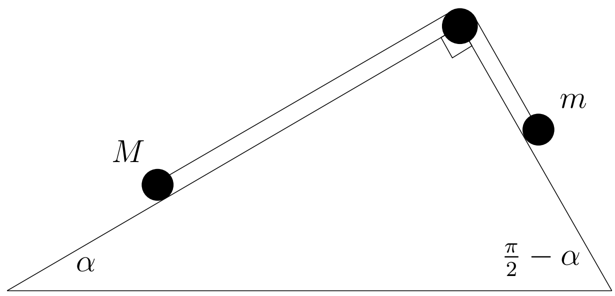
(i) Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2-1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0\,.$$

(ii) Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \leqslant (1+k)V^2\,.$$

10 A triangular wedge is fixed to a horizontal surface. The base angles of the wedge are α and $\frac{\pi}{2}-\alpha$. Two particles, of masses M and m, lie on different faces of the wedge, and are connected by a light inextensible string which passes over a smooth pulley at the apex of the wedge, as shown in the diagram. The contacts between the particles and the wedge are smooth.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ the particle of mass M will slide down the face of the wedge.

(ii) Given that $\tan\alpha = \dfrac{2m}{M}$, show that the magnitude of the acceleration of the particles is

$$\frac{g\sin\alpha}{\tan\alpha + 2}$$

and that this is maximised at $4m^3 = M^3$.

11 Two particles move on a smooth horizontal table and collide. The masses of the particles are m and M. Their velocities before the collision are $u\mathbf{i}$ and $v\mathbf{i}$, respectively, where $\mathbf{i}$ is a unit vector and $u > v$. Their velocities after the collision are $p\mathbf{i}$ and $q\mathbf{i}$, respectively. The coefficient of restitution between the two particles is e, where $e < 1$.

(i) Show that the loss of kinetic energy due to the collision is

$$\tfrac{1}{2}m(u-p)(u-v)(1-e)\,,$$

and deduce that $u \geqslant p$.

(ii) Given that each particle loses the same (non-zero) amount of kinetic energy in the collision, show that

$$u+v+p+q=0\,,$$

and that, if $m \neq M$,

$$e = \frac{(M+3m)u+(3M+m)v}{(M-m)(u-v)}\,.$$

Section C: Probability and Statistics

12 Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

(ii) Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is

$$\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

13 I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)}.$$

(iii) Find $\mathrm{E}(K)$.

Section A: Pure Mathematics

1 **(i)** What does it mean to say that a number x is *irrational*?

(ii) Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p+q$ is irrational, then at least one of p and q is irrational.

(iii) Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

(iv) If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

2 The variables t and x are related by $t = x + \sqrt{x^2+2bx+c}$, where b and c are constants and $b^2 < c$.

(i) Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b}\,,$$

and hence integrate $\dfrac{1}{\sqrt{x^2+2bx+c}}$.

(ii) Verify by direct integration that your result holds also in the case $b^2 = c$ if $x+b > 0$ but that your result does not hold in the case $b^2 = c$ if $x+b < 0$.

3 **(i)** Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad\,. \qquad (*)$$

(ii) If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy$.

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

(iii) Show similarly that the inequality

$$\frac{s}{t} + \frac{t}{r} + \frac{r}{s} + \frac{t}{s} + \frac{r}{t} + \frac{s}{r} \geqslant 6$$

holds for all positive r, s and t.

4 A function $\mathrm{f}(x)$ is said to be *convex* in the interval $a < x < b$ if $\mathrm{f}''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ is defined for $0 < x < 2\pi$ by

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ is convex.

(ii) The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

5 The polynomial $\mathrm{p}(x)$ is given by

$$\mathrm{p}(x) = x^n + \sum_{r=0}^{n-1} a_r x^r ,$$

where a_0, a_1, . . . , a_{n-1} are fixed real numbers and $n \geqslant 1$. Let M be the greatest value of $\left|\mathrm{p}(x)\right|$ for $|x| \leqslant 1$. Then *Chebyshev's theorem* states that $M \geqslant 2^{1-n}$.

(i) Prove Chebyshev's theorem in the case $n = 1$ and verify that Chebyshev's theorem holds in the following cases:

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$;

(b) $\mathrm{p}(x) = x^3 - x$.

(ii) Use Chebyshev's theorem to show that the curve $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.

6 The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g is the inverse function to f, so that $\mathrm{g}(\mathrm{f}(x)) = x$.

(i) Sketch $\mathrm{f}(x)$ and $\mathrm{g}(x)$ on the same axes.

(ii) Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}} + 1)} ,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}} + 1}$, and

(iii) explain this result by means of a diagram.

7 **(i)** The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by $60°$ clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$.

(ii) What is the new y-coordinate in the case of an anti-clockwise rotation by $60°$?

(iii) An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$ and find expressions for h_2 and h_3.

(iv) Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

8 **(i)** The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0\,. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

(ii) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0\,.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Section B: Mechanics

9 Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ.

(i) Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$.

(ii) Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

(iii) Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

10 On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2\sin^2\alpha > 2gh$.

(i) Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

(ii) Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'}\right) V \sin 2\alpha\,,$$

where $V' = \sqrt{V^2 - 2gh\,\mathrm{cosec}^2\alpha}$.

11 A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough rigid circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ.

(i) Given that the rod is in limiting equilibrium (i.e. on the point of slipping at both ends), show that

$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}\,,$$

where α is the angle between P_1O and the vertical ($0 < \alpha < 45°$).

(ii) Let θ be the acute angle between the rod and the horizontal.

Show that $\theta = 2\lambda$, where λ is defined by $\tan\lambda = \mu$ and $0 < \lambda < 22.5°$.

Section C: Probability and Statistics

12 In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\,.$$

The independent random variables X_1 and X_2 each take values $1, 2, \ldots, N$, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geqslant X_2 \\ X_2 & \text{if } X_2 \geqslant X_1\,. \end{cases}$$

(i) Show that $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ for $r = 1, 2, \ldots, N$.

(ii) Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.

(iii) The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. Find an expression for m in terms of N and give an explicit value for m in the case $N = 100$.

(iv) Show that when N is very large,

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}\,.$$

13 Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.

(ii) Find the probability that exactly two husbands sit next to their wives.

(iii) Find the probability that no husband sits next to his wife.

Section A: Pure Mathematics

1 A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits $0, 1, 2, 3$ and 4.

(ii) Show that $\frac{1}{6}k\left(k+1\right)\left(4k+5\right)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\displaystyle\sum_{r=0}^{n} r^2 \equiv \tfrac{1}{6}n\left(n+1\right)\left(2n+1\right)$.

2 **(i)** Given that $A = \arctan\frac{1}{2}$ and that $B = \arctan\frac{1}{3}$ (where A and B are acute) show, by considering $\tan\left(A+B\right)$, that $A + B = \frac{1}{4}\pi$.

The non-zero integers p and q satisfy

$$\arctan\frac{1}{p} + \arctan\frac{1}{q} = \frac{\pi}{4}\,.$$

Show that $\left(p-1\right)\left(q-1\right) = 2$ and hence determine p and q.

(ii) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan\frac{1}{r} + \arctan\frac{s}{s+t} = \frac{\pi}{4}\,,$$

then there are only two possible values for t, and give r in terms of s in each case.

3 **(i)** Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$.

(ii) Hence or otherwise evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^4\theta \, \mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^4\theta \, \mathrm{d}\theta\,.$$

(iii) Evaluate also

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \, \mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^6\theta \, \mathrm{d}\theta\,.$$

4 **(i)** Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x+b+c)\,\mathrm{Q}(x)$, where $\mathrm{Q}(x)$ is a quadratic expression.

(ii) Show that $2\mathrm{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

(iii) It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac - b^2\right)k = bc - a^2$$

and determine a similar expression involving k^2.

(iv) Hence show that

$$\left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0\,.$

(v) Deduce that either $k = 1$ or the two equations are identical.

5 *Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.*

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$.

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

6 **(i)** Given that $x^2 - y^2 = (x-y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = (\sqrt{m} - \sqrt{n})^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x-y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m-n)^4$.

7 **(i)** The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.

The line L_2 has vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Show that the distance D between a point on L_1 and a point on L_2 can be expressed in the form

$$D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36\,.$$

Hence determine the minimum distance between these two lines and find the coordinates of the points on the two lines that are the minimum distance apart.

(ii) The line L_3 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The line L_4 has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 4k \\ 1-k \\ -3k \end{pmatrix}$.

Determine the minimum distance between these two lines, explaining geometrically the two different cases that arise according to the value of k.

8 A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16)\ , \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2\,,\,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

Section B: Mechanics

9 A particle of weight W is placed on a rough plane inclined at an angle of θ to the horizontal. The coefficient of friction between the particle and the plane is μ. A horizontal force X acting on the particle is just sufficient to prevent the particle from sliding down the plane; when a horizontal force kX acts on the particle, the particle is about to slide up the plane. Both horizontal forces act in the vertical plane containing the line of greatest slope.

(i) Prove that

$$(k-1)\left(1+\mu^2\right)\sin\theta\cos\theta = \mu\,(k+1)$$

(ii) Hence show that $k \geqslant \dfrac{(1+\mu)^2}{(1-\mu)^2}$.

10 The Norman army is advancing with constant speed u towards the Saxon army, which is at rest. When the armies are d apart, a Saxon horseman rides from the Saxon army directly towards the Norman army at constant speed x. Simultaneously a Norman horseman rides from the Norman army directly towards the Saxon army at constant speed y, where $y > u$. The horsemen ride their horses so that $y - 2x < u < 2y - x$.

When each horseman reaches the opposing army, he immediately rides straight back to his own army without changing his speed.

(i) Represent this information on a displacement-time graph, and

(ii) show that the two horsemen pass each other at distances

$$\frac{xd}{x+y} \quad \text{and} \quad \frac{xd(2y-x-u)}{(u+x)(x+y)}$$

from the Saxon army.

(iii) Explain briefly what will happen in the cases (i) $u > 2y - x$ and (ii) $u < y - 2x$.

11 A smooth, straight, narrow tube of length L is fixed at an angle of 30° to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motion until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the tube.

(i) Show that D satisfies the equation

$$4gD^2 - 2\sqrt{3}\left(u^2 - Lg\right)D - 3L\left(u^2 - gL\right) = 0$$

and

(ii) hence that

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\,(u^2 - gL)}.$$

(iii) The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

Section C: Probability and Statistics

12 **(i)** A bag contains N sweets (where $N \geqslant 2$), of which a are red. Two sweets are drawn from the bag without replacement. Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

(ii) There are two bags, each containing N sweets (where $N \geqslant 2$). The first bag contains a red sweets, and the second bag contains b red sweets. There is also a biased coin, showing Heads with probability p and Tails with probability q, where $p + q = 1$.

The coin is tossed. If it shows Heads then a sweet is chosen from the first bag and transferred to the second bag; if it shows Tails then a sweet is chosen from the second bag and transferred to the first bag. The coin is then tossed a second time: if it shows Heads then a sweet is chosen from the first bag, and if it shows Tails then a sweet is chosen from the second bag.

Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

13 A bag contains eleven small discs, which are identical except that six of the discs are blank and five of the discs are numbered, using the numbers 1, 2, 3, 4 and 5. The bag is shaken, and four discs are taken one at a time without replacement.

Calculate the probability that:

(i) all four discs taken are numbered;

(ii) all four discs taken are numbered, given that the disc numbered "3" is taken first;

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first;

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken;

(v) exactly two numbered discs are taken, given that a numbered disc is taken first;

(vi) exactly two numbered discs are taken, given that a numbered disc is taken.

14 The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x+1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

It is known that $\mathrm{P}(X \geqslant 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathrm{P}\,(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathrm{P}\,(X = 1 \,|\, X \leqslant 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).

Section A: Pure Mathematics

1 **(i)** Find the integer, n, that satisfies $n^2 < 33127 < (n+1)^2$.

(ii) Find also a small integer m such that $(n+m)^2 - 33127$ is a perfect square.

(iii) Hence express 33127 in the form pq, where p and q are integers greater than 1.

(iv) By considering the possible factorisations of 33127, show that there are exactly two values of m for which $(n+m)^2 - 33127$ is a perfect square, and find the other value.

2 A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leqslant 14\pi a^2$ and determine the minimum value of A.

3 In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2+bx+c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

4 **(i)** By sketching on the same axes the graphs of $y = \sin x$ and $y = x$, show that, for $x > 0$:

(a) $x > \sin x$;

(b) $\dfrac{\sin x}{x} \approx 1$ for small x.

(ii) A regular polygon has n sides, and perimeter P. Show that the area of the polygon is

$$\frac{P^2}{4n\tan\left(\frac{\pi}{n}\right)} .$$

(iii) Show by differentiation (treating n as a continuous variable) that the area of the polygon increases as n increases with P fixed.

(iv) Show also that, for large n, the ratio of the area of the polygon to the area of the smallest circle which can be drawn around the polygon is approximately 1.

5 **(i)** Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K ,$$

where K is a constant.

(ii) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12} + \ln\frac{2}{3}$.

6 **(i)** Show that, if $(a\,,b)$ is **any** point on the curve $x^2 - 2y^2 = 1$, then $(3a+4b\,,2a+3b)$ also lies on the curve.

(ii) Determine the smallest positive integers M and N such that, if $(a\,,b)$ is **any** point on the curve $Mx^2 - Ny^2 = 1$, then $(5a+6b\,,4a+5b)$ also lies on the curve.

(iii) Given that the point $(a\,,b)$ lies on the curve $x^2 - 3y^2 = 1$, find positive integers P, Q, R and S such that the point $(Pa+Qb\,,Ra+Sb)$ also lies on the curve.

7 **(i)** Sketch on the same axes the functions $\operatorname{cosec} x$ and $2x/\pi$, for $0 < x < \pi$. Deduce that the equation $x \sin x = \pi/2$ has exactly two roots in the interval $0 < x < \pi$.

(ii) Show that

$$\int_{\pi/2}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \, \mathrm{d}x = 2 \sin \alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha \cos \alpha - 1$$

where α is the larger of the roots referred to above.

(iii) Show that the region bounded by the positive x-axis, the y-axis and the curve

$$y = \Big| |\mathrm{e}^x - 1| - 1 \Big|$$

has area $\ln 4 - 1$.

8 *Note that the volume of a tetrahedron is equal to* $\frac{1}{3} \times$ *the area of the base* $\times$ *the height.*

The points O, A, B and C have coordinates $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos \theta = \frac{c^2}{\sqrt{(a^2 + c^2)(b^2 + c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

(iii) Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Section B: Mechanics

9 A block of mass 4 kg is at rest on a smooth, horizontal table. A smooth pulley P is fixed to one edge of the table and a smooth pulley Q is fixed to the opposite edge. The two pulleys and the block lie in a straight line.

Two horizontal strings are attached to the block. One string runs over pulley P; a particle of mass x kg hangs at the end of this string. The other string runs over pulley Q; a particle of mass y kg hangs at the end of this string, where $x > y$ and $x + y = 6$.

The system is released from rest with the strings taut. When the 4 kg block has moved a distance d, the string connecting it to the particle of mass x kg is cut.

(i) Show that the time taken by the block from the start of the motion until it first returns to rest (assuming that it does not reach the edge of the table) is $\sqrt{d/(5g)}\,\mathrm{f}(y)$, where

$$\mathrm{f}(y) = \frac{10}{\sqrt{6-2y}} + \left(1 + \frac{4}{y}\right)\sqrt{6-2y}.$$

(ii) Calculate the value of y for which $\mathrm{f}'(y) = 0$.

10 **(i)** A particle P is projected in the x-y plane, where the y-axis is vertical and the x-axis is horizontal. The particle is projected with speed V from the origin at an angle of 45° above the positive x-axis. Determine the equation of the trajectory of P.

(ii) The point of projection (the origin) is on the floor of a barn. The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$ and α is an acute angle. Show that, if the particle just touches the roof, then $V(-1+\tan\alpha) = -2\sqrt{bg}$; you should justify the choice of the negative root. If this condition is satisfied, find, in terms of α, V and g, the time after projection at which touching takes place.

(iii) A particle Q can slide along a smooth rail fixed, in the x-y plane, to the under-side of the roof. It is projected from the point $(0, b)$ with speed U at the same time as P is projected from the origin. Given that the particles just touch in the course of their motions, show that

$$2\sqrt{2}\,U\cos\alpha = V\left(2 + \sin\alpha\cos\alpha - \sin^2\alpha\right).$$

11 Particles A_1, A_2, A_3, $\ldots$, A_n (where $n \geqslant 2$) lie at rest in that order in a smooth straight horizontal trough. The mass of A_{n-1} is m and the mass of A_n is λm, where $\lambda > 1$. Another particle, A_0, of mass m, slides along the trough with speed u towards the particles and collides with A_1. Momentum and energy are conserved in all collisions.

(i) Show that it is not possible for there to be exactly one particle moving after all collisions have taken place.

(ii) Show that it is not possible for A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iii) Show that it is not possible for A_{n-2}, A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iv) Given that there are exactly two particles moving after all collisions have taken place, find the speeds of these particles in terms of u and λ.

Section C: Probability and Statistics

12 Oxtown and Camville are connected by three roads, which are at risk of being blocked by flooding. On two of the three roads there are two sections which may be blocked. On the third road there is only one section which may be blocked. The probability that each section is blocked is p. Each section is blocked independently of the other four sections.

(i) Show that the probability that Oxtown is cut off from Camville is $p^3\left(2-p\right)^2$.

(ii) I want to travel from Oxtown to Camville. I choose one of the three roads at random and find that my road is not blocked. Find the probability that I would not have reached Camville if I had chosen either of the other two roads. You should factorise your answer as fully as possible.

(iii) Comment briefly on the value of this probability in the limit $p \to 1$.

13 A very generous shop-owner is hiding small diamonds in chocolate bars. Each diamond is hidden independently of any other diamond, and on average there is one diamond per kilogram of chocolate.

(i) I go to the shop and roll a fair six-sided die once. I decide that if I roll a score of N, I will buy $100N$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

Show also that the expected number of diamonds I find is 0.35.

(ii) Instead, I decide to roll a fair six-sided die repeatedly until I score a 6. If I roll my first 6 on my Tth throw, I will buy $100T$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6-5\mathrm{e}^{-0.1}}$$

Calculate also the expected number of diamonds that I find. (You may find it useful to consider the the binomial expansion of $(1-x)^{-2}$.)

14 **(i)** A bag of sweets contains one red sweet and n blue sweets. I take a sweet from the bag, note its colour, return it to the bag, then shake the bag. I repeat this until the sweet I take is the red one. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

(ii) Instead, I take sweets from the bag, without replacing them in the bag, until I take the red sweet. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

Section A: Pure Mathematics

1 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

2 The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q.

(i) Show that R has coordinates $(pq, p+q)$.

(ii) The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1,0)$ lies on the line PQ, show that S has coordinates $(p^2+q^2+1,\ p+q)$, and that the quadrilateral $PSQR$ is a rectangle.

3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that, if $c \neq 1$, the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if $c^2 = -\dfrac{4ab}{(a-b)^2}$. Show that this condition can be written $c^2 = 1 - \left(\dfrac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leqslant 1$.

4 **(i)** Given that $\cos\theta = \dfrac{3}{5}$ and that $\dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\dfrac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \dfrac{11}{2}$ and that $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

5 **(i)** Evaluate the integral

$$\int_0^1 (x+1)^{k-1}\ \mathrm{d}x$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

(ii) Evaluate the integral

$$\int_0^1 x\,(x+1)^m\ \mathrm{d}x$$

in the different cases that arise according to the value of m.

6 **(i)** The point A has coordinates $(5\,,16)$ and the point B has coordinates $(-4\,,4)$. The variable point P has coordinates $(x\,,y)$ and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100\ .$$

(ii) The point C has coordinates $(a\,,0)$ and the point D has coordinates $(b\,,0)$, where $a \neq b$. The variable point Q moves on a path such that

$$QC = k \times QD\ ,$$

where $k > 1$. Given that the path of Q is the same as the path of P, show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}\ .$$

Show further that $(a+7)(b+7) = 100$.

7 The notation $\prod_{r=1}^{n} \mathrm{f}(r)$ denotes the product $\mathrm{f}(1) \times \mathrm{f}(2) \times \mathrm{f}(3) \times \cdots \times \mathrm{f}(n)$.

Simplify the following products as far as possible:

(i) $\displaystyle\prod_{r=1}^{n} \left(\frac{r+1}{r}\right)$;

(ii) $\displaystyle\prod_{r=2}^{n} \left(\frac{r^2-1}{r^2}\right)$;

(iii) $\displaystyle\prod_{r=1}^{n} \left(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}\cot\frac{(2r-1)\,\pi}{n}\right)$, where n is even.

8 Show that, if $y^2 = x^k \mathrm{f}(x)$, then $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}\dfrac{\mathrm{df}}{\mathrm{d}x}$.

(i) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

(ii) Find the solution of the differential equation

$$2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln(x) - xy^2$$

for which $y = 1$ when $x = 1$.

Section B: Mechanics

9 A non-uniform rod AB has weight W and length $3l$. When the rod is suspended horizontally in equilibrium by vertical strings attached to the ends A and B, the tension in the string attached to A is T.

(i) When instead the rod is held in equilibrium in a horizontal position by means of a smooth pivot at a distance l from A and a vertical string attached to B, the tension in the string is T. Show that $5T = 2W$.

(ii) When instead the end B of the rod rests on rough horizontal ground and the rod is held in equilibrium at an angle θ to the horizontal by means of a string that is perpendicular to the rod and attached to A, the tension in the string is $\frac{1}{2}T$. Calculate θ and find the smallest value of the coefficient of friction between the rod and the ground that will prevent slipping.

10 Three collinear, non-touching particles A, B and C have masses a, b and c, respectively, and are at rest on a smooth horizontal surface. The particle A is given an initial velocity u towards B. These particles collide, giving B a velocity v towards C. These two particles then collide, giving C a velocity w.

The coefficient of restitution is e in both collisions. Determine an expression for v, and show that

$$w = \frac{abu\,(1+e)^2}{(a+b)\,(b+c)}\,.$$

Determine the final velocities of each of the three particles in the cases:

(i) $\dfrac{a}{b} = \dfrac{b}{c} = e\,;$

(ii) $\dfrac{b}{a} = \dfrac{c}{b} = e\,.$

11 A particle moves so that $\mathbf{r}$, its displacement from a fixed origin at time t, is given by

$$\mathbf{r} = (\sin 2t)\,\mathbf{i} + (2\cos t)\,\mathbf{j}\,,$$

where $0 \leqslant t < 2\pi$.

(i) Show that the particle passes through the origin exactly twice.

(ii) Determine the times when the velocity of the particle is perpendicular to its displacement.

(iii) Show that, when the particle is not at the origin, its velocity is never parallel to its displacement.

(iv) Determine the maximum distance of the particle from the origin, and sketch the path of the particle.

Section C: Probability and Statistics

12 **(i)** The probability that a hobbit smokes a pipe is 0.7 and the probability that a hobbit wears a hat is 0.4 . The probability that a hobbit smokes a pipe but does not wear a hat is p. Determine the range of values of p consistent with this information.

(ii) The probability that a wizard wears a hat is 0.7 ; the probability that a wizard wears a cloak is 0.8 ; and the probability that a wizard wears a ring is 0.4 . The probability that a wizard does not wear a hat, does not wear a cloak and does not wear a ring is 0.05 . The probability that a wizard wears a hat, a cloak and also a ring is 0.1 . Determine the probability that a wizard wears exactly two of a hat, a cloak, and a ring.

The probability that a wizard wears a hat but not a ring, **given** that he wears a cloak, is q. Determine the range of values of q consistent with this information.

13 The random variable X has mean μ and standard deviation σ. The distribution of X is symmetrical about μ and satisfies:

$$\mathrm{P}\left(X \leqslant \mu+\sigma\right)=a \qquad \text{and} \qquad \mathrm{P}\left(X \leqslant \mu+\tfrac{1}{2}\sigma\right)=b\,,$$

where a and b are fixed numbers. Do not assume that X is Normally distributed.

(i) Determine expressions (in terms of a and b) for

$$\mathrm{P}\left(\mu-\tfrac{1}{2}\sigma \leqslant X \leqslant \mu+\sigma\right) \qquad \text{and} \qquad \mathrm{P}\left(X \leqslant \mu+\tfrac{1}{2}\sigma \;\middle|\; X \geqslant \mu-\tfrac{1}{2}\sigma\right).$$

(ii) My local supermarket sells cartons of skimmed milk and cartons of full-fat milk: 60% of the cartons it sells contain skimmed milk, and the rest contain full-fat milk.

The volume of skimmed milk in a carton is modelled by X ml, with $\mu=500$ and $\sigma=10$. The volume of full-fat milk in a carton is modelled by X ml, with $\mu=495$ and $\sigma=10$.

(a) Today, I bought one carton of milk, chosen at random, from this supermarket. When I get home, I find that it contains less than 505 ml. Determine an expression (in terms of a and b) for the probability that this carton of milk contains more than 500 ml.

(b) Over the years, I have bought a very large number of cartons of milk, all chosen at random, from this supermarket. 70% of the cartons I have bought have contained at most 505 ml of milk. Of all the cartons that have contained at least 495 ml of milk, one third of them have contained full-fat milk. Use this information to estimate the values of a and b.

14 The random variable X can take the value $X = -1$, and also any value in the range $0 \leqslant X < \infty$. The distribution of X is given by

$$\mathrm{P}(X = -1) = m\,, \qquad \mathrm{P}(0 \leqslant X \leqslant x) = k(1 - \mathrm{e}^{-x})\,,$$

for any non-negative number x, where k and m are constants, and $m < \frac{1}{2}$.

(i) Find k in terms of m.

(ii) Show that $\mathrm{E}(X) = 1 - 2m$.

(iii) Find, in terms of m, $\mathrm{Var}\,(X)$ and the median value of X.

(iv) Given that

$$\int_0^\infty y^2 \mathrm{e}^{-y^2} \mathrm{d}y = \tfrac{1}{4}\sqrt{\pi}\,,$$

find $\mathrm{E}\big(|X|^{\frac{1}{2}}\big)$ in terms of m.

Section A: Pure Mathematics

1 **(i)** Express $\left(3+2\sqrt{5}\right)^3$ in the form $a+b\sqrt{5}$ where a and b are integers.

(ii) Find the positive integers c and d such that $\sqrt[3]{99-70\sqrt{2}}\ = c-d\sqrt{2}$.

(iii) Find the two real solutions of $x^6-198x^3+1=0$.

2 The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi]=3$, $[\sqrt{24}]=4$ and $[5]=5$.

(i) Sketch the graph of $y=\sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \sum_{r=0}^{a-1}\sqrt{r}$$

when a is a positive integer.

(ii) Show that $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x = 2^a-1$ when a is a positive integer.

(iii) Determine an expression for $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x$ when a is positive but not an integer.

3 **(i)** Show that $x-3$ is a factor of

$$x^3-5x^2+2x^2y+xy^2-8xy-3y^2+6x+6y\,. \qquad (*)$$

Express $(*)$ in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

(ii) Factorise $6y^3-y^2-21y+2x^2+12x-4xy+x^2y-5xy^2+10$ into three linear factors.

4 Differentiate $\sec t$ with respect to t.

(i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

(ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x$.

(iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x$.

5 The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned} A &: \ 1,\ 6,\ 11,\ 16,\ \dots \\ B &: \ 2,\ 7,\ 12,\ 17,\ \dots \\ C &: \ 3,\ 8,\ 13,\ 18,\ \dots \\ D &: \ 4,\ 9,\ 14,\ 19,\ \dots \\ E &: \ 5, 10,\ 15,\ 20,\ \dots \end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723 \,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974 \,.$$

6 **(i)** The three points A, B and C have coordinates $(p_1\,,\ q_1)$, $(p_2\,,\ q_2)$ and $(p_3\,,\ q_3)$, respectively. Find the point of intersection of the line joining A to the midpoint of BC, and the line joining B to the midpoint of AC.

(ii) Verify that this point lies on the line joining C to the midpoint of AB.

(iii) The point H has coordinates $(p_1+p_2+p_3\,,\ q_1+q_2+q_3)$. Show that if the line AH intersects the line BC at right angles, then $p_2^2+q_2^2=p_3^2+q_3^2$, and write down a similar result if the line BH intersects the line AC at right angles.

(iv) Deduce that if AH is perpendicular to BC and also BH is perpendicular to AC, then CH is perpendicular to AB.

7 **(i)** The function $\mathrm{f}(x)$ is defined for $|x|<\frac{1}{5}$ by

$$\mathrm{f}(x)=\sum_{n=0}^{\infty}a_nx^n\,,$$

where $a_0=2$, $a_1=7$ and $a_n=7a_{n-1}-10a_{n-2}$ for $n\geqslant 2$.

Simplify $\mathrm{f}(x)-7x\mathrm{f}(x)+10x^2\mathrm{f}(x)$, and hence show that $\mathrm{f}(x)=\dfrac{1}{1-2x}+\dfrac{1}{1-5x}$.

Hence show that $a_n=2^n+5^n$.

(ii) The function $\mathrm{g}(x)$ is defined for $|x|<\frac{1}{3}$ by

$$\mathrm{g}(x)=\sum_{n=0}^{\infty}b_nx^n\,,$$

where $b_0=5$, $b_1=10$, $b_2=40$, $b_3=100$ and $b_n=pb_{n-1}+qb_{n-2}$ for $n\geqslant 2$. Obtain an expression for $\mathrm{g}(x)$ as the sum of two algebraic fractions and determine b_n in terms of n.

8 A sequence t_0, t_1, t_2, ... is said to be *strictly increasing* if $t_{n+1} > t_n$ for all $n \geqslant 0$.

(i) The terms of the sequence x_0, x_1, x_2, ... satisfy

$$x_{n+1} = \frac{x_n^2 + 6}{5}$$

for $n \geqslant 0$. Prove that if $x_0 > 3$ then the sequence is strictly increasing.

(ii) The terms of the sequence y_0, y_1, y_2, ... satisfy

$$y_{n+1} = 5 - \frac{6}{y_n}$$

for $n \geqslant 0$. Prove that if $2 < y_0 < 3$ then the sequence is strictly increasing but that $y_n < 3$ for all n.

Section B: Mechanics

9 **(i)** A particle is projected over level ground with a speed u at an angle θ above the horizontal. Derive an expression for the greatest height of the particle in terms of u, θ and g.

(ii) A particle is projected from the floor of a horizontal tunnel of height $\frac{9}{10}d$. Point P is $\frac{1}{2}d$ metres vertically and d metres horizontally along the tunnel from the point of projection. The particle passes through point P and lands inside the tunnel without hitting the roof. Show that

$$\arctan \tfrac{3}{5} < \theta < \arctan 3 \,.$$

10 A particle is travelling in a straight line. It accelerates from its initial velocity u to velocity v, where $v > |u| > 0$, travelling a distance d_1 with uniform acceleration of magnitude $3a$. It then comes to rest after travelling a further distance d_2 with uniform deceleration of magnitude a. Show that

(i) if $u > 0$ then $3d_1 < d_2$;

(ii) if $u < 0$ then $d_2 < 3d_1 < 2d_2$.

Show also that the average speed of the particle (that is, the total distance travelled divided by the total time) is greater in the case $u > 0$ than in the case $u < 0$.

Note: In this question d_1 and d_2 are distances travelled by the particle which are not the same, in the second case, as displacements from the starting point.

11 Two uniform ladders AB and BC of equal length are hinged smoothly at B. The weight of AB is W and the weight of BC is $4W$. The ladders stand on rough horizontal ground with $\angle ABC = 60°$. The coefficient of friction between each ladder and the ground is μ.

A decorator of weight $7W$ begins to climb the ladder AB slowly. When she has climbed up $\frac{1}{3}$ of the ladder, one of the ladders slips. Which ladder slips, and what is the value of μ?

Section C: Probability and Statistics

12 In a certain factory, microchips are made by two machines. Machine A makes a proportion λ of the chips, where $0 < \lambda < 1$, and machine B makes the rest. A proportion p of the chips made by machine A are perfect, and a proportion q of those made by machine B are perfect, where $0 < p < 1$ and $0 < q < 1$. The chips are sorted into two groups: group 1 contains those that are perfect and group 2 contains those that are imperfect.

(i) In a large random sample taken from group 1, it is found that $\frac{2}{5}$ were made by machine A. Show that λ can estimated as

$$\frac{2q}{3p+2q}\,.$$

(ii) Subsequently, it is discovered that the sorting process is faulty: there is a probability of $\frac{1}{4}$ that a perfect chip is assigned to group 2 and a probability of $\frac{1}{4}$ that an imperfect chip is assigned to group 1. Taking into account this additional information, obtain a new estimate of λ.

13 **(i)** Three real numbers are drawn independently from the continuous rectangular distribution on $[0,1]$. The random variable X is the maximum of the three numbers. Show that the probability that $X \leqslant 0.8$ is 0.512, and calculate the expectation of X.

(ii) N real numbers are drawn independently from a continuous rectangular distribution on $[0,a]$. The random variable X is the maximum of the N numbers. A hypothesis test with a significance level of 5% is carried out using the value, x, of X. The null hypothesis is that $a = 1$ and the alternative hypothesis is that $a < 1$. The form of the test is such that H_0 is rejected if $x < c$, for some chosen number c.

Using the approximation $2^{10} \approx 10^3$, determine the smallest integer value of N such that if $x \leqslant 0.8$ the null hypothesis will be rejected.

With this value of N, write down the probability that the null hypothesis is rejected if $a = 0.8$, and find the probability that the null hypothesis is rejected if $a = 0.9$.

14 Three pirates are sharing out the contents of a treasure chest containing n gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

Find:

(i) the probability that the first pirate will have some gold coins;

(ii) the probability that the second pirate will have some gold coins;

(iii) the probability that all three pirates will have some gold coins.

Section A: Pure Mathematics

1 It is given that $\sum_{r=-1}^{n} r^2$ can be written in the form $pn^3 + qn^2 + rn + s$, where p, q, r and s are numbers.

(i) By setting $n = -1$, 0, 1 and 2, obtain four equations that must be satisfied by p, q, r and s and hence show that

$$\sum_{r=0}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\ .$$

(ii) Given that $\sum_{r=-2}^{n} r^3$ can be written in the form $an^4 + bn^3 + cn^2 + dn + e$, show similarly that

$$\sum_{r=0}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2\ .$$

2 **(i)** The first question on an examination paper is:

Solve for x the equation $\quad \dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$.

where (in the question) a and b are given non-zero real numbers. One candidate writes $x = a + b$ as the solution. Show that there are no values of a and b for which this will give the correct answer.

(ii) The next question on the examination paper is:

Solve for x the equation $\quad \dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

where (in the question) a, b and c are given non-zero numbers. The candidate uses the same technique, giving the answer as $x = a + b + c$. Show that the candidate's answer will be correct if and only if a, b and c satisfy at least one of the equations $a + b = 0$, $b + c = 0$ or $c + a = 0$.

3 **(i)** Show that $2\sin(\frac{1}{2}\theta) = \sin\theta$ if and only if $\sin(\frac{1}{2}\theta) = 0$.

(ii) Solve the equation $2\tan(\frac{1}{2}\theta) = \tan\theta$.

(iii) Show that $2\cos(\frac{1}{2}\theta) = \cos\theta$ if and only if $\theta = (4n+2)\pi \pm 2\phi$ where ϕ is defined by $\cos\phi = \frac{1}{2}(\sqrt{3}-1)$, $0 \leqslant \phi \leqslant \frac{1}{2}\pi$, and n is any integer.

4 Solve the inequality

$$\frac{\sin\theta + 1}{\cos\theta} \leqslant 1$$

where $0 \leqslant \theta < 2\pi$ and $\cos\theta \neq 0$.

5 **(i)** In the binomial expansion of $(2x+1/x^2)^6$ for $x \neq 0$, show that the term which is independent of x is 240.

Find the term which is independent of x in the binomial expansion of $(ax^3 + b/x^2)^{5n}$.

(ii) Let $\mathrm{f}(x) = (x^6 + 3x^5)^{1/2}$. By considering the expansion of $(1+3/x)^{1/2}$ show that the term which is independent of x in the expansion of $\mathrm{f}(x)$ in powers of $1/x$, for $|x| > 3$, is $27/16$.

Show that there is no term independent of x in the expansion of $\mathrm{f}(x)$ in powers of x, for $|x| < 3$.

6 Evaluate the following integrals, in the different cases that arise according to the value of the positive constant a:

(i) $$\int_0^1 \frac{1}{x^2 + (a+2)x + 2a}\,\mathrm{d}x\ ;$$

(ii) $$\int_1^2 \frac{1}{u^2 + au + a - 1}\,\mathrm{d}u\ .$$

7 **(i)** Let k be an integer satisfying $0 \leqslant k \leqslant 9$. Show that $0 \leqslant 10k - k^2 \leqslant 25$ and that $k(k-1)(k+1)$ is divisible by 3.

(ii) For each 3-digit number N, where $N \geqslant 100$, let S be the sum of the hundreds digit, the square of the tens digit and the cube of the units digit. Find the numbers N such that $S = N$.

[Hint: write $N = 100a + 10b + c$ where a, b and c are the digits of N.]

8 A liquid of fixed volume V is made up of two chemicals A and B. A reaction takes place in which A converts to B. The volume of A at time t is xV and the volume of B at time t is yV where x and y depend on t and $x + y = 1$. The rate at which A converts into B is given by $kVxy$, where k is a positive constant.

(i) Show that if both x and y are strictly positive at the start, then at time t

$$y = \frac{D\mathrm{e}^{kt}}{1 + D\mathrm{e}^{kt}},$$

where D is a constant.

(ii) Does A ever completely convert to B? Justify your answer.

Section B: Mechanics

9 A particle is projected with speed V at an angle θ above the horizontal. The particle passes through the point P which is a horizontal distance d and a vertical distance h from the point of projection.

(i) Show that

$$T^2 - 2kT + \frac{2kh}{d} + 1 = 0\,,$$

where $T = \tan\theta$ and $k = \dfrac{V^2}{gd}$.

(ii) Show that, if $kd > h + \sqrt{h^2 + d^2}$, there are two distinct possible angles of projection.

(iii) Let these two angles be α and β. Show that $\alpha + \beta = \pi - \arctan(d/h)$.

10 $ABCD$ is a uniform rectangular lamina and X is a point on BC . The lengths of AD, AB and BX are p , q and r respectively. The triangle ABX is cut off the lamina. Let (a, b) be the position of the centre of gravity of the lamina, where the axes are such that the coordinates of A , D and C are $(0, 0)$, $(p, 0)$ and (p, q) respectively.

(i) Derive equations for a and b in terms of p , q and r .

(ii) When the resulting trapezium is freely suspended from the point A , the side AD is inclined at 45° below the horizontal. Show that $r = q - \sqrt{q^2 - 3pq + 3p^2}$. You should justify carefully the choice of sign in front of the square root.

11 A smooth plane is inclined at an angle α to the horizontal. A and B are two points a distance d apart on a line of greatest slope of the plane, with B higher than A. A particle is projected up the plane from A towards B with initial speed u, and simultaneously another particle is released from rest at B .

(i) Show that they collide after a time d/u .

(ii) The coefficient of restitution between the two particles is e and both particles have mass m . Show that the loss of kinetic energy in the collision is $\frac{1}{4}mu^2\big(1 - e^2\big)$.

Section C: Probability and Statistics

12 In a bag are n balls numbered 1, 2, ..., n. When a ball is taken out of the bag, each ball is equally likely to be taken.

(i) A ball is taken out of the bag. The number on the ball is noted and the ball is replaced in the bag. The process is repeated once. Explain why the expected value of the product of the numbers on the two balls is

$$\frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n} rs$$

and simplify this expression.

(ii) A ball is taken out of the bag. The number on the ball is noted and the ball is *not* replaced in the bag. Another ball is taken out of the bag and the number on this ball is noted. Show that the expected value of the product of the two numbers is

$$\frac{(n+1)(3n+2)}{12}\,.$$

Note: $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

13 If a football match ends in a draw, there may be a "penalty shoot-out". Initially the teams each take 5 shots at goal. If one team scores more times than the other, then that team wins. If the scores are level, the teams take shots alternately until one team scores and the other team does not score, both teams having taken the same number of shots. The team that scores wins.

Two teams, Team A and Team B, take part in a penalty shoot-out. Their probabilities of scoring when they take a single shot are p_A and p_B respectively.

(i) Explain why the probability α of neither side having won at the end of the initial 10-shot period is given by

$$\alpha = \sum_{i=0}^{5}\binom{5}{i}^2 (1-p_A)^i(1-p_B)^i\,p_A^{5-i}p_B^{5-i}.$$

(ii) Show that the expected number of shots taken is $10 + \dfrac{2\alpha}{\beta}$, where $\beta = p_A + p_B - 2p_Ap_B$.

14 Jane goes out with any of her friends who call, except that she never goes out with more than two friends in a day. The number of her friends who call on a given day follows a Poisson distribution with parameter 2.

(i) Show that the average number of friends she sees in a day is $2 - 4e^{-2}$.

(ii) Now Jane has a new friend who calls on any given day with probability p. Her old friends call as before, independently of the new friend. She never goes out with more than two friends in a day. Find the average number of friends she now sees in a day.

Section A: Pure Mathematics

1 Show that the equation of any circle passing through the points of intersection of the ellipse $(x+2)^2 + 2y^2 = 18$ and the ellipse $9(x-1)^2 + 16y^2 = 25$ can be written in the form

$$x^2 - 2ax + y^2 = 5 - 4a\ .$$

2 Let $\mathrm{f}(x) = x^m(x-1)^n$, where m and n are both integers greater than 1.

(i) Show that the curve $y = \mathrm{f}(x)$ has a stationary point with $0 < x < 1$.

(ii) By considering $\mathrm{f}''(x)$, show that this stationary point is a maximum if n is even and a minimum if n is odd.

(iii) Sketch the graphs of $\mathrm{f}(x)$ in the four cases that arise according to the values of m and n.

3 **(i)** Show that $(a+b)^2 \leqslant 2a^2 + 2b^2$.

(ii) Find the stationary points on the curve $y = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}}$, where a and b are constants. State, with brief reasons, which points are maxima and which are minima.

(iii) Hence prove that

$$|a| + |b| \leqslant \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}} \leqslant \left(2a^2 + 2b^2\right)^{\frac{1}{2}}\ .$$

4 **(i)** Give a sketch of the curve $y = \dfrac{1}{1+x^2}$, for $x \geqslant 0$.

(ii) Find the equation of the line that intersects the curve at $x = 0$ and is tangent to the curve at some point with $x > 0$.

(iii) Prove that there are no further intersections between the line and the curve. Draw the line on your sketch.

(iv) By considering the area under the curve for $0 \leqslant x \leqslant 1$, show that $\pi > 3$.

Show also, by considering the volume formed by rotating the curve about the y axis, that $\ln 2 > 2/3$.

[**Note**: $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$.]

5 Let

$$\mathrm{f}(x) = x^n + a_1 x^{n-1} + \cdots + a_n \ ,$$

where a_1, a_2, ..., a_n are given numbers. It is given that $\mathrm{f}(x)$ can be written in the form

$$\mathrm{f}(x) = (x + k_1)(x + k_2) \cdots (x + k_n) \ .$$

(i) By considering $\mathrm{f}(0)$, or otherwise, show that $k_1 k_2 \ldots k_n = a_n$.

(ii) Show also that

$$(k_1 + 1)(k_2 + 1) \cdots (k_n + 1) = 1 + a_1 + a_2 + \cdots + a_n$$

and give a corresponding result for $(k_1 - 1)(k_2 - 1) \cdots (k_n - 1)$.

(iii) Find the roots of the equation

$$x^4 + 22x^3 + 172x^2 + 552x + 576 = 0 \ ,$$

given that they are all integers.

6 A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three sides of the pyramid are of length b and its volume is V.

(i) Given that the formula for the volume of any pyramid is $\frac{1}{3}\times$ area of base $\times$ height, show that

$$V = \frac{1}{12}a^2(3b^2 - a^2)^{\frac{1}{2}} .$$

(ii) The pyramid is then placed so that a non-equilateral face lies on the ground. Show that the new height, h, of the pyramid is given by

$$h^2 = \frac{a^2(3b^2 - a^2)}{4b^2 - a^2} .$$

(iii) Find, in terms of a and b, the angle between the equilateral triangle and the horizontal.

7 Let

$$I = \int_0^a \frac{\cos x}{\sin x + \cos x}\,\mathrm{d}x \quad \text{and} \quad J = \int_0^a \frac{\sin x}{\sin x + \cos x}\,\mathrm{d}x ,$$

where $0 \leqslant a < \frac{3}{4}\pi$. By considering $I + J$ and $I - J$, show that $2I = a + \ln(\sin a + \cos a)$.

Find also:

(i) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x}{p\sin x + q\cos x}\,\mathrm{d}x$, where p and q are positive numbers;

(ii) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x + 4}{3\sin x + 4\cos x + 25}\,\mathrm{d}x$.

8 I borrow C pounds at interest rate $100\alpha\,\%$ per year. The interest is added at the end of each year. Immediately after the interest is added, I make a repayment. The amount I repay at the end of the kth year is R_k pounds and the amount I owe at the beginning of kth year is C_k pounds (with $C_1 = C$).

(i) Express C_{n+1} in terms of R_k ($k = 1, 2, \ldots, n$), α and C and show that, if I pay off the loan in N years with repayments given by $R_k = (1+\alpha)^k r$, where r is constant, then $r = C/N$.

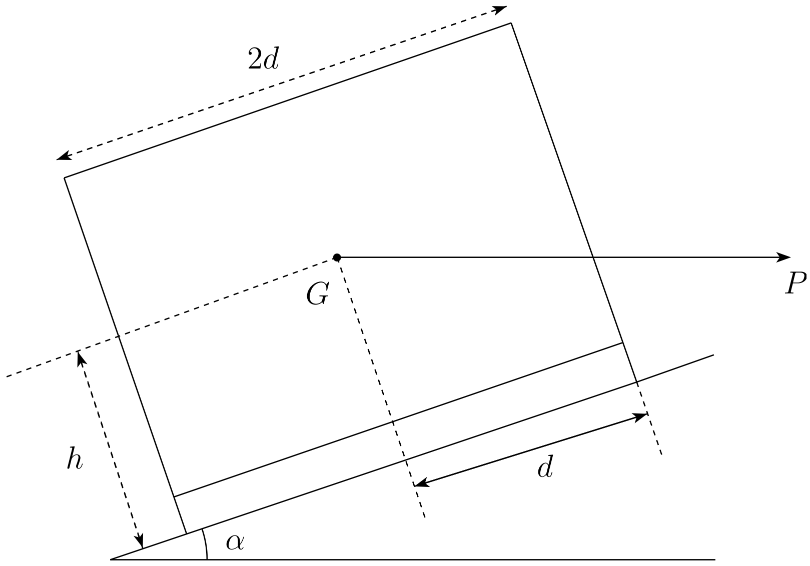
(ii) If instead I pay off the loan in N years with N equal repayments of R pounds, show that

$$\frac{R}{C} = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1} ,$$

and that $R/C \approx 27/103$ in the case $\alpha = 1/50$, $N = 4$.

Section B: Mechanics

9



A lorry of weight W stands on a plane inclined at an angle α to the horizontal. Its wheels are a distance $2d$ apart, and its centre of gravity G is at a distance h from the plane, and halfway between the sides of the lorry. A horizontal force P acts on the lorry through G, as shown.

(i) If the normal reactions on the lower and higher wheels of the lorry are equal, show that the sum of the frictional forces between the wheels and the ground is zero.

(ii) If P is such that the lorry does not tip over (but the normal reactions on the lower and higher wheels of the lorry need not be equal), show that

$$W\tan(\alpha-\beta) \leqslant P \leqslant W\tan(\alpha+\beta) ,$$

where $\tan\beta = d/h$.

10 A bicycle pump consists of a cylinder and a piston. The piston is pushed in with steady speed u. A particle of air moves to and fro between the piston and the end of the cylinder, colliding perfectly elastically with the piston and the end of the cylinder, and always moving parallel with the axis of the cylinder. Initially, the particle is moving towards the piston at speed v.

(i) Show that the speed, v_n, of the particle just after the nth collision with the piston is given by $v_n = v + 2nu$.

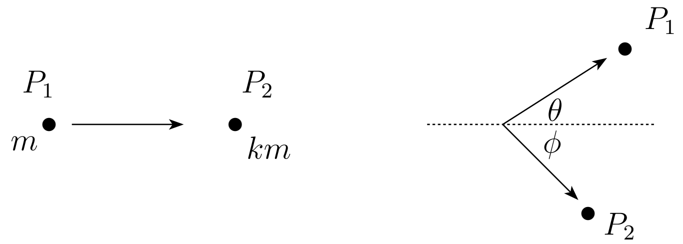
(ii) Let d_n be the distance between the piston and the end of the cylinder at the nth collision, and let t_n be the time between the nth and $(n+1)$th collisions. Express $d_n - d_{n+1}$ in terms of u and t_n, and show that

$$d_{n+1} = \frac{v + (2n-1)u}{v + (2n+1)u}\, d_n \ .$$

(iii) Express d_n in terms of d_1, u, v and n.

(iv) In the case $v = u$, show that $ut_n = \dfrac{d_1}{n(n+1)}$.

11



A particle P_1 of mass m collides with a particle P_2 of mass km which is at rest. No energy is lost in the collision. The direction of motion of P_1 and P_2 after the collision make non-zero angles of θ and ϕ, respectively, with the direction of motion of P_1 before the collision, as shown.

(i) Show that

$$\sin^2\theta + k\sin^2\phi = k\sin^2(\theta + \phi) \ .$$

(ii) Show that, if the angle between the particles after the collision is a right angle, then $k = 1$.

Section C: Probability and Statistics

12 Harry the Calculating Horse will do any mathematical problem I set him, providing the answer is 1, 2, 3 or 4. When I set him a problem, he places a hoof on a large grid consisting of unit squares and his answer is the number of squares partly covered by his hoof. Harry has circular hoofs, of radius $1/4$ unit.

After many years of collaboration, I suspect that Harry no longer bothers to do the calculations, instead merely placing his hoof on the grid completely at random. I often ask him to divide 4 by 4, but only about $1/4$ of his answers are right; I often ask him to add 2 and 2, but disappointingly only about $\pi/16$ of his answers are right.

(i) Is this consistent with my suspicions?

(ii) I decide to investigate further by setting Harry many problems, the answers to which are 1, 2, 3, or 4 with equal frequency. If Harry is placing his hoof at random, find the expected value of his answers. The average of Harry's answers turns out to be 2. Should I get a new horse?

13 The random variable U takes the values $+1$, 0 and -1, each with probability $\frac{1}{3}$. The random variable V takes the values $+1$ and -1 as follows:

if $U = 1$, then $\mathrm{P}(V=1) = \frac{1}{3}$ and $\mathrm{P}(V=-1) = \frac{2}{3}$;

if $U = 0$, then $\mathrm{P}(V=1) = \frac{1}{2}$ and $\mathrm{P}(V=-1) = \frac{1}{2}$;

if $U = -1$, then $\mathrm{P}(V=1) = \frac{2}{3}$ and $\mathrm{P}(V=-1) = \frac{1}{3}$.

(i) Show that the probability that both roots of the equation $x^2 + Ux + V = 0$ are real is $\frac{1}{2}$.

(ii) Find the expected value of the larger root of the equation $x^2 + Ux + V = 0$, given that both roots are real.

(iii) Find the probability that the roots of the equation

$$x^3 + (U - 2V)x^2 + (1 - 2UV)x + U = 0$$

are all positive.

14 In order to get money from a cash dispenser I have to punch in an identification number. I have forgotten my identification number, but I do know that it is equally likely to be any one of the integers 1, 2, $\dots$, n. I plan to punch in integers in order until I get the right one. I can do this at the rate of r integers per minute. As soon as I punch in the first wrong number, the police will be alerted. The probability that they will arrive within a time t minutes is $1 - \mathrm{e}^{-\lambda t}$, where λ is a positive constant.

(i) If I follow my plan, show that the probability of the police arriving before I get my money is

$$\sum_{k=1}^{n} \frac{1 - \mathrm{e}^{-\lambda(k-1)/r}}{n} \, .$$

Simplify the sum.

(ii) On past experience, I know that I will be so flustered that I will just punch in possible integers at random, without noticing which I have already tried. Show that the probability of the police arriving before I get my money is

$$1 - \frac{1}{n - (n-1)\mathrm{e}^{-\lambda/r}} \, .$$

Section A: Pure Mathematics

1 The points A, B and C lie on the sides of a square of side 1 cm and no two points lie on the same side. Show that the length of at least one side of the triangle ABC must be less than or equal to $(\sqrt{6} - \sqrt{2})$ cm.

2 Solve the inequalities

(i) $1 + 2x - x^2 > 2/x \qquad (x \neq 0)$,

(ii) $\sqrt{3x+10} > 2 + \sqrt{x+4} \qquad (x \geqslant -10/3)$.

3 **(i)** Sketch, without calculating the stationary points, the graph of the function $\mathrm{f}(x)$ given by

$$\mathrm{f}(x) = (x-p)(x-q)(x-r) ,$$

where $p < q < r$.

(ii) By considering the quadratic equation $\mathrm{f}'(x) = 0$, or otherwise, show that

$$(p+q+r)^2 > 3(qr + rp + pq) .$$

(iii) By considering $(x^2 + gx + h)(x - k)$, or otherwise, show that $g^2 > 4h$ is a sufficient condition but not a necessary condition for the inequality

$$(g-k)^2 > 3(h - gk)$$

to hold.

4 **(i)** Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(ii) Given that $\theta = \cos^{-1}(2/\sqrt{5})$ and $0 < \theta < \pi/2$, show that $\tan 3\theta = 11/2$. Hence, or otherwise, find all solutions of the equations

(iii) $\tan(3\cos^{-1} x) = 11/2$,

(iv) $\cos(\frac{1}{3}\tan^{-1} y) = 2/\sqrt{5}$.

5 Show that (for $t > 0$)

(i) $\displaystyle\int_0^1 \frac{1}{(1+tx)^2}\mathrm{d}x = \frac{1}{(1+t)}$,

(ii) $\displaystyle\int_0^1 \frac{-2x}{(1+tx)^3}\mathrm{d}x = -\frac{1}{(1+t)^2}$.

Noting that the right hand side of (ii) is the derivative of the right hand side of (i), conjecture the value of

$$\int_0^1 \frac{6x^2}{(1+x)^4}\mathrm{d}x \ .$$

(You need not verify your conjecture.)

6 **(i)** A spherical loaf of bread is cut into parallel slices of equal thickness. Show that, after any number of the slices have been eaten, the area of crust remaining is proportional to the number of slices remaining.

(ii) A European ruling decrees that a parallel-sliced spherical loaf can only be referred to as 'crusty' if the ratio of volume V (in cubic metres) of bread remaining to area A (in square metres) of crust remaining after any number of slices have been eaten satisfies $V/A < 1$. Show that the radius of a crusty parallel-sliced spherical loaf must be less than $2\frac{2}{3}$ metres.

[*The area* A *and volume* V *formed by rotating a curve in the* x–y *plane round the* x*-axis from* $x = -a$ *to* $x = -a + t$ *are given by*

$$A = 2\pi\int_{-a}^{-a+t} y\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}} \mathrm{d}x \ , \qquad V = \pi\int_{-a}^{-a+t} y^2\mathrm{d}x \ . \ ]$$

7 In a cosmological model, the radius R of the universe is a function of the age t of the universe. The function R satisfies the three conditions:

$$\mathrm{R}(0) = 0, \qquad \mathrm{R}'(t) > 0 \text{ for } t > 0, \qquad \mathrm{R}''(t) < 0 \text{ for } t > 0, \qquad (*)$$

where R'' denotes the second derivative of R. The function H is defined by

$$\mathrm{H}(t) = \frac{\mathrm{R}'(t)}{\mathrm{R}(t)} \, .$$

(i) Sketch a graph of $\mathrm{R}(t)$. By considering a tangent to the graph, show that $t < 1/\mathrm{H}(t)$.

(ii) Observations reveal that $\mathrm{H}(t) = a/t$, where a is constant. Derive an expression for $\mathrm{R}(t)$. What range of values of a is consistent with the three conditions $(*)$?

(iii) Suppose, instead, that observations reveal that $\mathrm{H}(t) = bt^{-2}$, where b is constant. Show that this is not consistent with conditions $(*)$ for any value of b.

8 **(i)** Given that $y = x$ and $y = 1 - x^2$ satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{p}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{q}(x)y = 0 \, , \qquad (*)$$

show that $\mathrm{p}(x) = -2x(1 + x^2)^{-1}$ and $\mathrm{q}(x) = 2(1 + x^2)^{-1}$.

(ii) Show also that $ax + b(1 - x^2)$ satisfies the differential equation for any constants a and b.

(iii) Given instead that $y = \cos^2(\frac{1}{2}x^2)$ and $y = \sin^2(\frac{1}{2}x^2)$ satisfy the equation $(*)$, find $\mathrm{p}(x)$ and $\mathrm{q}(x)$.

Section B: Mechanics

9 A ship sails at 20 kilometres/hour in a straight line which is, at its closest, 1 kilometre from a port. A tug-boat with maximum speed 12 kilometres/hour leaves the port and intercepts the ship, leaving the port at the latest possible time for which the interception is still possible. How far does the tug-boat travel?

10 A gun is sited on a horizontal plain and can fire shells in any direction and at any elevation at speed v. The gun is a distance d from a straight railway line which crosses the plain, where $v^2 > gd$. The gunner aims to hit the line, choosing the direction and elevation so as to maximize the time of flight of the shell. Show that the time of flight, T, of the shell satisfies

$$g^2T^2 = 2v^2 + 2\left(v^4 - g^2d^2\right)^{\frac{1}{2}} \ .$$

11 A smooth cylinder with circular cross-section of radius a is held with its axis horizontal. A light elastic band of unstretched length $2\pi a$ and modulus of elasticity λ is wrapped round the circumference of the cylinder, so that it forms a circle in a plane perpendicular to the axis of the cylinder. A particle of mass m is then attached to the rubber band at its lowest point and released from rest.

(i) Given that the particle falls to a distance $2a$ below the below the axis of the cylinder, but no further, show that

$$\lambda = \frac{9\pi mg}{(3\sqrt{3} - \pi)^2} \ .$$

(ii) Given instead that the particle reaches its maximum speed at a distance $2a$ below the axis of the cylinder, find a similar expression for λ.

Section C: Probability and Statistics

12 Four students, Arthur, Bertha, Chandra and Delilah, exchange gossip. When Arthur hears a rumour, he tells it to one of the other three without saying who told it to him. He decides whom to tell by choosing at random amongst the other three, omitting the ones that he knows have already heard the rumour. When Bertha, Chandra or Delilah hear a rumour, they behave in exactly the same way (even if they have already heard it themselves). The rumour stops being passed round when it is heard by a student who knows that the other three have already heard it.

(i) Arthur starts a rumour and tells it to Chandra. By means of a tree diagram, or otherwise, show that the probability that Arthur rehears it is $3/4$.

(ii) Find also the probability that Bertha hears it twice and the probability that Chandra hears it twice.

13 Four students, one of whom is a mathematician, take turns at washing up over a long period of time. The number of plates broken by any student in this time obeys a Poisson distribution, the probability of any given student breaking n plates being $\mathrm{e}^{-\lambda}\lambda^n/n!$ for some fixed constant λ, independent of the number of breakages by other students. Given that five plates are broken, find the probability that three or more were broken by the mathematician.

14 On the basis of an interview, the N candidates for admission to a college are ranked in order according to their mathematical potential. The candidates are interviewed in random order (that is, each possible order is equally likely).

(i) Find the probability that the best amongst the first n candidates interviewed is the best overall.

(ii) Find the probability that the best amongst the first n candidates interviewed is the best or second best overall.

Verify your answers for the case $N = 4$, $n = 2$ by listing the possibilities.

Section A: Pure Mathematics

1 To nine decimal places, $\log_{10}(2) = 0.301029996$ and $\log_{10}(3) = 0.477121255$.

(i) Calculate $\log_{10}(5)$ and $\log_{10}(6)$ to three decimal places. By taking logs, or otherwise, show that

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}.$$

Hence write down the first digit of 3^{100}.

(ii) Find the first digit of each of the following numbers: 2^{1000}; $2^{10\,000}$; and $2^{100\,000}$.

2 **(i)** Show that the coefficient of x^{-12} in the expansion of

$$\left(x^4 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^6$$

is -15, and calculate the coefficient of x^2.

(ii) Hence, or otherwise, calculate the coefficients of x^4 and x^{38} in the expansion of

$$(x^2 - 1)^{11}(x^4 + x^2 + 1)^5.$$

3 For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

(i) Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

$$\int_0^5 [x]\,\mathrm{d}x.$$

(ii) Sketch the graph of $y = [\mathrm{e}^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [\mathrm{e}^x]\,\mathrm{d}x = n \ln n - \ln(n!).$$

4 **(i)** Show that, for $0 \leqslant x \leqslant 1$, the largest value of $\dfrac{x^6}{(x^2+1)^4}$ is $\dfrac{1}{16}$.

(ii) Find constants A, B, C and D such that, for all x,

$$\frac{1}{(x^2+1)^4} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}.$$

(iii) Hence, or otherwise, prove that

$$\frac{11}{24} \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \frac{11}{24} + \frac{1}{16}\,.$$

5 Arthur and Bertha stand at a point O on an inclined plane. The steepest line in the plane through O makes an angle θ with the horizontal. Arthur walks uphill at a steady pace in a straight line which makes an angle α with the steepest line. Bertha walks uphill at the same speed in a straight line which makes an angle β with the steepest line (and is on the same side of the steepest line as Arthur).

(i) Show that, when Arthur has walked a distance d, the distance between Arthur and Bertha is $2d|\sin\frac{1}{2}(\alpha-\beta)|$.

(ii) Show also that, if $\alpha \neq \beta$, the line joining Arthur and Bertha makes an angle ϕ with the vertical, where

$$\cos\phi = \sin\theta\sin\tfrac{1}{2}(\alpha+\beta).$$

6 **(i)** Show that

$$x^2-y^2+x+3y-2 = (x-y+2)(x+y-1)$$

and hence, or otherwise, indicate by means of a sketch the region of the x-y plane for which

$$x^2-y^2+x+3y > 2.$$

(ii) Sketch also the region of the x-y plane for which

$$x^2-4y^2+3x-2y < -2.$$

(iii) Give the coordinates of a point for which both inequalities are satisfied or explain why no such point exists.

7 Let

$$\mathrm{f}(x) = ax - \frac{x^3}{1+x^2},$$

where a is a constant. Show that, if $a \geqslant 9/8$, then $\mathrm{f}'(x) \geqslant 0$ for all x.

8 **(i)** Show that

$$\int_{-1}^{1} |\, x\mathrm{e}^x \,| \mathrm{d}x = -\int_{-1}^{0} x\mathrm{e}^x \mathrm{d}x + \int_{0}^{1} x\mathrm{e}^x \mathrm{d}x$$

and hence evaluate the integral.

(ii) Evaluate $\displaystyle\int_0^4 |\, x^3 - 2x^2 - x + 2 \,| \, \mathrm{d}x\,;$

(iii) Evaluate $\displaystyle\int_{-\pi}^{\pi} |\, \sin x + \cos x \,| \, \mathrm{d}x\,.$

Section B: Mechanics

9 A child is playing with a toy cannon on the floor of a long railway carriage. The carriage is moving horizontally in a northerly direction with acceleration a. The child points the cannon southward at an angle θ to the horizontal and fires a toy shell which leaves the cannon at speed V.

(i) Find, in terms of a and g, the value of $\tan 2\theta$ for which the cannon has maximum range (in the carriage).

(ii) If a is small compared with g, show that the value of θ which gives the maximum range is approximately

$$\frac{\pi}{4} + \frac{a}{2g},$$

(iii) and show that the maximum range is approximately $\dfrac{V^2}{g} + \dfrac{V^2 a}{g^2}$.

10 Three particles P_1, P_2 and P_3 of masses m_1, m_2 and m_3 respectively lie at rest in a straight line on a smooth horizontal table. P_1 is projected with speed v towards P_2 and brought to rest by the collision. After P_2 collides with P_3, the latter moves forward with speed v. The coefficients of restitution in the first and second collisions are e and e', respectively.

(i) Show that

$$e' = \frac{m_2 + m_3 - m_1}{m_1}.$$

(ii) Show that $2m_1 \geqslant m_2 + m_3 \geqslant m_1$ for such collisions to be possible.

(iii) If m_1, m_3 and v are fixed, find, in terms of m_1, m_3 and v, the largest and smallest possible values for the final energy of the system.

11 A rod AB of length 0.81 m and mass 5 kg is in equilibrium with the end A on a rough floor and the end B against a very rough vertical wall. The rod is in a vertical plane perpendicular to the wall and is inclined at 45° to the horizontal. The centre of gravity of the rod is at G, where $AG = 0.21$ m. The coefficient of friction between the rod and the floor is 0.2, and the coefficient of friction between the rod and the wall is 1.0.

(i) Show that the friction cannot be limiting at both A and B.

(ii) A mass of 5 kg is attached to the rod at the point P such that now the friction is limiting at both A and B. Determine the length of AP.

Section C: Probability and Statistics

12 I have k different keys on my key ring. When I come home at night I try one key after another until I find the key that fits my front door. What is the probability that I find the correct key in exactly n attempts in each of the following three cases?

(i) At each attempt, I choose a key that I have not tried before but otherwise each choice is equally likely.

(ii) At each attempt, I choose a key from all my keys and each of the k choices is equally likely.

(iii) At the first attempt, I choose from all my keys and each of the k choices is equally likely. Thereafter, I choose from the keys that I did not try the previous time but otherwise each choice is equally likely.

13 Every person carries two genes which can each be either of type A or of type B. It is known that 81% of the population are AA (i.e. both genes are of type A), 18% are AB (i.e. there is one gene of type A and one of type B) and 1% are BB. A child inherits one gene from each of its parents. If one parent is AA, the child inherits a gene of type A from that parent; if the parent is BB, the child inherits a gene of type B from that parent; if the parent is AB, the inherited gene is equally likely to be A or B.

(i) Given that two AB parents have four children, show that the probability that two of them are AA and two of them are BB is $3/128$.

(ii) My mother is AB and I am AA. Find the probability that my father is AB.

14 **(i)** The random variable X is uniformly distributed on the interval $[-1,1]$. Find $\mathrm{E}(X^2)$ and $\mathrm{Var}\,(X^2)$.

(ii) A second random variable Y, independent of X, is also uniformly distributed on $[-1,1]$, and $Z = Y - X$. Find $\mathrm{E}(Z^2)$ and show that $\mathrm{Var}\,(Z^2) = 7\mathrm{Var}\,(X^2)$.

Section A: Pure Mathematics

1 **(i)** How many integers greater than or equal to zero and less than a million are not divisible by 2 or 5? What is the average value of these integers?

(ii) How many integers greater than or equal to zero and less than 4179 are not divisible by 3 or 7? What is the average value of these integers?

2 A point moves in the x-y plane so that the sum of the squares of its distances from the three fixed points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is always a^2.

(i) Find the equation of the locus of the point and interpret it geometrically.

(ii) Explain why a^2 cannot be less than the sum of the squares of the distances of the three points from their centroid.

[The *centroid* has coordinates $(\bar{x}, \bar{y})$ where $3\bar{x} = x_1 + x_2 + x_3$, $3\bar{y} = y_1 + y_2 + y_3$.]

3 The n positive numbers $x_1, x_2, \ldots, x_n$, where $n \geqslant 3$, satisfy

$$x_1 = 1 + \frac{1}{x_2}, \quad x_2 = 1 + \frac{1}{x_3}, \quad \ldots, \quad x_{n-1} = 1 + \frac{1}{x_n},$$

and also

$$x_n = 1 + \frac{1}{x_1}.$$

Show that

(i) $x_1, x_2, \ldots, x_n > 1$,

(ii) $x_1 - x_2 = -\dfrac{x_2 - x_3}{x_2 x_3}$,

(iii) $x_1 = x_2 = \cdots = x_n$.

(iv) Hence find the value of x_1.

4 Sketch the following subsets of the x-y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x-1| + |y-1| \leqslant 1$;

(iii) $|x-1| - |y+1| \leqslant 1$;

(iv) $|x|\,|y-2| \leqslant 1$.

5 For this question, you may use the following approximations, valid if θ is small: $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.

A satellite X is directly above the point Y on the Earth's surface and can just be seen (on the horizon) from another point Z on the Earth's surface. The radius of the Earth is R and the height of the satellite above the Earth is h.

(i) Find the distance d of Z from Y along the Earth's surface.

(ii) If the satellite is in low orbit (so that h is small compared with R), show that

$$d \approx k(Rh)^{1/2},$$

where k is to be found.

(iii) If the satellite is very distant from the Earth (so that R is small compared with h), show that

$$d \approx aR + b(R^2/h),$$

where a and b are to be found.

6 **(i)** Find the greatest and least values of $bx + a$ for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.

(ii) Find the greatest and least values of $cx^2 + bx + a$, where $c \geqslant 0$, for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases that can arise for different values of b and c.

7 **(i)** Show that $\sin(k\sin^{-1}x)$, where k is a constant, satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + k^2y = 0. \qquad (*)$$

(ii) In the particular case when $k = 3$, find the solution of equation $(*)$ of the form

$$y = Ax^3 + Bx^2 + Cx + D,$$

that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ at $x = 0$.

(iii) Use this result to express $\sin 3\theta$ in terms of powers of $\sin\theta$.

8 The function f satisfies $0 \leqslant \mathrm{f}(t) \leqslant K$ when $0 \leqslant t \leqslant x$.

(i) Explain by means of a sketch, or otherwise, why

$$0 \leqslant \int_0^x \mathrm{f}(t)\,\mathrm{d}t \leqslant Kx.$$

(ii) By considering $\displaystyle\int_0^1 \frac{t}{n(n-t)}\,\mathrm{d}t$, or otherwise, show that, if $n > 1$,

$$0 \leqslant \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leqslant \frac{1}{n-1} - \frac{1}{n}$$

and deduce that

$$0 \leqslant \ln N - \sum_{n=2}^{N}\frac{1}{n} \leqslant 1.$$

(iii) Deduce that as $N \to \infty$

$$\sum_{n=1}^{N}\frac{1}{n} \to \infty.$$

(iv) Noting that $2^{10} = 1024$, show also that if $N < 10^{30}$ then

$$\sum_{n=1}^{N}\frac{1}{n} < 101.$$

Section B: Mechanics

9 A tortoise and a hare have a race to the vegetable patch, a distance X kilometres from the starting post, and back. The tortoise sets off immediately, at a steady v kilometers per hour. The hare goes to sleep for half an hour and then sets off at a steady speed V kilometres per hour. The hare overtakes the tortoise half a kilometre from the starting post, and continues on to the vegetable patch, where she has another half an hour's sleep before setting off for the return journey at her previous pace. One and quarter kilometres from the vegetable patch, she passes the tortoise, still plodding gallantly and steadily towards the vegetable patch.

(i) Show that

$$V = \frac{10}{4X - 9}$$

and find v in terms of X.

(ii) Find X if the hare arrives back at the starting post one and a half hours after the start of the race.

10 A particle is attached to a point P of an unstretched light uniform spring AB of modulus of elasticity λ in such a way that AP has length a and PB has length b. The ends A and B of the spring are now fixed to points in a vertical line a distance l apart, The particle oscillates along this line.

(i) Show that the motion is simple harmonic.

(ii) Show also that the period is the same whatever the value of l and whichever end of the string is uppermost.

11 The force of attraction between two stars of masses m_1 and m_2 a distance r apart is $\gamma m_1 m_2 / r^2$. The Starmakers of Kryton place three stars of equal mass m at the corners of an equilateral triangle of side a.

(i) Show that it is possible for each star to revolve round the centre of mass of the system with angular velocity $(3\gamma m/a^3)^{1/2}$.

(ii) Find a corresponding result if the Starmakers place a fourth star, of mass λm, at the centre of mass of the system.

Section C: Probability and Statistics

12 **(i)** Prove that if $x > 0$ then $x + x^{-1} \geqslant 2$.

I have a pair of six-faced dice, each with faces numbered from 1 to 6. The probability of throwing i with the first die is q_i and the probability of throwing j with the second die is r_j $(1 \leqslant i, j \leqslant 6)$. The two dice are thrown independently and the sum noted. By considering the probabilities of throwing 2, 12 and 7, show the sums 2, 3, . . . , 12 are not equally likely.

(ii) The first die described above is thrown twice and the two numbers on the die noted. Is it possible to find values of q_j so that the probability that the numbers are the same is less than $1/36$?

13 Bar magnets are placed randomly end-to-end in a straight line. If adjacent magnets have ends of opposite polarities facing each other, they join together to form a single unit. If they have ends of the same polarity facing each other, they stand apart. Find the expectation and variance of the number of separate units in terms of the total number N of magnets.

14 When I throw a dart at a target, the probability that it lands a distance X from the centre is a random variable with density function

$$\mathrm{f}(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

I score points according to the position of the dart as follows: if $0 \leqslant X < \frac{1}{4}$, my score is 4; if $\frac{1}{4} \leqslant X < \frac{1}{2}$, my score is 3; if $\frac{1}{2} \leqslant X < \frac{3}{4}$, my score is 2; if $\frac{3}{4} \leqslant X \leqslant 1$, my score is 1.

(i) Show that my expected score from one dart is 15/8.

(ii) I play a game with the following rules. I start off with a total score 0, and each time I throw a dart my score on that throw is added to my total. Then:

if my new total is greater than 3, I have lost and the game ends;
if my new total is 3, I have won and the game ends;
if my new total is less than 3, I throw again.

Show that, if I have won such a game, the probability that I threw the dart three times is 343/2231.

Section A: Pure Mathematics

1 How many integers between $10\,000$ and $100\,000$ (inclusive) contain exactly two different digits? ($23\,332$ contains exactly two different digits but neither of $33\,333$ and $12\,331$ does.)

2 **(i)** Show, by means of a suitable change of variable, or otherwise, that

$$\int_0^\infty \mathrm{f}((x^2+1)^{1/2}+x)\,\mathrm{d}x = \frac{1}{2}\int_1^\infty (1+t^{-2})\mathrm{f}(t)\,\mathrm{d}t.$$

(ii) Hence, or otherwise, show that

$$\int_0^\infty ((x^2+1)^{1/2}+x)^{-3}\,\mathrm{d}x = \frac{3}{8}.$$

3 Which of the following statements are true and which are false? Justify your answers.

(i) $a^{\ln b} = b^{\ln a}$ for all $a, b > 0$.

(ii) $\cos(\sin\theta) = \sin(\cos\theta)$ for all real θ.

(iii) There exists a polynomial P such that $|\mathrm{P}(\theta) - \cos\theta| \leqslant 10^{-6}$ for all real θ.

(iv) $x^4 + 3 + x^{-4} \geqslant 5$ for all $x > 0$.

4 **(i)** Prove that the rectangle of greatest perimeter which can be inscribed in a given circle is a square.

(ii) The result changes if, instead of maximising the sum of lengths of sides of the rectangle, we seek to maximise the sum of nth powers of the lengths of those sides for $n \geqslant 2$. What happens if $n = 2$?

(iii) What happens if $n = 3$? Justify your answers.

5 **(i)** In the Argand diagram, the points Q and A represent the complex numbers $4+6i$ and $10+2i$. If A, B, C, D, E, F are the vertices, taken in clockwise order, of a regular hexagon (regular six-sided polygon) with centre Q, find the complex number which represents B.

(ii) Let a, b and c be real numbers. Find a condition of the form $Aa + Bb + Cc = 0$, where A, B and C are integers, which ensures that

$$\frac{a}{1+i} + \frac{b}{1+2i} + \frac{c}{1+3i}$$

is real.

6 Let $a_1 = \cos x$ with $0 < x < \pi/2$ and let $b_1 = 1$.

(i) Given that

$$\begin{aligned} a_{n+1} &= \tfrac{1}{2}(a_n + b_n), \\ b_{n+1} &= (a_{n+1}b_n)^{1/2}, \end{aligned}$$

find a_2 and b_2 and show that

$$a_3 = \cos\frac{x}{2}\cos^2\frac{x}{4} \quad \text{and} \quad b_3 = \cos\frac{x}{2}\cos\frac{x}{4}.$$

(ii) Guess general expressions for a_n and b_n (for $n \geqslant 2$) as products of cosines and verify that they satisfy the given equations.

7 My bank pays ρ% interest at the end of each year. I start with nothing in my account. Then for m years I deposit £a in my account at the beginning of each year. After the end of the mth year, I neither deposit nor withdraw for l years.

(i) Show that the total amount in my account at the end of this period is

$$£a\frac{r^{l+1}(r^m - 1)}{r-1}$$

where $r = 1 + \dfrac{\rho}{100}$.

(ii) At the beginning of each of the n years following this period I withdraw £b and this leaves my account empty after the nth withdrawal. Find an expression for a/b in terms of r, l, m and n.

8 Fluid flows steadily under a constant pressure gradient along a straight tube of circular cross-section of radius a. The velocity v of a particle of the fluid is parallel to the axis of the tube and depends only on the distance r from the axis. The equation satisfied by v is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = -k,$$

where k is constant.

(i) Find the general solution for v.

(ii) Show that $|v| \to \infty$ as $r \to 0$ unless one of the constants in your solution is chosen to be 0.

(iii) Suppose that this constant is, in fact, 0 and that $v = 0$ when $r = a$.

Find v in terms of k, a and r.

(iv) The volume F flowing through the tube per unit time is given by

$$F = 2\pi \int_0^a rv\,\mathrm{d}r.$$

Find F.

Section B: Mechanics

9 Two small spheres A and B of equal mass m are suspended in contact by two light inextensible strings of equal length so that the strings are vertical and the line of centres is horizontal. The coefficient of restitution between the spheres is e. The sphere A is drawn aside through a very small distance in the plane of the strings and allowed to fall back and collide with the other sphere B, its speed on impact being u.

(i) Explain briefly why the succeeding collisions will all occur at the lowest point. (Hint: Consider the periods of the two pendulums involved.)

(ii) Show that the speed of sphere A immediately after the second impact is $\frac{1}{2}u(1+e^2)$ and find the speed, then, of sphere B.

10 A shell explodes on the surface of horizontal ground. Earth is scattered in all directions with varying velocities.

(i) Show that particles of earth with initial speed v landing a distance r from the centre of explosion will do so at times t given by

$$\tfrac{1}{2}g^2t^2 = v^2 \pm \sqrt{v^4 - g^2r^2}.$$

(ii) Find an expression in terms of v, r and g for the greatest height reached by such particles.

11 Hank's Gold Mine has a very long vertical shaft of height l. A light chain of length l passes over a small smooth light fixed pulley at the top of the shaft. To one end of the chain is attached a bucket A of negligible mass and to the other a bucket B of mass m. The system is used to raise ore from the mine as follows. When bucket A is at the top it is filled with mass $2m$ of water and bucket B is filled with mass λm of ore, where $0 < \lambda < 1$. The buckets are then released, so that bucket A descends and bucket B ascends. When bucket B reaches the top both buckets are emptied and released, so that bucket B descends and bucket A ascends. The time to fill and empty the buckets is negligible.

(i) Find the time taken from the moment bucket A is released at the top until the first time it reaches the top again.

(ii) This process goes on for a very long time. Show that, if the greatest amount of ore is to be raised in that time, then λ must satisfy the condition $\mathrm{f}'(\lambda) = 0$ where

$$\mathrm{f}(\lambda) = \frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}.$$

Section C: Probability and Statistics

12 Suppose that a solution (X, Y, Z) of the equation

$$X + Y + Z = 20,$$

with X, Y and Z non-negative integers, is chosen at random (each such solution being equally likely).

(i) Are X and Y independent? Justify your answer.

(ii) Show that the probability that X is divisible by 5 is $5/21$.

(iii) What is the probability that XYZ is divisible by 5?

13 I have a bag initially containing r red fruit pastilles (my favourites) and b fruit pastilles of other colours. From time to time I shake the bag thoroughly and remove a pastille at random. (It may be assumed that all pastilles have an equal chance of being selected.) If the pastille is red I eat it but otherwise I replace it in the bag. After n such drawings, I find that I have only eaten one pastille. Show that the probability that I ate it on my last drawing is

$$\frac{(r+b-1)^{n-1}}{(r+b)^n - (r+b-1)^n}.$$

14 To celebrate the opening of the financial year the finance minister of Genland flings a Slihing, a circular coin of radius a cm, where $0 < a < 1$, onto a large board divided into squares by two sets of parallel lines 2 cm apart. If the coin does not cross any line, or if the coin covers an intersection, the tax on yaks remains unchanged. Otherwise the tax is doubled.

(i) Show that, in order to raise most tax, the value of a should be

$$\left(1 + \frac{\pi}{4}\right)^{-1}.$$

(ii) If, indeed, $a = \left(1 + \frac{\pi}{4}\right)^{-1}$ and the tax on yaks is 1 Slihing per yak this year, show that its expected value after n years will have passed is

$$\left(\frac{8+\pi}{4+\pi}\right)^n.$$

Section A: Pure Mathematics

1 **(i)** Show that you can make up 10 pence in eleven ways using 10p, 5p, 2p and 1p coins.

(ii) In how many ways can you make up 20 pence using 20p, 10p, 5p, 2p and 1p coins?

[You are reminded that no credit will be given for unexplained answers.]

2 **(i)** If

$$\mathrm{f}(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1-x}{1+x}\right),$$

find $\mathrm{f}'(x)$. Hence, or otherwise, find a simple expression for $\mathrm{f}(x)$.

(ii) Suppose that y is a function of x with $0 < y < (\pi/2)^{1/2}$ and

$$x = y \sin y^2$$

for $0 < x < (\pi/2)^{1/2}$. Show that (for this range of x)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x + 2y^2\sqrt{y^2 - x^2}}.$$

3 Let $a_1 = 3$, $a_{n+1} = a_n^3$ for $n \geqslant 1$. (Thus $a_2 = 3^3$, $a_3 = (3^3)^3$ and so on.)

(i) What digit appears in the unit place of a_7?

(ii) Show that $a_7 \geqslant 10^{100}$.

(iii) What is $\dfrac{a_7 + 1}{2a_7}$ correct to two places of decimals? Justify your answer.

4 Find all the solutions of the equation

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2.$$

5 Four rigid rods AB, BC, CD and DA are freely jointed together to form a quadrilateral in the plane.

(i) Show that if P, Q, R, S are the mid-points of the sides AB, BC, CD, DA, respectively, then

$$|AB|^2 + |CD|^2 + 2|PR|^2 = |AD|^2 + |BC|^2 + 2|QS|^2.$$

(ii) Deduce that $|PR|^2 - |QS|^2$ remains constant however the vertices move.

(Here $|PR|$ denotes the length of PR.)

6 **(i)** Find constants a_0, a_1, a_2, a_3, a_4, a_5, a_6 and b such that

$$x^4(1-x)^4 = (a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(x^2+1) + b.$$

(ii) Hence, or otherwise, prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \frac{22}{7} - \pi.$$

(iii) Evaluate $\displaystyle\int_0^1 x^4(1-x)^4\mathrm{d}x$ and deduce that

$$\frac{22}{7} > \pi > \frac{22}{7} - \frac{1}{630}.$$

7 Find constants a_1, a_2, u_1 and u_2 such that, whenever P is a cubic polynomial,

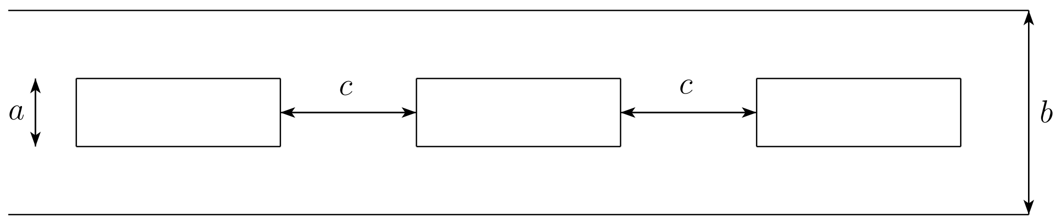
$$\int_{-1}^{1} \mathrm{P}(t)\,\mathrm{d}t = a_1\mathrm{P}(u_1) + a_2\mathrm{P}(u_2).$$

8 **(i)** By considering the maximum of $\ln x - x\ln a$, or otherwise, show that the equation $x = a^x$ has no real roots if $a > e^{1/e}$.

(ii) How many real roots does the equation have if $0 < a < 1$? Justify your answer.

Section B: Mechanics

9 A single stream of cars, each of width a and exactly in line, is passing along a straight road of breadth b with speed V. The distance between the successive cars is c.



A chicken crosses the road in safety at a constant speed u in a straight line making an angle θ with the direction of traffic.

(i) Show that

$$u \geqslant \frac{Va}{c\sin\theta + a\cos\theta}.$$

(ii) Show also that if the chicken chooses θ and u so that it crosses the road at the least possible uniform speed, it crosses in time

$$\frac{b}{V}\left(\frac{c}{a}+\frac{a}{c}\right).$$

10 The point A is vertically above the point B. A light inextensible string, with a smooth ring P of mass m threaded onto it, has its ends attached at A and B. The plane APB rotates about AB with constant angular velocity ω so that P describes a horizontal circle of radius r and the string is taut. The angle BAP has value θ and the angle ABP has value ϕ.

(i) Show that

$$\tan\frac{\phi-\theta}{2} = \frac{g}{r\omega^2}.$$

(ii) Find the tension in the string in terms of m, g, r, ω and $\sin\frac{1}{2}(\theta+\phi)$.

11 A particle of unit mass is projected vertically upwards in a medium whose resistance is k times the square of the velocity of the particle.

(i) If the initial velocity is u, prove that the velocity v after rising through a distance s satisfies

$$v^2 = u^2 e^{-2ks} + \frac{g}{k}(e^{-2ks} - 1). \qquad (*)$$

(ii) Find an expression for the maximum height of the particle above the point of projection.

(iii) Does equation $(*)$ still hold on the downward path? Justify your answer.

Section C: Probability and Statistics

12 An experiment produces a random number T uniformly distributed on $[0,1]$. Let X be the larger root of the equation

$$x^2 + 2x + T = 0.$$

(i) What is the probability that $X > -1/3$?

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 1/18$.

(iii) The experiment is repeated independently 800 times generating the larger roots X_1, X_2, $\ldots$, X_{800}. If

$$Y = X_1 + X_2 + \cdots + X_{800}.$$

find an approximate value for K such that

$$\mathrm{P}(Y \leqslant K) = 0.08.$$

13 Mr Blond returns to his flat to find it in complete darkness. He knows that this means that one of four assassins Mr 1, Mr 2, Mr 3 or Mr 4 has set a trap for him. His trained instinct tells him that the probability that Mr i has set the trap is $i/10$. His knowledge of their habits tells him that Mr i uses a deadly trained silent anaconda with probability $(i+1)/10$, a bomb with probability $i/10$ and a vicious attack canary with probability $(9-2i)/10$ $[i = 1, 2, 3, 4]$.

He now listens carefully and, hearing no singing, concludes correctly that no canary is involved. If he switches on the light and the trap is a bomb he has probability $1/2$ of being killed but if the trap is an anaconda he has probability $2/3$ of survival. If he does not switch on the light and the trap is a bomb he is certain to survive but, if the trap is an anaconda, he has a probability $1/2$ of being killed. His professional pride means that he must enter the flat. Advise Mr Blond, giving reasons for your advice.

14 The maximum height X of flood water each year on a certain river is a random variable with density function

$$f(x) = \begin{cases} \exp(-x) & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

It costs y megadollars each year to prepare for flood water of height y or less. If $X \leqslant y$ no further costs are incurred but if $X \geqslant y$ the cost of flood damage is $r + s(X - y)$ megadollars where $r, s > 0$. The total cost T megadollars is thus given by

$$T = \begin{cases} y & \text{if } X \leqslant y, \\ y + r + s(X - y) & \text{if } X > y. \end{cases}$$

Show that we can minimise the expected total cost by taking

$$y = \ln(r + s).$$

Section A: Pure Mathematics

1 A cylindrical biscuit tin has volume V and surface area S (including the ends).

(i) Show that the minimum possible surface area for a given value of V is $S = 3(2\pi V^2)^{1/3}$.

(ii) For this value of S show that the volume of the largest sphere which can fit inside the tin is $\frac{2}{3}V$,

(iii) and find the volume of the smallest sphere into which the tin fits.

2 **(i)** Show that

$$\int_0^1 (1 + (\alpha - 1)x)^n \, \mathrm{d}x = \frac{\alpha^{n+1} - 1}{(n+1)(\alpha - 1)}$$

when $\alpha \neq 1$ and n is a positive integer.

(ii) Show that if $0 \leqslant k \leqslant n$ then the coefficient of α^k in the polynomial

$$\int_0^1 (\alpha x + (1 - x))^n \, \mathrm{d}x$$

is

$$\binom{n}{k} \int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x \, .$$

(iii) Hence, or otherwise, show that

$$\int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x = \frac{k!(n - k)!}{(n + 1)!} \, .$$

3 Let n be a positive integer.

(i) Factorise $n^5 - n^3$, and show that it is divisible by 24.

(ii) Prove that $2^{2n} - 1$ is divisible by 3.

(iii) If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

4 Show that

$$\int_0^1 \frac{1}{x^2+2ax+1}\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-a^2}}\tan^{-1}\sqrt{\dfrac{1-a}{1+a}} & \text{if } |a|<1, \\ \dfrac{1}{2\sqrt{a^2-1}}\ln\left|a+\sqrt{a^2-1}\right| & \text{if } |a|>1. \end{cases}$$

5 **(i)** Find all rational numbers r and s which satisfy

$$(r+s\sqrt{3})^2 = 4-2\sqrt{3}.$$

(ii) Find all real numbers p and q which satisfy

$$(p+q\mathrm{i})^2 = (3-2\sqrt{3}) + 2(1-\sqrt{3})\mathrm{i}.$$

(iii) Solve the equation

$$(1+\mathrm{i})z^2 - 2z + 2\sqrt{3} - 2 = 0,$$

writing your solutions in as simple a form as possible.

[No credit will be given to answers involving use of calculators.]

6 Let $\mathrm{f}(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$ for $0 < x \leqslant \pi$.

(i) Using the formula

$$2\sin\tfrac{1}{2}x\cos kx = \sin(k+\tfrac{1}{2})x - \sin(k-\tfrac{1}{2})x$$

(which you may assume), or otherwise, show that

$$\mathrm{f}(x) = 1 + 2\sum_{k=1}^{n}\cos kx\,.$$

(ii) Find $\displaystyle\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^\pi \mathrm{f}(x)\cos x\,\mathrm{d}x$.

7 **(i)** At time $t = 0$ a tank contains one unit of water. Water flows out of the tank at a rate proportional to the amount of water in the tank. The amount of water in the tank at time t is y. Show that there is a constant $b < 1$ such that $y = b^t$.

(ii) Suppose instead that the tank contains one unit of water at time $t = 0$, but that in addition to water flowing out as described, water is added at a steady rate $a > 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y \ln b = a,$$

and hence find y in terms of a, b and t.

8 **(i)** By using the formula for the sum of a geometric series, or otherwise, express the number $0.38383838\ldots$ as a fraction in its lowest terms.

(ii) Let x be a real number which has a recurring decimal expansion

$$x = 0 \cdot a_1 a_2 a_2 \cdots ,$$

so that there exists positive integers N and k such that $a_{n+k} = a_n$ for all $n > N$. Show that

$$x = \frac{b}{10^N} + \frac{c}{10^N(10^k - 1)},$$

where b and c are integers to be found. Deduce that x is rational.

Section B: Mechanics

9 A bungee-jumper of mass m is attached by means of a light rope of natural length l and modulus of elasticity mg/k, where k is a constant, to a bridge over a ravine. She jumps from the bridge and falls vertically towards the ground.

(i) If she only just avoids hitting the ground, show that the height h of the bridge above the floor of the ravine satisfies

$$h^2 - 2hl(k+1) + l^2 = 0,$$

(ii) Hence find h. Show that the maximum speed v which she attains during her fall satisfies

$$v^2 = (k+2)gl.$$

10 A spaceship of mass M is at rest. It separates into two parts in an explosion in which the total kinetic energy released is E. Immediately after the explosion the two parts have masses m_1 and m_2 and speeds v_1 and v_2 respectively. Show that the minimum possible relative speed $v_1 + v_2$ of the two parts of the spaceship after the explosion is $(8E/M)^{1/2}$.

11 A particle is projected under the influence of gravity from a point O on a level plane in such a way that, when its horizontal distance from O is c, its height is h. It then lands on the plane at a distance $c + d$ from O.

(i) Show that the angle of projection α satisfies

$$\tan\alpha = \frac{h(c+d)}{cd}$$

(ii) and that the speed of projection v satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c+d)^2 h}{cd}\right).$$

Section C: Probability and Statistics

12 An examiner has to assign a mark between 1 and m inclusive to each of n examination scripts ($n \leqslant m$). He does this randomly, but never assigns the same mark twice.

(i) If K is the highest mark that he assigns, explain why

$$\mathrm{P}(K = k) = \binom{k-1}{n-1} \Big/ \binom{m}{n}$$

for $n \leqslant k \leqslant m$,

(ii) and deduce that

$$\sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}.$$

Find the expected value of K.

13 I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess correctly, adding another cup each time. The price he pays for the stamp is $\pounds N$, where N is the number of cups present when Jim fails to guess correctly.

(i) Find $\mathrm{P}(N = k)$.

(ii) Show that $\mathrm{E}(N) = \mathrm{e}$ and calculate $\mathrm{Var}(N)$.

14 A biased coin, with a probability p of coming up heads and a probability $q = 1 - p$ of coming up tails, is tossed repeatedly. Let A be the event that the first run of r successive heads occurs before the first run of s successive tails. If H is the even that on the first toss the coin comes up heads and T is the event that it comes up tails.

(i) Show that

$$\mathrm{P}(A|H) = p^{\alpha} + (1 - p^{\alpha})\mathrm{P}(A|T),$$
$$\mathrm{P}(A|T) = (1 - q^{\beta})\mathrm{P}(A|H),$$

where α and β are to be determined.

(ii) Use these two equations to find $\mathrm{P}(A|H)$, $\mathrm{P}(A|T)$, and hence $\mathrm{P}(A)$.

Section A: Pure Mathematics

1 **(i)** Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \geqslant 0.$$

(ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

(iii) On a sketch shade the regions of the (x, y) plane for which

$$x^3 - 4x^2y - xy^2 + 4y^3 \geqslant 0.$$

2 **(i)** Suppose that

$$S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x \quad \text{and} \quad T = \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

By considering $S + T$ and $S - T$ determine S and T.

(ii) Evaluate $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} (1 - 4x)\sqrt{\frac{1}{x} - 1}\,\mathrm{d}x$ by using the substitution $x = \sin^2 t$.

3 **(i)** If $\mathrm{f}(r)$ is a function defined for $r = 0, 1, 2, 3, \ldots$, show that

$$\sum_{r=1}^{n} \{\mathrm{f}(r) - \mathrm{f}(r-1)\} = \mathrm{f}(n) - \mathrm{f}(0).$$

(ii) If $\mathrm{f}(r) = r^2(r+1)^2$, evaluate $\mathrm{f}(r) - \mathrm{f}(r-1)$ and hence determine $\displaystyle\sum_{r=1}^{n} r^3$.

(iii) Find the sum of the series $1^3 - 2^3 + 3^3 - 4^3 + \cdots + (2n+1)^3$.

4 By applying de Moivre's theorem to $\cos 5\theta + \mathrm{i}\sin 5\theta$, expanding the result using the binomial theorem, and then equating imaginary parts, show that

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

Use this identity to evaluate $\cos^2 \frac{1}{5}\pi$, and deduce that $\cos\frac{1}{5}\pi = \frac{1}{4}(1+\sqrt{5})$.

5 If

$$\mathrm{f}(x) = nx - \binom{n}{2}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} - \cdots + (-1)^{r+1}\binom{n}{r}\frac{x^r}{r} + \cdots + (-1)^{n+1}\frac{x^n}{n},$$

show that

$$\mathrm{f}'(x) = \frac{1-(1-x)^n}{x}.$$

Deduce that

$$\mathrm{f}(x) = \int_{1-x}^{1} \frac{1-y^n}{1-y}\,\mathrm{d}y.$$

Hence show that

$$\mathrm{f}(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

6 **(i)** In the differential equation

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y} = \mathrm{e}^{2x}$$

make the substitution $u = 1/y$, and hence show that the general solution of the original equation is

$$y = \frac{1}{A\mathrm{e}^{x} - \mathrm{e}^{2x}}.$$

(ii) Use a similar method to solve the equation

$$\frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y^2} = \mathrm{e}^{2x}.$$

7 Let A, B, C be three non-collinear points in the plane. Explain briefly why it is possible to choose an origin equidistant from the three points. Let O be such an origin, let G be the centroid of the triangle ABC, let Q be a point such that $\overrightarrow{GQ} = 2\overrightarrow{OG}$, and let N be the midpoint of OQ.

(i) Show that $\overrightarrow{AQ}$ is perpendicular to $\overrightarrow{BC}$ and deduce that the three altitudes of $\triangle ABC$ are concurrent.

(ii) Show that the midpoints of AQ, BQ and CQ, and the midpoints of the sides of $\triangle ABC$ are all equidistant from N.

[The *centroid* of $\triangle ABC$ is the point G such that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. The *altitudes* of the triangle are the lines through the vertices perpendicular to the opposite sides.]

8 Find functions f, g and h such that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = \mathrm{h}(x) \qquad (*)$$

is satisfied by all three of the solutions $y = x, y = 1$ and $y = x^{-1}$ for $0 < x < 1$.

If f, g and h are the functions you have found in the first paragraph, what condition must the real numbers a, b and c satisfy in order that

$$y = ax + b + \frac{c}{x}$$

should be a solution of $(*)$?

Section B: Mechanics

9 A particle is projected from a point O with speed $\sqrt{2gh}$, where g is the acceleration due to gravity. Show that it is impossible, whatever the angle of projection, for the particle to reach a point above the parabola

$$x^2 = 4h(h - y),$$

where x is the horizontal distance from O and y is the vertical distance above O. State briefly the simplifying assumptions which this solution requires.

10 A small ball of mass m is suspended in equilibrium by a light elastic string of natural length l and modulus of elasticity λ.

(i) Show that the total length of the string in equilibrium is $l(1 + mg/\lambda)$.

(ii) If the ball is now projected downwards from the equilibrium position with speed u_0, show that the speed v of the ball at distance x below the equilibrium position is given by

$$v^2 + \frac{\lambda}{lm}x^2 = u_0^2.$$

(iii) At distance h, where $\lambda h^2 < lmu_0^2$, below the equilibrium position is a horizontal surface on which the ball bounces with a coefficient of restitution e. Show that after one bounce the velocity u_1 at $x = 0$ is given by

$$u_1^2 = e^2u_0^2 + \frac{\lambda}{lm}h^2(1 - e^2),$$

and that after the second bounce the velocity u_2 at $x = 0$ is given by

$$u_2^2 = e^4u_0^2 + \frac{\lambda}{lm}h^2(1 - e^4).$$

11 Two identical uniform cylinders, each of mass m, lie in contact with one another on a horizontal plane and a third identical cylinder rests symmetrically on them in such a way that the axes of the three cylinders are parallel. Assuming that all the surfaces in contact are equally rough, show that the minimum possible coefficient of friction is $2 - \sqrt{3}$.

Section C: Probability and Statistics

12 A school has n pupils, of whom r play hocket, where $n \geqslant r \geqslant 2$. All n pupils are arranged in a row at random.

(i) What is the probability that there is a hockey player at each end of the row?

(ii) What is the probability that all the hockey players are standing together?

(iii) By considering the gaps between the non-hockey-players, find the probability that no two hockey players are standing together, distinguishing between cases when the probability is zero and when it is non-zero.

13 A scientist is checking a sequence of microscope slides for cancerous cells, marking each cancerous cell that she detects with a red dye. The number of cancerous cells on a slide is random and has a Poisson distribution with mean μ. The probability that the scientist spots any one cancerous cell is p, and is independent of the probability that she spots any other one.

(i) Show that the number of cancerous cells which she marks on a single slide has a Poisson distribution of mean $p\mu$.

(ii) Show that the probability Q that the second cancerous cell which she marks is on the kth slide is given by

$$Q = \mathrm{e}^{-\mu p(k-1)} \left\{(1 + k\mu p)(1 - \mathrm{e}^{-\mu p}) - \mu p\right\}.$$

14 **(i)** Find the maximum value of $\sqrt{p(1-p)}$ as p varies between 0 and 1.

(ii) Suppose that a proportion p of the population is female. In order to estimate p we pick a sample of n people at random and find the proportion of them who are female. Find the value of n which ensures that the chance of our estimate of p being more than 0.01 in error is less than 1%.

(iii) Discuss how the required value of n would be affected if (a) p were the proportion of people in the population who are left-handed; (b) p were the proportion of people in the population who are millionaires.

Section A: Pure Mathematics

1 My house has an attic consisting of a horizontal rectangular base of length $2q$ and breadth $2p$ (where $p < q$) and four plane roof sections each at angle θ to the horizontal. Show that the length of the roof ridge is independent of θ and find the volume of the attic and the surface area of the roof.

2 Given that a is constant, differentiate the following expressions with respect to x:

(i) x^a;

(ii) a^x;

(iii) x^x;

(iv) $x^{(x^x)}$;

(v) $(x^x)^x$.

3 By considering the coefficient of x^n in the identity $(1-x)^n(1+x)^n = (1-x^2)^{2n}$, or otherwise, simplify

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + (-1)^n\binom{n}{n}^2$$

in the cases **(i)** when n is even, **(ii)** when n is odd.

4 Show that

(i) $\dfrac{1-\cos\alpha}{\sin\alpha} = \tan\frac{1}{2}\alpha$,

(ii) if $|k| < 1$ then $\displaystyle\int \frac{\mathrm{d}x}{1-2kx+x^2} = \frac{1}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{x-k}{\sqrt{1-k^2}}\right) + C$, where C is a constant of integration.

Hence, or otherwise, show that if $0 < \alpha < \pi$ then

$$\int_0^1 \frac{\sin\alpha}{1-2x\cos\alpha+x^2}\,\mathrm{d}x = \frac{\pi-\alpha}{2}.$$

5 A parabola has the equation $y = x^2$. The points P and Q with coordinates (p, p^2) and (q, q^2) respectively move on the parabola in such a way that $\angle POQ$ is always a right angle.

(i) Find and sketch the locus of the midpoint R of the chord PQ.

(ii) Find and sketch the locus of the point T where the tangents to the parabola at P and Q intersect.

6 The function f is defined, for any complex number z, by

$$\mathrm{f}(z) = \frac{\mathrm{i}z - 1}{\mathrm{i}z + 1}.$$

Suppose throughout that x is a real number.

(i) Show that

$$\mathrm{Re}\,\mathrm{f}(x) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{and} \qquad \mathrm{Im}\,\mathrm{f}(x) = \frac{2x}{x^2 + 1}.$$

(ii) Show that $\mathrm{f}(x)\mathrm{f}(x)^* = 1$, where $\mathrm{f}(x)^*$ is the complex conjugate of $\mathrm{f}(x)$.

(iii) Find expressions for $\mathrm{Re}\,\mathrm{f}(\mathrm{f}(x))$ and $\mathrm{Im}\,\mathrm{f}(\mathrm{f}(x))$.

(iv) Find $\mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$.

7 From the facts

$$\begin{aligned} 1 &= 1 \\ 2+3+4 &= 1+8 \\ 5+6+7+8+9 &= 8+27 \\ 10+11+12+13+14+15+16 &= 27+64 \end{aligned}$$

guess a general law. Prove it.

Hence, or otherwise, prove that

$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \tfrac{1}{4}N^2(N+1)^2$$

for every positive integer N.

[**Hint.** *You may assume that* $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$.]

8 By means of the change of variable $\theta = \frac{1}{4}\pi - \phi$, or otherwise, show that

$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan\theta)\,\mathrm{d}\theta = \tfrac{1}{8}\pi\ln 2.$$

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{1}{2}\pi} \ln\left(\frac{1+\sin x}{1+\cos x}\right)\mathrm{d}x.$$

Section B: Mechanics

9 A cannon-ball is fired from a cannon at an initial speed u. After time t it has reached height h and is at a distance $\sqrt{x^2+h^2}$ from the cannon.

(i) Ignoring air resistance, show that

$$\tfrac{1}{4}g^2t^4 - (u^2 - gh)t^2 + h^2 + x^2 = 0.$$

(ii) Hence show that if $u^2 > 2gh$ then the horizontal range for a given height h and initial speed u is less than or equal to

$$\frac{u\sqrt{u^2-2gh}}{g}.$$

(iii) Show that there is always an angle of firing for which this value is attained.

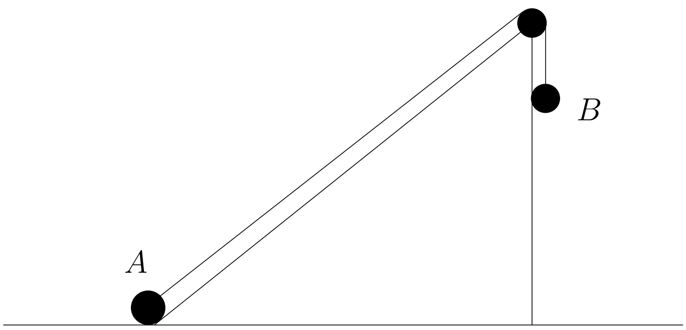
10 One end A of a light elastic string of natural length l and modulus of elasticity λ is fixed and a particle of mass m is attached to the other end B. The particle moves in a horizontal circle with centre on the vertical through A with angular velocity ω.

(i) If θ is the angle AB makes with the downward vertical, find an expression for $\cos\theta$ in terms of m, g, l, λ and ω.

(ii) Show that the motion described is possible only if

$$\frac{g\lambda}{l(\lambda+mg)} < \omega^2 < \frac{\lambda}{ml}.$$

11



The diagram shows a small railway wagon A of mass m standing at the bottom of a smooth railway track of length d inclined at an angle θ to the horizontal. A light inextensible string, also of length d, is connected to the wagon and passes over a light frictionless pulley at the top of the incline. On the other end of the string is a ball B of mass M which hangs freely. The system is initially at rest and is then released.

(i) Find the condition which m, M and θ must satisfy to ensure that the ball will fall to the ground. Assuming that this condition is satisfied, show that the velocity v of the ball when it hits the ground satisfies

$$v^2 = \frac{2g(M - m\sin\theta)d\sin\theta}{M+m}.$$

(ii) Find the condition which m, M and θ must satisfy if the wagon is not to collide with the pulley at the top of the incline.

Section C: Probability and Statistics

12 There are 28 colleges in Cambridge, of which two (New Hall and Newnham) are for women only; the others admit both men and women. Seven women, Anya, Betty, Celia, Doreen, Emily, Fariza and Georgina, are all applying to Cambridge. Each has picked three colleges at random to enter on her application form.

(i) What is the probability that Anya's first choice college is single-sex?

(ii) What is the probability that Betty has picked Newnham?

(iii) What is the probability that Celia has picked at least one single-sex college?

(iv) Doreen's first choice is Newnham. What is the probability that one of her other two choices is New Hall?

(v) Emily has picked Newnham. What is the probability that she has also picked New Hall?

(vi) Fariza's first choice college is single-sex. What is the probability that she has also chosen the other single-sex college?

(vii) One of Georgina's choices is a single-sex college. What is the probability that she has also picked the other single-sex college?

13 I have a bag containing M tokens, m of which are red. I remove n tokens from the bag at random without replacement. Let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th token I remove is red;} \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the total number of red tokens I remove.

(i) Explain briefly why $X = X_1 + X_2 + \cdots + X_n$.

(ii) Find the expectation $\mathrm{E}(X_i)$.

(iii) Show that $\mathrm{E}(X) = mn/M$.

(iv) Find $\mathrm{P}(X = k)$ for $k = 0, 1, 2, \ldots, n$.

(v) Deduce that

$$\sum_{k=1}^{n} k \binom{m}{k} \binom{M-m}{n-k} = m \binom{M-1}{n-1}.$$

14 Each of my n students has to hand in an essay to me. Let T_i be the time at which the ith essay is handed in and suppose that $T_1, T_2, \ldots, T_n$ are independent, each with probability density function $\lambda \mathrm{e}^{-\lambda t}$ $(t \geqslant 0)$. Let T be the time I receive the first essay to be handed in and let U be the time I receive the last one.

(i) Find the mean and variance of T_i.

(ii) Show that $\mathrm{P}(U \leqslant u) = (1 - \mathrm{e}^{-\lambda u})^n$ for $u \geqslant 0$, and hence find the probability density function of U.

(iii) Obtain $\mathrm{P}(T > t)$, and hence find the probability density function of T.

(iv) Write down the mean and variance of T.

Section A: Pure Mathematics

1 I have two dice whose faces are all painted different colours. I number the faces of one of them $1, 2, 2, 3, 3, 6$ and the other $1, 3, 3, 4, 5, 6$. I can now throw a total of 3 in two different ways using the two number 2's on the first die once each. Show that there are seven different ways of throwing a total of 6.

I now renumber the dice (again only using integers in the range 1 to 6) with the results shown in the following table

Total shown by the two dice	2	3	4	5	6	7	8	9	10	11	12
Different ways of obtaining the total	0	2	1	1	4	3	8	6	5	6	0

Find how I have numbered the dice explaining your reasoning.

[You will only get high marks if the examiner can follow your argument.]

2 **(i)** If $|r| \neq 1$, show that

$$1 + r^2 + r^4 + \cdots + r^{2n} = \frac{1 - r^{2n+2}}{1 - r^2}\,.$$

(ii) If $r \neq 1$, find an expression for $\mathrm{S}_n(r)$, where

$$\mathrm{S}_n(r) = r + r^2 + r^4 + r^5 + r^7 + r^8 + r^{10} + \cdots + r^{3n-1}.$$

(iii) Show that, if $|r| < 1$, then, as $n \to \infty$,

$$\mathrm{S}_n(r) \to \frac{1}{1 - r} - \frac{1}{1 - r^3}\,.$$

(iv) If $|r| \neq 1$, find an expression for $\mathrm{T}_n(r)$, where

$$\mathrm{T}_n(r) = 1 + r^2 + r^3 + r^4 + r^6 + r^8 + r^9 + r^{10} + r^{12} + r^{14} + r^{15} + r^{16} + \cdots + r^{6n}.$$

(v) If $|r| < 1$, find the limit of $\mathrm{T}_n(r)$ as $n \to \infty$.

What happens to $\mathrm{T}_n(r)$ as $n \to \infty$ in the three cases $r > 1, r = 1$ and $r = -1$? In each case give reasons for your answer.

3 **(i)** Find all the integer solutions with $1 \leqslant p \leqslant q \leqslant r$ of the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\,,$$

showing that there are no others.

(ii) The integer solutions with $1 \leqslant p \leqslant q \leqslant r$ of

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\,,$$

include $p = 1$, $q = n$, $r = m$ where n and m are any integers satisfying $1 \leqslant m \leqslant n$. Find all the other solutions, showing that you have found them all.

4 By making the change of variable $t = \pi - x$ in the integral

$$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x,$$

or otherwise, show that, for any function f,

$$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x\,.$$

Evaluate

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x \quad \text{and} \quad \int_0^{2\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x\,.$$

5 If $z = x + \mathrm{i}y$ where x and y are real, define $|z|$ in terms of x and y.

(i) Show, using your definition, that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1|\,|z_2|$.

(ii) Explain, by means of a diagram, or otherwise, why $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

(iii) Suppose that $a_j \in \mathbb{C}$ and $|a_j| \leqslant 1$ for $j = 1, 2, \ldots, n$. Show that, if $|z| \leqslant \frac{1}{2}$, then

$$\left|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z\right| < 1,$$

and deduce that any root w of the equation

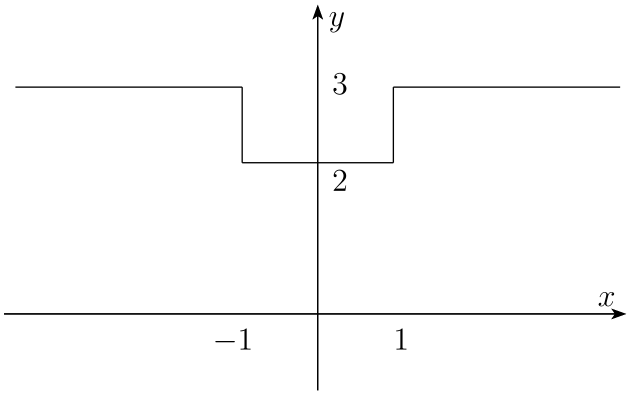
$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + 1 = 0$$

must satisfy $|x| > \frac{1}{2}$.

6 Let $N = 10^{100}$. The graph of

$$f(x) = \frac{x^N}{1+x^N} + 2$$

for $-3 \leqslant x \leqslant 3$ is sketched in the following diagram.



Explain the main features of the sketch.

Sketch the graphs for $-3 \leqslant x \leqslant 3$ of the two functions

$$g(x) = \frac{x^{N+1}}{1+x^N}$$

and

$$h(x) = 10^N \sin(10^{-N}x).$$

In each case explain briefly the main features of your sketch.

7 **(i)** Sketch the curve

$$f(x) = x^3 + Ax^2 + B$$

first in the case $A > 0$ and $B > 0$, and then in the case $A < 0$ and $B > 0$.

(ii) Show that the equation

$$x^3 + ax^2 + b = 0,$$

where a and b are real, will have three distinct real roots if

$$27b^2 + 3a^3b < 0,$$

but will have fewer than three if

$$27b^2 + 4a^3b < 0.$$

8 **(i)** Prove that the intersection of the surface of a sphere with a plane is always a circle, a point or the empty set. Prove that the intersection of the surfaces of two spheres with distinct centres is always a circle, a point or the empty set.

[If you use coordinate geometry, a careful choice of origin and axes may help.]

(ii) The parish council of Little Fitton have just bought a modern sculpture entitled 'Truth, Love and Justice pouring forth their blessings on Little Fitton.' It consists of three vertical poles AD, BE and CF of heights 2 metres, 3 metres and 4 metres respectively. Show that $\angle DEF = \cos^{-1}\frac{1}{5}$.

Vandals now shift the pole AD so that A is unchanged and the pole is still straight but D is vertically above AB with $\angle BAD = \frac{1}{4}\pi$ (in radians). Find the new angle $\angle DEF$ in radians correct to four figures.

9 In the manufacture of Grandma's Home Made Ice-cream, chemicals A and B pour at constant rates a and $b-a$ litres per second ($0 < a < b$) into a mixing vat which mixes the chamicals rapidly and empties at a rate b litres per second into a second mixing vat. At time $t = 0$ the first vat contains K litres of chemical B only.

(i) Show that the volume $V(t)$ (in litres) of the chemical A in the first vat is governed by the differential equation

$$\dot{V}(t) = -\frac{bV(t)}{K} + a,$$

and that

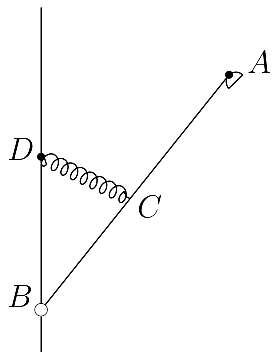
$$V(t) = \frac{aK}{b}(1 - \mathrm{e}^{-bt/K})$$

for $t \geqslant 0$.

(ii) The second vat also mixes chemicals rapidly and empties at the rate of b litres per second. If at time $t = 0$ it contains L litres of chemical C only (where $L \neq K$), how many litres of chemical A will it contain at a later time t?

Section B: Mechanics

10 A small lamp of mass m is at the end A of a light rod AB of length $2a$ attached at B to a vertical wall in such a way that the rod can rotate freely about B in a vertical plane perpendicular to the wall. A spring CD of natural length a and modulus of elasticity λ is joined to the rod at its mid-point C and to the wall at a point D a distance a vertically above B. The arrangement is sketched below.



Show that if $\lambda > 4mg$ the lamp can hang in equilibrium away from the wall and calculate the angle $\angle DBA$.

11 A piece of uniform wire is bent into three sides of a square $ABCD$ so that the side AD is missing. Show that if it is first hung up by the point A and then by the point B then the angle between the two directions of BC is $\tan^{-1} 18$.

12 In a clay pigeon shoot the target is launched vertically from ground level with speed v. At a time T later the competitor fires a rifle inclined at angle α to the horizontal. The competitor is also at ground level and is a distance l from the launcher. The speed of the bullet leaving the rifle is u.

(i) Show that, if the competitor scores a hit, then

$$l \sin\alpha - \left(vT - \tfrac{1}{2}gT^2\right)\cos\alpha = \frac{v - gT}{u} l.$$

(ii) Suppose now that $T = 0$. Show that if the competitor can hit the target before it hits the ground then $v < u$ and

$$\frac{2v\sqrt{u^2 - v^2}}{g} > l.$$

13 A train starts from a station. The tractive force exerted by the engine is at first constant and equal to F. However, after the speed attains the value u, the engine works at constant rate P, where $P = Fu$. The mass of the engine and the train together is M. Forces opposing motion may be neglected.

(i) Show that the engine will attain a speed v, with $v \geqslant u$, after a time

$$t = \frac{M}{2P}\left(u^2 + v^2\right).$$

(ii) Show also that it will have travelled a distance

$$\frac{M}{6P}(2v^3 + u^2)$$

in this time.

Section C: Probability and Statistics

14 When he sets out on a drive Mr Toad selects a speed V kilometres per minute where V is a random variable with probability density

$$\alpha v^{-2} \mathrm{e}^{-\alpha v^{-1}}$$

and α is a strictly positive constant. He then drives at constant speed, regardless of other drivers, road conditions and the Highway Code. The traffic lights at the Wild Wood cross-roads change from red to green when Mr Toad is exactly 1 kilometre away in his journey towards them.

(i) If the traffic light is green for g minutes, then red for r minutes, then green for g minutes, and so on, show that the probability that he passes them after $n(g+r)$ minutes but before $n(g+r)+g$ minutes, where n is a positive integer, is

$$\mathrm{e}^{-\alpha n(g+r)} - \mathrm{e}^{-\alpha(n(g+r)+g)}.$$

(ii) Find the probability $\mathrm{P}(\alpha)$ that he passes the traffic lights when they are green.

(iii) Show that $\mathrm{P}(\alpha) \to 1$ as $\alpha \to \infty$ and, by noting that $(\mathrm{e}^x - 1)/x \to 1$ as $x \to 0$, or otherwise, show that

$$\mathrm{P}(\alpha) \to \frac{g}{r+g} \quad \text{as } \alpha \to 0.$$

[NB: the traffic light show only green and red - not amber.]

15 Captain Spalding is on a visit to the idyllic island of Gambriced. The population of the island consists of the two lost tribes of Frodox and the latest census shows that $11/16$ of the population belong to the Ascii who tell the truth $3/4$ of the time and $5/16$ to the Biscii who always lie. The answers of an Ascii to each question (even if it is the same as one before) are independent.

(i) Show that the probability that an Ascii gives the same answer twice in succession to the same question is $5/8$. Show that the probability that an Ascii gives the same answer twice is telling the truth is $9/10$.

(ii) Captain Spalding addresses one of the natives as follows.

Spalding: My good man, I'm afraid I'm lost. Should I go left or right to reach the nearest town?

Native: Left.

Spalding: I am a little deaf. Should I go left or right to reach the nearest town?

Native (patiently): Left.

Show that, on the basis of this conversation, Captain Spalding should go left to try and reach the nearest town and that there is a probability $99/190$ that this is the correct direction.

(iii) The conversation resumes as follows.

Spalding: I'm sorry I didn't quite hear that. Should I go left or right to reach the nearest town?

Native (loudly and clearly): Left.

Shouls Captain Spalding go left or right and why? Show that if he follows your advice the probability that this is the correct direction is $331/628$.

16 **(i)** By making the substitution $y = \cos^{-1} t$, or otherwise, show that

$$\int_0^1 \cos^{-1} t \, \mathrm{d}t = 1.$$

(ii) A pin of length $2a$ is thrown onto a floor ruled with parallel lines equally spaced at a distance $2b$ apart. The distance X of its centre from the nearest line is a uniformly distributed random variable taking values between 0 and b and the acute angle Y the pin makes with a direction perpendicular to the line is a uniformly distributed random variable taking values between 0 and $\pi/2$. X and Y are independent. If $X = x$ what is the probability that the pin crosses the line?

(iii) If $a < b$ show that the probability that the pin crosses a line for a general throw is $\dfrac{2a}{\pi b}$.

Section A: Pure Mathematics

1 Today's date is June 26th 1992 and the day of the week is Friday. Find which day of the week was April 3rd 1905, explaining your method **carefully.**

[30 days hath September, April, June and November. All the rest have 31, excepting February alone which has 28 days clear and 29 in each leap year.]

2 A 3×3 magic square is a 3×3 array

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & k \end{matrix}$$

whose entries are the nine distinct integers $1, 2, 3, 4, 5, 6, 7, 8, 9$ and which has the property that all its rows, columns and main diagonals add up to the same number n. (Thus $a+b+c = d+e+f = g+h+k = a+d+g = b+e+h = c+f+k = a+e+k = c+e+g = n$.)

(i) Show that $n = 15$.

(ii) Show that $e = 5$.

(iii) Show that one of b, d, h or f must have value 9.

(iv) Find all 3×3 magic squares with $b = 9$.

(v) How many different 3×3 magic squares are there? Why?

[Two magic squares are different if they have different entries in any place of the array.]

3 Evaluate

(i) $\displaystyle\int_{-\pi}^{\pi} |\sin x| \; \mathrm{d}x,$

(ii) $\displaystyle\int_{-\pi}^{\pi} \sin |x| \; \mathrm{d}x,$

(iii) $\displaystyle\int_{-\pi}^{\pi} x \sin x \, \mathrm{d}x,$

(iv) $\displaystyle\int_{-\pi}^{\pi} x^{10} \sin x \, \mathrm{d}x.$

4 Sketch the following subsets of the complex plane using Argand diagrams. Give reasons for your answers.

(i) $\{z : \mathrm{Re}((1+\mathrm{i})z) \geqslant 0\}$.

(ii) $\{z : |z^2| \leqslant 2, \mathrm{Re}(z^2) \geqslant 0\}$.

(iii) $\{z = z_1 + z_2 : |z_1| = 2, |z_2| = 1\}$.

5 Let $\mathrm{p}_0(x) = (1-x)(1-x^2)(1-x^4)$.

(i) Show that $(1-x)^3$ is a factor of $\mathrm{p}_0(x)$. If $\mathrm{p}_1(x) = x\mathrm{p}_0'(x)$ show, by considering factors of the polynomials involved, that $\mathrm{p}_0'(1) = 0$ and $\mathrm{p}_1'(1) = 0$.

(ii) By writing $\mathrm{p}_0(x)$ in the form

$$\mathrm{p}_0(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7,$$

deduce that

$$\begin{aligned} 1+2+4+7 &= 3+5+6 \\ 1^2+2^2+4^2+7^2 &= 3^2+5^2+6^2. \end{aligned}$$

(iii) Show that we can write the integers $1, 2, \ldots, 15$ in some order as $a_1, a_2, \ldots, a_{15}$ in such a way that

$$a_1^r + a_2^r + \cdots + a_8^r = a_9^r + a_{10}^r + \cdots + a_{15}^r$$

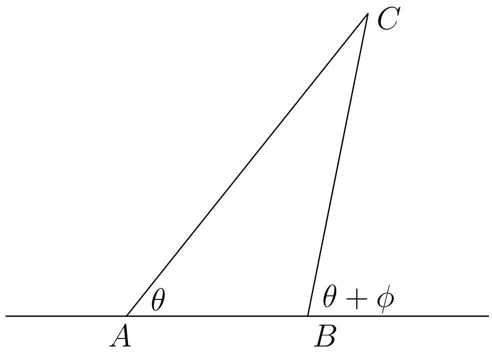
for $r = 1, 2, 3$.

6 **(i)** Explain briefly, by means of a diagram, or otherwise, why

$$\mathrm{f}(\theta+\delta\theta)\approx \mathrm{f}(\theta)+\mathrm{f}'(\theta)\delta\theta,$$

when $\delta\theta$ is small.

(ii) Two powerful telescopes are placed at points A and B which are a distance a apart. A very distant point C is such that AC makes an angle θ with AB and BC makes an angle $\theta+\phi$ with AB produced. (A sketch of the arrangement is given in the diagram.)



If the perpendicular distance h of C from AB is very large compared with a show that h is approximately $(a\sin^2\theta)/\phi$ and find the approximate value of AC in terms of a,θ and ϕ.

(iii) It is easy to show (but you are not asked to show it) that errors in measuring ϕ are much more important than errors in measuring θ. If we make an error of $\delta\phi$ in measuring ϕ (but measure θ correctly) what is the approximate error in our estimate of AC and, roughly, in what proportion is it reduced by doubling the distance between A and B?

7 **(i)** Let $\mathrm{g}(x)=ax+b$. Show that, if $\mathrm{g}(0)$ and $\mathrm{g}(1)$ are integers, then $\mathrm{g}(n)$ is an integer for all integers n.

(ii) Let $\mathrm{f}(x)=Ax^2+Bx+C$. Show that, if $\mathrm{f}(-1),\mathrm{f}(0)$ and $\mathrm{f}(1)$ are integers, then $\mathrm{f}(n)$ is an integer for all integers n.

(iii) Show also that, if α is any real number and $\mathrm{f}(\alpha-1)$, $\mathrm{f}(\alpha)$ and $\mathrm{f}(\alpha+1)$ are integers, then $\mathrm{f}(\alpha+n)$ is an integer for all integers n.

8 **(i)** Explain diagrammatically, or otherwise, why

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \mathrm{f}(t)\,\mathrm{d}t = \mathrm{f}(x).$$

(ii) Show that, if

$$\mathrm{f}(x) = \int_0^x \mathrm{f}(t)\,\mathrm{d}t + 1,$$

then $\mathrm{f}(x) = \mathrm{e}^x$.

(iii) What is the solution of

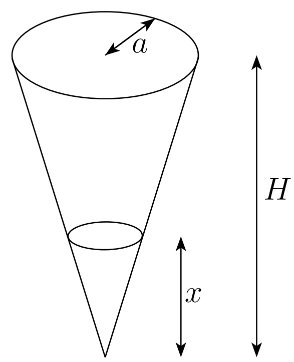
$$\mathrm{f}(x) = \int_0^x \mathrm{f}(t)\,\mathrm{d}t?$$

(iv) Given that

$$\int_0^x \mathrm{f}(t)\,\mathrm{d}t = \int_x^1 t^2\mathrm{f}(t)\,\mathrm{d}t + x - \frac{x^5}{5} + C,$$

find $\mathrm{f}(x)$ and show that $C = -2/15$.

9 The diagram shows a coffee filter consisting of an inverted hollow right circular cone of height H cm and base radius a cm.



When the water level is x cm above the vertex, water leaves the cone at a rate $Ax\ \mathrm{cm}^3\mathrm{sec}^{-1}$, where A is a positive constant. Suppose that the cone is initially filled to a height h cm with $0 < h < H$.

(i) Show that it will take $\pi a^2h^2/(2AH^2)$ seconds to empty.

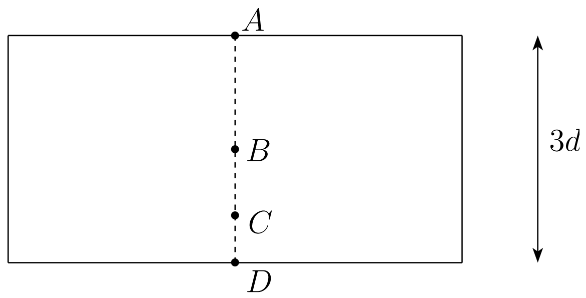
(ii) Suppose now that the cone is initially filled to a height h cm, but that water is poured in at a constant rate $B\ \mathrm{cm}^3\mathrm{sec}^{-1}$ and continues to drain as before. Establish, by considering the sign of $\mathrm{d}x/\mathrm{d}t$, or otherwise, what will happen subsequently to the water level in the different cases that arise. (You are not asked to find an explicit formula for x.)

Section B: Mechanics

10 A projectile of mass m is fired horizontally from a toy cannon of mass M which slides freely on a horizontal floor. The length of the barrel is l and the force exerted on the projectile has the constant value P for so long as the projectile remains in the barrel. Initially the cannon is at rest. Show that the projectile emerges from the barrel at a speed relative to the ground of

$$\sqrt{\frac{2PMl}{m(M+m)}}.$$

11 Three light elastic strings AB, BC and CD, each of natural length a and modulus of elasticity λ, are joined together as shown in the diagram.

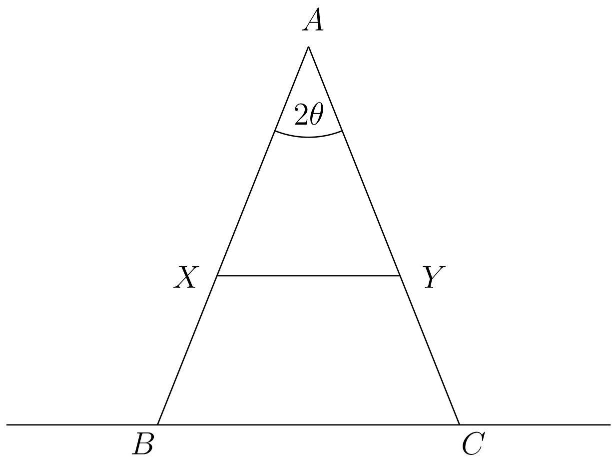


A is attached to the ceiling and D to the floor of a room of height $3d$ in such a way that A, B, C and D are in a vertical line. Particles of mass m are attached at B and C.

(i) Find the heights of B and C above the floor.

(ii) Find the set of values of d for which it is possible, by choosing m suitably, to have $CD = a$?

12 The diagram shows a crude step-ladder constructed by smoothly hinging-together two light ladders AB and AC, each of length l, at A. A uniform rod of wood, of mass m, is pin-jointed to X on AB and to Y on AC, where $AX = \frac{3}{4}l = AY$. The angle $\angle XAY$ is 2θ.



The rod XY will break if the tension in it exceeds T. The step-ladder stands on rough horizontal ground (coefficient of friction μ). Given that $\tan\theta > \mu$, find how large a mass M can safely be placed at A and show that if

$$\tan\theta > \frac{6T}{mg} + 4\mu$$

the step-ladder will fail under its own weight.

[You may assume that friction is limiting at the moment of collapse.]

13 A comet, which may be regarded as a particle of mass m, moving in the sun's gravitational field, at a distance x from the sun, experiences a force Gm/x^2 (where G is a constant) directly towards the sun.

(i) Show that if, at some time, $x = h$ and the comet is travelling directly away from the sun with speed V, then x cannot become arbitrarily large unless $V^2 \geqslant 2G/h$.

(ii) A comet is initially motionless at a great distance from the sun. If, at some later time, it is at a distance h from the sun, how long after that will it take to fall into the sun?

Section C: Probability and Statistics

14 The average number of pedestrians killed annually in road accidents in Poldavia during the period 1974-1989 was 1080 and the average number killed annually in commercial flight accidents during the same period was 180. Discuss the following newspaper headlines which appeared in 1991. (The percentage figures in square brackets give a rough indication of the weight of marks attached to each discussion.)

(i) [10%] Six Times Safer To Fly Than To Walk. 1974-1989 Figures Prove It.

(ii) [10%] Our Skies Are Safer. Only 125 People Killed In Air Accidents In 1990.

(iii) [30%] Road Carnage Increasing. 7 People Killed On Tuesday.

(iv) [50%] Alarming Rise In Pedestrian Casualties. 1350 Pedestrians Killed In Road Accidents During 1990.

15 Trains leave Barchester Station for London at 12 minutes past the hour, taking 60 minutes to complete the journey and at 48 minutes past the hour taking 75 minutes to complete the journey. The arrival times of passengers for London at Barchester Station are uniformly distributed over the day and all passengers take the first available train.

(i) Show that their average journey time from arrival at Barchester Station to arrival in London is 84.6 minutes.

(ii) Suppose that British Rail decide to retime the fast 60 minute train so that it leaves at x minutes past the hour. What choice of x will minimise the average journey time?

16 The four towns A, B, C and D are linked by roads AB, AC, CB, BD and CD. The probability that any one road will be blocked by snow on the 1st of January is p, independent of what happens to any other $[0 < p < 1]$.

(i) Show that the probability that any open route from A to D is $ABCD$ is

$$p^2(1-p)^3.$$

(ii) In order to increase the probability that it is possible to get from A to D by a sequence of unblocked roads the government proposes either to snow-proof the road AB (so that it can never be blocked) or to snow-proof the road CB. Because of the high cost it cannot do both. Which road should it choose (or are both choices equally advantageous)?

(iii) In fact, $p = \frac{1}{10}$ and the government decides that it is only worth going ahead if the present probability of A being cut off from D is greater than $\frac{1}{100}$. Will it go ahead?

Section A: Pure Mathematics

1 If $\theta + \phi + \psi = \frac{1}{2}\pi$, show that

$$\sin^2\theta + \sin^2\phi + \sin^2\psi + 2\sin\theta\sin\phi\sin\psi = 1.$$

By taking $\theta = \phi = \frac{1}{5}\pi$ in this equation, or otherwise, show that $\sin\frac{1}{10}\pi$ satisfies the equation

$$8x^3 + 8x^2 - 1 = 0.$$

2 Frosty the snowman is made from two uniform spherical snowballs, of initial radii $2R$ and $3R$. The smaller (which is his head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for both snowballs. During melting each snowball remains spherical and uniform.

(i) When Frosty is half his initial height, find the ratio of his volume to his initial volume.

(ii) If V and S denote his total volume and surface area respectively, find the maximum value of $\dfrac{\mathrm{d}V}{\mathrm{d}S}$ up to the moment when his head disappears.

3 A path is made up in the Argand diagram of a series of straight line segments $P_1P_2, P_2P_3, P_3P_4, \ldots$ such that each segment is d times as long as the previous one, $(d \neq 1)$, and the angle between one segment and the next is always θ (where the segments are directed from P_j towards P_{j+1}, and all angles are measured in the anticlockwise direction).

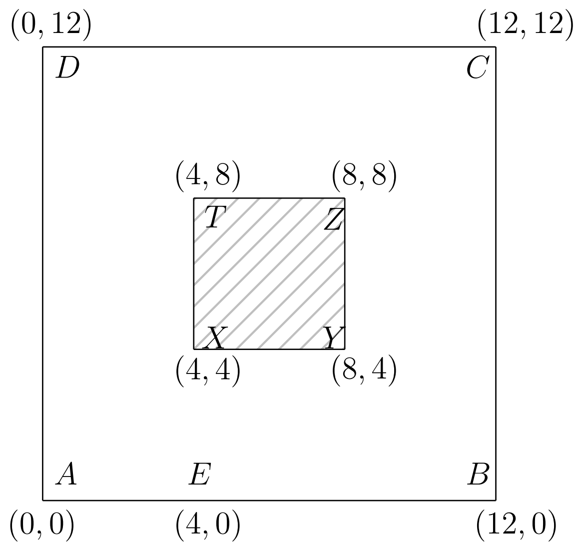
(i) If P_j represents the complex number z_j, express

$$\frac{z_{n+1} - z_n}{z_n - z_{n-1}}$$

as a complex number (for each $n \geqslant 2$), briefly justifying your answer.

(ii) If $z_1 = 0$ and $z_2 = 1$, obtain an expression for z_{n+1} when $n \geqslant 2$. By considering its imaginary part, or otherwise, show that if $\theta = \frac{1}{3}\pi$ and $d = 2$, then the path crosses the real axis infinitely often.

4



The above diagram is a plan of a prison compound. The outer square $ABCD$ represents the walls of the compound (whose height may be neglected), while the inner square $XYZT$ is the Black Tower, a solid stone structure. A guard patrols along segment AE of the walls, for a distance of up to 4 units from A. Determine the distance from A of points at which the area of the courtyard that he can see is

(i) as small as possible,

(ii) as large as possible.

[**Hint.** It is suggested that you express the area he *cannot* see in terms of p, his distance from A.]

5 A set of n distinct vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, where $n \geqslant 2$, is called *regular* if it satisfies the following two conditions:

(a) there are constants α and β, with $\alpha > 0$, such that for any i and j,

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} \alpha^2 & \text{when } i = j \\ \beta & \text{when } i \neq j, \end{cases}$$

(b) the centroid of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is the origin $\mathbf{0}$.
[The centroid of vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ is the vector $\frac{1}{m}(\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_m)$.]

(i) Prove that **(a)** and **(b)** imply that $(n-1)\beta = -\alpha^2$.

(ii) If $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is a regular set of vectors in 2-dimensional space, show that either $n = 2$ or $n = 3$, and in each case find the other vectors in the set.

(iii) Hence, or otherwise, find all regular sets of vectors in 3-dimensional space for which $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{a}_2$ lies in the x-y plane.

6 Criticise each step of the following arguments. You should correct the arguments where necessary and possible, and say (with justification) whether you think the conclusion are true even though the argument is incorrect.

(i) The function g defined by

$$\mathrm{g}(x) = \frac{2x^3 + 3}{x^4 + 4}$$

satisfies $\mathrm{g}'(x) = 0$ only for $x = 0$ or $x = \pm 1$. Hence the stationary values are given by $x = 0$, $\mathrm{g}(x) = \frac{3}{4}$ and $x = \pm 1$, $\mathrm{g}(x) = 1$. Since $\frac{3}{4} < 1$, there is a minimum at $x = 0$ and maxima at $x = \pm 1$. Thus we must have $\frac{3}{4} \leqslant \mathrm{g}(x) \leqslant 1$ for all x.

(ii) $\displaystyle\int (1-x)^{-3}\,\mathrm{d}x = -3(1-x)^{-4}$ and so $\displaystyle\int_{-1}^{3} (1-x)^{-3}\,\mathrm{d}x = 0.$

7 According to the Institute of Economic Modelling Sciences, the Slakan economy has alternate years of growth and decline, as in the following model. The number V of vloskan (the unit of currency) in the Slakan Treasury is assumed to behave as a continuous variable, as follows. In a year of growth it increases continuously at an annual rate $aV_0\left(1+(V/V_0)\right)^2$. During a year of decline, as long as there is still money in the Treasury, the amount decreases continuously at an annual rate $bV_0\left(1+(V/V_0)\right)^2$; but if V becomes zero, it remains zero until the end of the year. Here a, b and V_0 are positive constants. A year of growth has just begun and there are k_0V_0 vloskan in the Treasury, where $0 \leqslant k_0 < a^{-1} - 1$.

(i) Explain the significance of these inequalities for the model to be remotely sensible.

(ii) If k_0 is as above and at the end of one year there are k_1V_0 vloskan in the Treasury, where $k_1 > 0$, find the condition involving b which k_1 must satisfy so that there will be some vloskan left after a further year. Under what condition (involving a, b and k_0) does the model predict that unlimited growth will take place in the third year (but not before)?

8 **(i)** By a substitution of the form $y = k - x$ for suitable k, prove that, for any function f,

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \pi \int_0^{\frac{1}{2}\pi} \mathrm{f}(\sin x)\,\mathrm{d}x.$$

Hence or otherwise evaluate

$$\int_0^{\pi} \frac{x}{2+\sin x}\,\mathrm{d}x.$$

(ii) Evaluate

$$\int_0^1 \frac{(\sin^{-1} t)\cos\left[(\sin^{-1} t)^2\right]}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

[No credit will be given for numerical answers obtained by use of a calculator.]

9 **(i)** Suppose that the real number x satisfies the n inequalities

$$\begin{aligned} 1 &< x &< 2 \\ 2 &< x^2 &< 3 \\ 3 &< x^3 &< 4 \\ &\vdots& \\ n &< x^n &< n+1 \end{aligned}$$

Prove without the use of a calculator that $n \leqslant 4$.

(ii) If n is an integer strictly greater than 1, by considering how many terms there are in

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2},$$

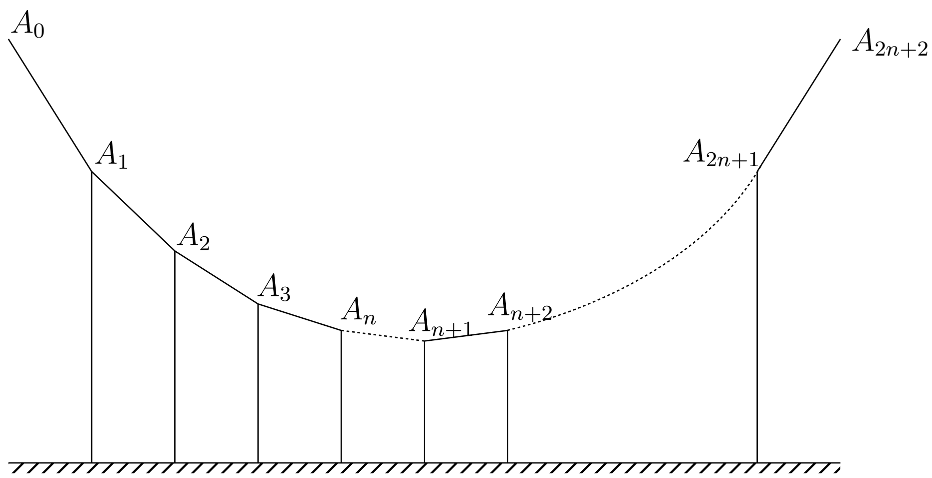
or otherwise, show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1.$$

Hence or otherwise find, with justification, an integer N such that $\displaystyle\sum_{n=1}^{N} \frac{1}{n} > 10$.

Section B: Mechanics

10



The above diagram represents a suspension bridge. A heavy uniform horizontal roadway is attached by vertical struts to a light flexible chain at points $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2), \ldots,$ $A_{2n+1} = (x_{2n+1}, y_{2n+1})$, where the coordinates are referred to horizontal and vertically upward axes Ox, Oy. The chain is fixed to external supports at points

$$A_0 = (x_0, y_0) \quad \text{and} \quad A_{2n+2} = (x_{2n+2}, y_{2n+2})$$

at the same height. The weight of the chain and struts may be neglected. Each strut carries the same weight w. The horizontal spacing h between A_i and A_{i+1} (for $0 \leqslant i \leqslant 2n+1$) is constant.

(i) Write down equations satisfied by the tensions T_i in the portion $A_{i-1}A_i$ of the chain for $1 \leqslant i \leqslant n+1$. Hence or otherwise show that

$$\frac{h}{y_n - y_{n+1}} = \frac{3h}{y_{n-1} - y_n} = \cdots = \frac{(2n+1)y}{y_0 - y_1}.$$

(ii) Verify that the points $A_0, A_1, \ldots, A_{2n+1}, A_{2n+2}$ lie on a parabola.

11 A piledriver consists of a weight of mass M connected to a lighter counterweight of mass m by a light inextensible string passing over a smooth light fixed pulley.

(i) By considerations of energy or otherwise, show that if the weights are released from rest, and move vertically, then as long as the string remains taut and no collisions occur, the weights experience a constant acceleration of magnitude

$$g\left(\frac{M-m}{M+m}\right).$$

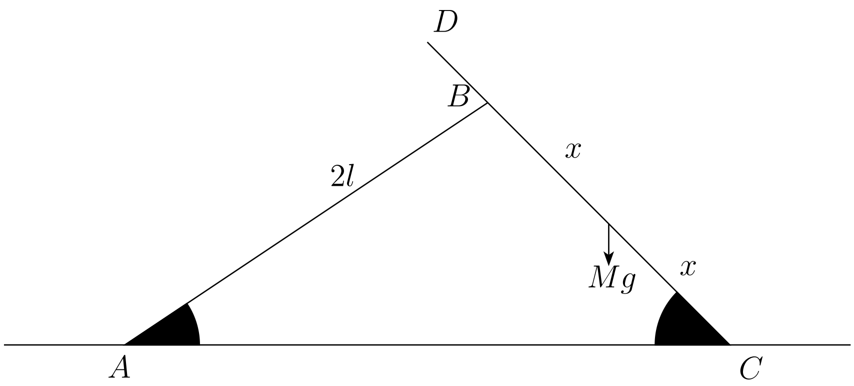
(ii) Initially the weight is held vertically above the pile, and is released from rest. During the subsequent motion both weights move vertically and the only collisions are between the weight and the pile. Treating the pile as fixed and the collisions as completely inelastic, show that, if just before a collision the counterweight is moving with speed v, then just before the next collision it will be moving with speed $mv/\left(M+m\right)$.

[You may assume that when the string becomes taut, the momentum lost by one weight equals that gained by the other.]

(iii) Further show that the times between successive collisions with the pile form a geometric progression.

(iv) Show that the total time before the weight finally comes to rest is three times the time from the start to the first impact.

12



The above diagram illustrates a makeshift stepladder, made from two equal light planks AB and CD, each of length $2l$. The plank AB is smoothly hinged to the ground at A and makes an angle of α with the horizontal. The other plank CD has its bottom end C resting on the same horizontal ground and makes an angle β with the horizontal. It is pivoted smoothly to B at a point distance $2x$ from C. The coefficient of friction between CD and the ground is μ. A painter of mass M stands on CD, half between C and B.

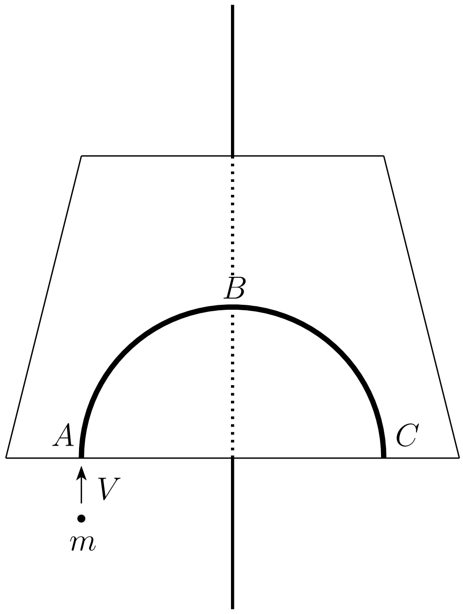
(i) Show that, for equilibrium to be possible,

$$\mu \geqslant \frac{\cot\alpha\cot\beta}{2\cot\alpha + \cot\beta}.$$

(ii) Suppose now that B coincides with D. Show that, as α varies, the maximum distance from A at which the painter will be standing is

$$l\sqrt{\frac{1+81\mu^2}{1+9\mu^2}}.$$

13



A heavy smooth lamina of mass M is free to slide without rotation along a straight line on a fixed smooth horizontal table. A smooth groove ABC is inscribed in the lamina, as indicated in the above diagram. The tangents to the groove at A and at B are parallel to the line. When the lamina is stationary, a particle of mass m (where $m < M$) enters the groove at A. The particle is travelling, with speed V, parallel to the line and in the plane of the lamina and table.

(i) Calculate the speeds of the particle and of the lamina, when the particle leaves the groove at C.

(ii) Suppose now that the lamina is held fixed by a peg attached to the line. Supposing that the groove ABC is a semicircle of radius r, obtain the value of the average force per unit time exerted on the peg by the lamina between the instant that the particle enters the groove and the instant that it leaves it.

Section C: Probability and Statistics

14 A set of $2N+1$ rods consists of one of each length $1, 2, \ldots, 2N, 2N+1$, where N is an integer greater than 1. Three different rods are selected from the set. Suppose their lengths are a, b and c, where $a > b > c$.

(i) Given that a is even and fixed, show, by considering the possible values of b, that the number of selections in which a triangle can then be formed from the three rods is

$$1 + 3 + 5 + \cdots + (a - 3),$$

where we allow only non-degenerate triangles (i.e. triangles with non-zero area).

(ii) Similarly obtain the number of selections in which a triangle may be formed when a takes some fixed odd value. Write down a formula for the number of ways of forming a non-degenerate triangle and verify it for $N = 3$.

(iii) Hence show that, if three rods are drawn at random without replacement, then the probability that they can form a non-degenerate triangle is

$$\frac{(N-1)(4N+1)}{2(4N^2-1)}.$$

15 A fair coin is thrown n times. On each throw, 1 point is scored for a head and 1 point is lost for a tail. Let S_n be the points total for the series of n throws, i.e. $S_n = X_1 + X_2 + \cdots + X_n$, where

$$X_j = \begin{cases} 1 & \text{if the } j\text{th throw is a head} \\ -1 & \text{if the } j\text{th throw is a tail.} \end{cases}$$

(i) If $n = 10\,000$, find an approximate value for the probability that $S_n > 100$.

(ii) Find an approximate value for the least n for which $\mathrm{P}(S_n > 0.01n) < 0,01$.

(iii) Suppose that instead no points are scored for the first throw, but that on each successive threw, 2 points are scored if both it and the first throw are heads, two points are deducted if both are tails, and no points are scored or lost if the throws differ. Let Y_k be the score on the kth throw, where $2 \leqslant k \leqslant n$. Show that $Y_k = X_1 + X_k$.

(iv) Calculate the mean and variance of each Y_k and determine whether it is true that

$$\mathrm{P}(Y_2 + Y_3 + \cdots + Y_n > 0.01(n-1)) \to 0 \quad \text{as } n \to \infty.$$

16 At any instant the probability that it is safe to cross a busy road is 0.1. A toad is waiting to cross this road. Every minute she looks at the road. If it is safe, she will cross; if it is not safe, she will wait for a minute before attempting to cross again.

(i) Find the probability that she eventually crosses the road without mishap.

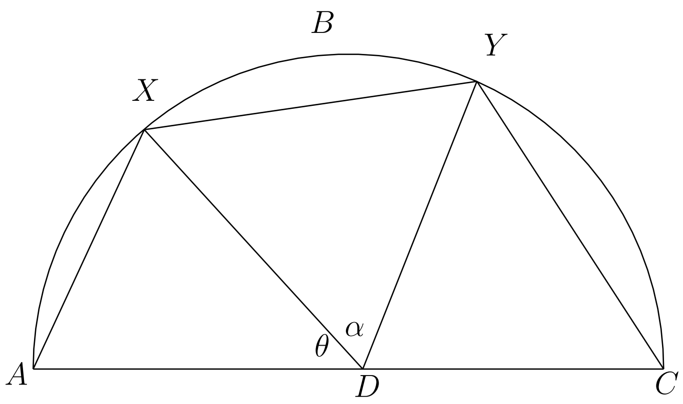
(ii) Later on, a frog is also trying to cross the same road. He also inspects the traffic at one minute intervals and crosses if it is safe. Being more impatient than the toad, he may also attempt to cross when it is not safe. The probability that he will attempt to cross when it is not safe is $n/3$ if $n \leqslant 3$, where n minutes have elapsed since he firrst inspected the road. If he attempts to cross when it is not safe, he is run over with probability 0.8, but otherwise he reaches the other side safely. Find the probability that he eventually crosses the road without mishap.

(iii) What is the probability that both reptiles safely cross the road with the frog taking less time than the toad? If the frog has not arrived at the other side 2 minutes after he began his attempt to cross, what is the probability that the frog is run over (at some stage) in his attempt to cross?

[Once moving, the reptiles spend a negligible time on their attempt to cross the road.]

Section A: Pure Mathematics

1



In the above diagram, $ABCD$ represents a semicircular window of fixed radius r and centre D, and $AXYC$ is a quadrilateral blind.

(i) If $\angle XDY = \alpha$ is fixed and $\angle ADX = \theta$ is variable, determine the value of θ which gives the blind **maximum** area.

(ii) If now α is allowed to vary but r remains fixed, find the maximum possible area of the blind.

2 **(i)** Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. Show that $1+\omega+\omega^2 = 0$ and calculate the modulus and argument of $1+\omega^2$.

(ii) Let n be a positive integer. By evaluating $(1+\omega^r)^n$ in two ways, taking $r = 1, 2$ and 3, or otherwise, prove that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{k} = \frac{1}{3}\left(2^n + 2\cos\left(\frac{n\pi}{3}\right)\right),$$

where k is the largest multiple of 3 less than or equal to n.

(iii) Without using a calculator, evaluate

$$\binom{25}{0} + \binom{25}{3} + \cdots + \binom{25}{24}$$

and

$$\binom{24}{2} + \binom{24}{5} + \cdots + \binom{24}{23}.$$

$[2^{25} = 33554432.]$

3 Given a curve described by $y = \mathrm{f}(x)$, and such that $y \geqslant 0$, a *push-off* of the curve is a new curve obtained as follows: for each point $(x, \mathrm{f}(x))$ with position vector $\mathbf{r}$ on the original curve, there is a point with position vector $\mathbf{s}$ on the new curve such that $\mathbf{s} - \mathbf{r} = \mathrm{p}(x)\mathbf{n}$, where p is a given function and $\mathbf{n}$ is the downward-pointing unit normal to the original curve at $\mathbf{r}$.

(i) For the curve $y = x^k$, where $x > 0$ and k is a positive integer, obtain the function p for which the push-off is the positive x-axis, and find the value of k such that, for all points on the original curve, $|\mathbf{r}| = |\mathbf{r} - \mathbf{s}|$.

(ii) Suppose that the original curve is $y = x^2$ and p is such that the gradient of the curves at the points with position vectors $\mathbf{r}$ and $\mathbf{s}$ are equal (for every point on the original curve). By writing $\mathrm{p}(x) = \mathrm{q}(x)\sqrt{1+4x^2}$, where q is to be determined, or otherwise, find the form of p.

4 **(i)** The sequence $a_1, a_2, \ldots, a_n, \ldots$ forms an arithmetic progression. Establish a formula, involving n, a_1 and a_2 for the sum $a_1 + a_2 + \cdots + a_n$ of the first n terms.

(ii) A sequence $b_1, b_2, \ldots, b_n, \ldots$ is called a *double arithmetic progression* if the sequence of differences

$$b_2 - b_1, b_3 - b_2, \ldots, b_{n+1} - b_n, \ldots$$

is an arithmetic progression. Establish a formula, involving n, b_1, b_2 and b_3, for the sum $b_1 + b_2 + b_3 + \cdots + b_n$ of the first n terms of such a progression.

(iii) A sequence $c_1, c_2, \ldots, c_n, \ldots$ is called a *factorial progression* if $c_{n+1} - c_n = n!d$ for some non-zero d and every $n \geqslant 1$. Suppose $1, b_2, b_3, \ldots$ is a double arithmetic progression, and also that b_2, b_4, b_6 and 220 are the first four terms in a factorial progression. Find the sum $1 + b_2 + b_3 + \cdots + b_n$.

5 **(i)** Evaluate

$$\int_1^3 \frac{1}{6x^2 + 19x + 15}\,\mathrm{d}x\,.$$

(ii) Sketch the graph of the function f, where $\mathrm{f}(x) = x^{1760} - x^{220} + q$, and q is a constant. Find the possible numbers of *distinct* roots of the equation $\mathrm{f}(x) = 0$, and state the inequalities satisfied by q.

6 Let $ABCD$ be a parallelogram. By using vectors, or otherwise, prove that:

(i) $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$;

(ii) $|AC^2 - BD^2|$ is 4 times the area of the rectangle whose sides are *any* side of the parallelogram and the projection of an adjacent side on that side.

(iii) State and prove a result like **(ii)** about $|AB^2 - AD^2|$ and the diagonals.

7 **(i)** Let y, u, v, P and Q all be functions of x. Show that the substitution $y = uv$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

leads to an equation for $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ in terms of x, Q and u, provided that u satisfies a suitable first order differential equation.

(ii) Hence or otherwise solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}},$$

given that $y(1) = 0$. For what set of values of x is the solution valid?

8 **(i)** Show that

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right) = \frac{\sin\alpha}{4\sin\left(\frac{\alpha}{4}\right)},$$

where $\alpha \neq k\pi$, k is an integer.

(ii) Prove that, for such α,

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right) = \frac{\sin\alpha}{2^n\sin\left(\frac{\alpha}{2^n}\right)},$$

where n is a positive integer.

(iii) Deduce that

$$\alpha = \frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\cdots},$$

and hence that

$$\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

9 **(i)** Let A and B be the points $(1, 1)$ and $(b, 1/b)$ respectively, where $b > 1$. The tangents at A and B to the curve $y = 1/x$ intersect at C. Find the coordinates of C.

(ii) Let A', B' and C' denote the projections of A, B and C, respectively, to the x-axis. Obtain an expression for the sum of the areas of the quadrilaterals $ACC'A'$ and $CBB'C'$.

(iii) Hence or otherwise prove that, for $z > 0$,

$$\frac{2z}{2+z} \leqslant \ln(1+z) \leqslant z.$$

Section B: Mechanics

10 In a certain race, runners run 5 km in a straight line to a fixed point and then turn and run back to the starting point. A steady wind of 3 ms^{-1} is blowing from the start to the turning point. At steady racing pace, a certain runner expends energy at a constant rate of 300 W. Two resistive forces act. One is of constant magnitude 50 N. The other, arising from air resistance, is of magnitude $2w\,\mathrm{N}$, where w ms^{-1} is the runner's speed relative to the air.

(i) Give a careful argument to derive formulae from which the runner's steady speed in each half of the race may be found. Calculate, to the nearest second, the time the runner will take for the whole race.

[Effects due to acceleration and deceleration at the start and turn may be ignored.]

(ii) The runner may use alternative tactics, expending the same total energy during the race as a whole, but applying different constant powers, x_1 W in the outward trip, and x_2 W on the return trip. Prove that, with the wind as above, if the outward and return speeds are v_1 ms^{-1} and v_2 ms^{-1} respectively, then $v_1 + v_2$ is independent of the choices for x_1 and x_2.

(iii) Hence show that these alternative tactics allow the runner to run the whole race approximately 15 seconds faster.

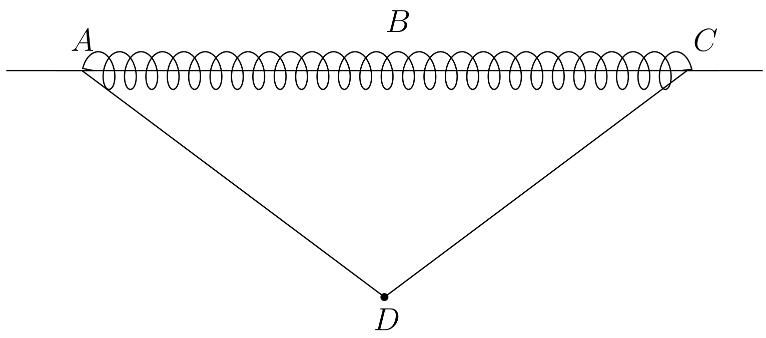
11 A shell of mass m is fired at elevation $\pi/3$ and speed v. Superman, of mass $2m$, catches the shell at the top of its flight, by gliding up behind it in the same horizontal direction with speed $3v$. As soon as Superman catches the shell, he instantaneously clasps it in his cloak, and immediately pushes it vertically downwards, without further changing its horizontal component of velocity, but giving it a downward vertical component of velocity of magnitude $3v/2$.

(i) Calculate the total time of flight of the shell in terms of v and g.

(ii) Calculate also, to the nearest degree, the angle Superman's flight trajectory initially makes with the horizontal after releasing the shell, as he soars upwards like a bird.

[Superman and the shell may be regarded as particles.]

12



In the above diagram, ABC represents a light spring of natural length $2l$ and modulus of elasticity λ, which is coiled round a smooth fixed horizontal rod. B is the midpoint of AC. The two ends of a light inelastic string of length $2l$ are attached to the spring at A and C. A particle of mass m is fixed to the string at D, the midpoint of the string. The system can be in equilibrium with the angle CAD equal to $\pi/6$.

(i) Show that

$$mg = \lambda\left(\frac{2}{\sqrt{3}} - 1\right).$$

(ii) Write the length AC as $2xl$, obtain an expression for the potential energy of the system as a function of x.

(iii) The particle is held at B, and the spring is restored to its natural length $2l$. The particle is then released and falls vertically. Obtain an equation satisfied by x when the particle next comes to rest. Verify numerically that a possible solution for x is approximately 0.66.

13 A rough circular cylinder of mass M and radius a rests on a rough horizontal plane. The curved surface of the cylinder is in contact with a smooth rail, parallel to the axis of the cylinder, which touches the cylinder at a height $a/2$ above the plane. Initially the cylinder is held at rest. A particle of mass m rests in equilibrium on the cylinder, and the normal reaction of the cylinder on the particle makes an angle of θ with the upward vertical. The particle is on the same side of the centre of the cylinder as the rail. The coefficient of friction between the cylinder and the particle and between the cylinder and the plane are both μ.

(i) Obtain the condition on θ for the particle to rest in equilibrium.

(ii) Show that, if the cylinder is released, equilibrium of both particle and cylinder is possible provided another inequality involving μ and θ (which should be found explicitly) is satisfied.

(iii) Determine the largest possible value of θ for equilibrium, if $m = 7M$ and $\mu = 0.75$.

Section C: Probability and Statistics

14 A bag contains 5 white balls, 3 red balls and 2 black balls. In the game of Blackball, a player draws a ball at random from the bag, looks at it and replaces it. If he has drawn a white ball, he scores one point, while for a red ball he scores two points, these scores being added to his total score before he drew the ball. If he has drawn a black ball, the game is over and his final score is zero. After drawing a red or white ball, he can either decide to stop, when his final score for the game is the total so far, or he may elect to draw another ball. The starting score is zero.

(i) Juggins' strategy is to continue drawing until either he draws a black ball (when of course he must stop, with final score zero), or until he has drawn three (non-black) balls, when he elects to stop. Find the probability that in any game he achieves a final score of zero by employing this strategy. Find also his expected final score.

(ii) Muggins has so far scored N points, and is deciding whether to draw another ball. Find the expected score if another ball is drawn, and suggest a strategy to achieve the greatest possible average final score in each game.

15 **(i)** A coin has probability p $(0 < p < 1)$ of showing a head when tossed. Give a careful argument to show that the kth head in a series of consecutive tosses is achieved after *exactly* n tosses with probability

$$\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad (n \geqslant k).$$

(ii) Given that it took an even number of tosses to achieve exactly $k-1$ heads, find the probability that exactly k heads are achieved after an even number of tosses.

(iii) If this coin is tossed until exactly 3 heads are obtained, what is the probability that *exactly* 2 of the heads occur on even-numbered tosses?

16 A bus is supposed to stop outside my house every hour on the hour. From long observation I know that a bus will always arrive some time between 10 minutes before and ten minutes after the hour. The probability it arrives at a given instant increases linearly (from zero at 10 minutes before the hour) up to a maximum value at the hour, and then decreases linearly at the same rate after the hour.

(i) Obtain the probability density function of T, the time in minutes after the scheduled time at which a bus arrives.

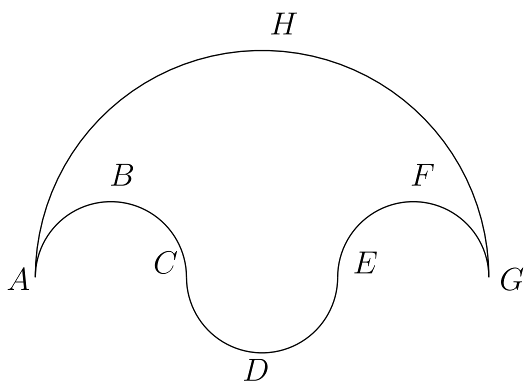
(ii) If I get up when my alarm clock goes off, I arrive at the bus stop at 7.55am. However, with probability 0.5, I doze for 3 minutes before it rings again. In that case with probability 0.8 I get up then and reach the bus stop at 7.58am, or, with probability 0.2, I sleep a little longer, not reaching the stop until 8.02am. What is the probability that I catch a bus by 8.10am?

(iii) I buy a louder alarm clock which ensures that I reach the stop at exactly the same time each morning. This clock keeps perfect time, but may be set to an incorrect time. If it is correct, the alarm goes off so that I should reach the stop at 7.55am. After 100 mornings I find that I have had to wait for a bus until *after* 9am (according to the new clock) on 5 occasions. Is this evidence that the new clock is incorrectly set?

[The time of arrival of different buses are independent of each other.]

Section A: Pure Mathematics

1



In the above diagram, ABC, CDE, EFG and AHG are semicircles and A, C, E, G lie on a straight line. The radii of ABC, EFG, AHG are h, h and r respectively, where $2h < r$.

(i) Show that the area enclosed by $ABCDEFGH$ is equal to that of a circle with diameter HD.

(ii) Each semicircle is now replaced by a portion of a parabola, with vertex at the midpoint of the semicircle arc, passing through the endpoints (so, for example, ABC is replaced by part of a parabola passing through A and C and with vertex at B). Find a formula in terms of r and h for the area enclosed by $ABCDEFGH$.

2 **(i)** For $x > 0$ find $\int x \ln x \, \mathrm{d}x$.

(ii) By approximating the area corresponding to $\int_0^1 x \ln(1/x) \, \mathrm{d}x$ by n rectangles of equal width and with their top right-hand vertices on the curve $y = x \ln(1/x)$, show that, as $n \to \infty$,

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\ln n - \frac{1}{n^2}\left[\ln\left(\frac{n!}{0!}\right) + \ln\left(\frac{n!}{1!}\right) + \ln\left(\frac{n!}{2!}\right) + \cdots + \ln\left(\frac{n!}{(n-1)!}\right)\right] \to \frac{1}{4}.$$

[You may assume that $x \ln x \to 0$ as $x \to 0$.]

3 In the triangle OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $OA = OB = 1$. Points C and D trisect AB (i.e. $AC = CD = DB = \frac{1}{3}AB$). X and Y lie on the line-segments OA and OB respectively, in such a way that CY and DX are perpendicular, and $OX + OY = 1$.

(i) Denoting OX by x, obtain a condition relating x and $\mathbf{a}\cdot\mathbf{b}$, and prove that

$$\frac{8}{17} \leqslant \mathbf{a}\cdot\mathbf{b} \leqslant 1.$$

(ii) If the angle AOB is as large as possible, determine the distance OE, where E is the point of intersection of CY and DX.

4 Six points A, B, C, D, E and F lie in three dimensional space and are in general positions, that is, no three are collinear and no four lie on a plane. All possible line segments joining pairs of points are drawn and coloured either gold or silver.

(i) Prove that there is a triangle whose edges are entirely of one colour.
[**Hint**: consider segments radiating from A.]

(ii) Give a sketch showing that the result is false for five points in general positions.

5 **(i)** Write down the binomial expansion of $(1+x)^n$, where n is a positive integer.

(ii) By substituting particular values of x in the above expression, or otherwise, show that, if n is an even positive integer,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{n-1} = 2^{n-1}.$$

(iii) Show that, if n is any positive integer, then

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(iv) Hence evaluate

$$\sum_{r=0}^{n} \left(r + (-1)^r\right)\binom{n}{r}.$$

6 The normal to the curve $y = \mathrm{f}(x)$ at the point P with coordinates $(x, \mathrm{f}(x))$ cuts the y-axis at the point Q.

(i) Derive an expression in terms of x, $\mathrm{f}(x)$ and $\mathrm{f}'(x)$ for the y-coordinate of Q.

(ii) If, for all x, $PQ = \sqrt{\mathrm{e}^{x^2} + x^2}$, find a differential equation satisfied by $\mathrm{f}(x)$. If the curve also has a minimum point $(0, -2)$, find its equation.

7 **(i)** Sketch the curve $y^2 = 1 - |x|$. A rectangle, with sides parallel to the axes, is inscribed within this curve.

(ii) Show that the largest possible area of the rectangle is $8/\sqrt{27}$.

(iii) Find the maximum area of a rectangle similarly inscribed within the curve given by $y^{2m} = (1 - |x|)^n$, where m and n are positive integers, with n odd.

8 By using de Moivre's theorem, or otherwise, show that

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$;

(ii) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

(iii) Hence, or otherwise, find all the real roots of the equation

$$16x^6 - 28x^4 + 13x^2 - 1 = 0.$$

[No credit will be given for numerical approximations.]

9 **(i)** Sketch the graph of $8y = x^3 - 12x$ for $-4 \leqslant x \leqslant 4$, marking the coordinates of the turning points.

(ii) Similarly marking the turning points, sketch the corresponding graphs in the (X, Y)-plane, if

$$\begin{array}{lll} \text{(a)} & X = \frac{1}{2}x, & Y = y, \\ \text{(b)} & X = x, & Y = \frac{1}{2}y, \\ \text{(c)} & X = \frac{1}{2}x + 1, & Y = y, \\ \text{(d)} & X = x, & Y = \frac{1}{2}y + 1. \end{array}$$

(iii) Find values for a, b, c, d such that, if $X = ax + b$, $Y = cy + d$, then the graph in the (X, Y)-plane corresponding to $8y = x^3 - 12x$ has turning points at $(X, Y) = (0, 0)$ and $(X, Y) = (1, 1)$.

Section B: Mechanics

10 A spaceship of mass M is travelling at constant speed V in a straight line when it enters a force field which applies a resistive force acting directly backwards and of magnitude $M\omega(v^2 + V^2)/v$, where v is the instantaneous speed of the spaceship, and ω is a positive constant. No other forces act on the spaceship.

(i) Find the distance travelled from the edge of the force field until the speed is reduced to $\frac{1}{2}V$.

(ii) As soon as the spaceship has travelled this distance within the force field, the field is altered to a constant resistive force, acting directly backwards, whose magnitude is within 10% of that of the force acting on the spaceship immediately before the change. If z is the extra distance travelled by the spaceship before coming instantaneously to rest, determine limits between which z must lie.

11 A shot-putter projects a shot at an angle θ above the horizontal, releasing it at height h above the level ground, with speed v.

(i) Show that the distance R travelled horizontally by the shot from its point of release until it strikes the ground is given by

$$R = \frac{v^2}{2g}\sin 2\theta\left(1+\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}\right).$$

(ii) The shot-putter's style is such that currently $\theta = 45^\circ$. Determine (with justification) whether a small decrease in θ will increase R.

[Air resistance may be neglected.]

12 A regular tetrahedron $ABCD$ of mass M is made of 6 identical uniform rigid rods, each of length $2a$. Four light elastic strings XA, XB, XC and XD, each of natural length a and modulus of elasticity λ, are fastened together at X, the other end of each string being attached to the corresponding vertex.

(i) Given that X lies at the centre of mass of the tetrahedron, find the tension in each string.

(ii) The tetrahedron is at rest on a smooth horizontal table, with B, C and D touching the table, and the ends of the strings at X attached to a point O fixed in space. Initially the centre of mass of the tetrahedron coincides with O. Suddenly the string XA breaks, and the tetrahedron as a result rises vertically off the table. If the maximum height subsequently attained is such that BCD is level with the fixed point O, show that (to 2 significant figures)

$$\frac{Mg}{\lambda} = 0.098.$$

13 A uniform ladder of mass M rests with its upper end against a smooth vertical wall, and with its lower end on a rough slope which rises upwards towards the wall and makes an angle of ϕ with the horizontal. The acute angle between the ladder and the wall is θ.

(i) If the ladder is in equilibrium, show that N and F, the normal reaction and frictional force at the foot of the ladder are given by

$$N = Mg\left(\cos\phi - \frac{\tan\theta\sin\phi}{2}\right),$$

$$F = Mg\left(\sin\phi + \frac{\tan\theta\cos\phi}{2}\right).$$

(ii) If the coefficient of friction between the ladder and the slope is 2, and $\phi = 45°$, what is the largest value of θ for which the ladder can rest in equilibrium?

Section C: Probability and Statistics

14 The prevailing winds blow in a constant southerly direction from an enchanted castle. Each year, according to an ancient tradiction, a princess releases 96 magic seeds from the castle, which are carried south by the wind before falling to rest. South of the castle lies one league of grassy parkland, then one league of lake, then one league of farmland, and finally the sea. If a seed falls on land it will immediately grow into a fever tree. (Fever trees do not grow in water). Seeds are blown independently of each other. The random variable L is the distance in leagues south of the castle at which a seed falls to rest (either on land or water). It is known that the probability density function f of L is given by

$$\mathrm{f}(x) = \begin{cases} \frac{1}{2} - \frac{1}{8}x & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

(i) What is the mean number of fever trees which begin to grow each year?

(ii) The random variable Y is defined as the distance in leagues south of the castle at which a new fever tree grows from a seed carried by the wind. Sketch the probability density function of Y, and find the mean of Y.

(iii) One year messengers bring the king the news that 23 new fever trees have grown in the farmland. The wind never varies, and so the king suspects that the ancient tradition have not been followed properly. Is he justified in his suspicions?

15 I can choose one of three routes to cycle to school. Via Angle Avenue the distance is 5 km, and I am held up at a level crossing for A minutes, where A is a continuous random variable uniformly distributed between 0 and 10. Via Bend Boulevard the distance is 4 km, and I am delayed, by talking to each of B friends for 3 minutes, for a total of $3B$ minutes, where B is a random variable whose distribution is Poisson with mean 4. Via Detour Drive the distance should be only 2 km, but in addition, due to never-ending road works, there are five places at each of which, with probability $\frac{4}{5}$, I have to make a detour that increases the distance by 1 km. Except when delayed by talking to friends or at the level crossing, I cycle at a steady 12 km h^{-1}.

(i) For each of the three routs, calculate the probability that a journey lasts at least 27 minutes.

(ii) Each day I choose one of the three routes at random, and I am equally likely to choose any of the three alternatives. One day I arrive at school after a journey of at least 27 minutes.

What is the probability that I came via Bend Boulevard?

(iii) Which route should I use all the time:

(a) if I wish my average journey time to be as small as possible;

(b) if I wish my journey time to be less than 32 minutes as often as possible?

Justify your answers.

16 A and B play a guessing game. Each simultaneously names one of the numbers $1, 2, 3$. If the numbers differ by 2, whoever guessed the *smaller* pays the opponent $\pounds 2$. If the numbers differ by 1, whoever guessed the *larger* pays the opponent $\pounds 1$. Otherwise no money changes hands. Many rounds of the game are played.

(i) If A says he will always guess the same number N, explain (for each value of N) how B can maximise his winnings.

(ii) In an attempt to improve his play, A announces that he will guess each number at random with probability $\frac{1}{3}$, guesses on different rounds being independent. To counter this, B secretly decides to guess j with probability b_j $(j = 1, 2, 3,\ b_1 + b_2 + b_3 = 1)$, guesses on different rounds being independent. Derive an expression for B's expected winnings on any round. How should the probabilities b_j be chosen so as to maximize this expression?

(iii) A now announces that he will guess j with probability a_j $(j = 1, 2, 3,\ a_1 + a_2 + a_3 = 1)$. If B guesses j with probability b_j $(j = 1, 2, 3,\ b_1 + b_2 + b_3 = 1)$, obtain an expression for his expected winnings in the form

$$Xa_1 + Ya_2 + Za_3.$$

Show that he can choose b_1, b_2 and b_3 such that X, Y and Z are all non-negative. Deduce that, whatever values for a_j are chosen by A, B can ensure that in the long run he loses no money.

Section A: Pure Mathematics

1 **(i)** Sketch the graph of the function h, where

$$\mathrm{h}(x) = \frac{\ln x}{x}, \qquad (x > 0).$$

(ii) Hence, or otherwise, find all pairs of distinct positive integers m and n which satisfy the equation

$$n^m = m^n.$$

2 The function f and g are related (for all real x) by

$$\mathrm{g}(x) = \mathrm{f}(x) + \frac{1}{\mathrm{f}(x)}\,.$$

(i) Express $\mathrm{g}'(x)$ and $\mathrm{g}''(x)$ in terms of $\mathrm{f}(x)$ and its derivatives.

(ii) If $\mathrm{f}(x) = 4 + \cos 2x + 2\sin x$, find the stationary points of g for $0 \leqslant x \leqslant 2\pi$, and determine which are maxima and which are minima.

3 Two points P and Q lie within, or on the boundary of, a square of side 1cm, one corner of which is the point O. Show that the length of at least one of the lines OP, PQ and QO must be less than or equal to $(\sqrt{6} - \sqrt{2})$ cm.

4 Each of m distinct points on the positive y-axis is joined by a line segment to each of n distinct points on the positive x-axis. Except at the endpoints, no three of these segments meet in a single point. Derive formulae for

(i) the number of such line segments;

(ii) the number of points of intersections of the segments, ignoring intersections at the endpoints of the segments.

(iii) If $m = n \geqslant 3$, and the two segments with the greatest number of points of intersection, and the two segments with the least number of points of intersection, are excluded, prove that the average number of points of intersection per segment on the remaining segments is

$$\frac{n^3 - 7n + 2}{4(n+2)}\,.$$

5 **(i)** Given that $b > a > 0$, find, by using the binomial theorem, coefficients c_m $(m = 0, 1, 2, \ldots)$ such that

$$\frac{1}{(1-ax)\,(1-bx)} = c_0 + c_1x + c_2x^2 + \ldots + c_mx^m + \cdots$$

for $b\,|x| < 1$.

(ii) Show that

$$c_m^2 = \frac{a^{2m+2} - 2(ab)^{m+1} + b^{2m+2}}{(a-b)^2}\,.$$

(iii) Hence, or otherwise, show that

$$c_0^2 + c_1^2x + c_2^2x^2 + \cdots + c_m^2x^m + \cdots = \frac{1 + abx}{(1-abx)\,(1-a^2x)\,(1-b^2x)}\,,$$

for x in a suitable interval which you should determine.

6 The complex numbers $z_1, z_2, \ldots, z_6$ are represented by six distinct points $P_1, P_2, \ldots, P_6$ in the Argand diagram. Express the following statements in terms of complex numbers:

(a) $\overrightarrow{P_1P_2} = \overrightarrow{P_5P_4}$ and $\overrightarrow{P_2P_3} = \overrightarrow{P_6P_5}$;

(b) $\overrightarrow{P_2P_4}$ is perpendicular to $\overrightarrow{P_3P_6}$.

(i) If **(a)** holds, show that $\overrightarrow{P_3P_4} = \overrightarrow{P_1P_6}$.

(ii) Suppose that the statements **(a)** and **(b)** both hold, and that $z_1 = 0$, $z_2 = 1$, $z_3 = z$, $z_5 = \mathrm{i}$ and $z_6 = w$. Determine the conditions which $\mathrm{Re}(z)$ and $\mathrm{Re}(w)$ must satisfy in order that $P_1P_2P_3P_4P_5P_6$ should form a convex hexagon.

(iii) Find the distance between P_3 and P_6 when $\tan(\angle P_3P_2P_6) = -2/3$.

7 The function f is defined by

$$\mathrm{f}(x) = ax^2 + bx + c.$$

(i) Show that

$$\mathrm{f}'(x) = \mathrm{f}(1)\left(x + \tfrac{1}{2}\right) + \mathrm{f}(-1)\left(x - \tfrac{1}{2}\right) - 2\mathrm{f}(0)x.$$

(ii) If a, b and c are real and such that $|\mathrm{f}(x)| \leqslant 1$ for $|x| \leqslant 1$, show that $|\mathrm{f}'(x)| \leqslant 4$ for $|x| \leqslant 1$.

(iii) Find particular values of a, b and c such that, for the corresponding function f of the above form $|\mathrm{f}(x)| \leqslant 1$ for all x with $|x| \leqslant 1$ and $|\mathrm{f}'(x)| = 4$ for some x satisfying $|x| \leqslant 1$.

8 $ABCD$ is a skew (non-planar) quadrilateral, and its pairs of opposite sides are equal, i.e. $AB = CD$ and $BC = AD$. Prove that the line joining the midpoints of the diagonals AC and BD is perpendicular to each diagonal.

9 Find the following integrals:

(i) $$\int_1^{\mathrm{e}} \frac{\ln x}{x^2}\,\mathrm{d}x\,,$$

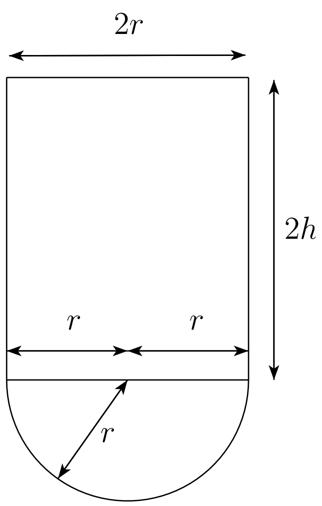
(ii) $$\int \frac{\cos x}{\sin x\sqrt{1+\sin x}}\,\mathrm{d}x.$$

Section B: Mechanics

10 A sniper at the top of a tree of height h is hit by a bullet fired from the undergrowth covering the horizontal ground below. The position and elevation of the gun which fired the shot are unknown, but it is known that the bullet left the gun with speed v. Show that it must have been fired from a point within a circle centred on the base of the tree and of radius $(v/g)\sqrt{v^2 - 2gh}$.

[Neglect air resistance.]

11 **(i)** Derive a formula for the position of the centre of mass of a uniform circular arc of radius r which subtends an angle 2θ at the centre.



(ii) A plane framework consisting of a rectangle and a semicircle, as in the above diagram, is constructed of uniform thin rods. It can stand in equilibrium if it is placed in a vertical plane with any point of the semicircle in contact with a horizontal floor. Express h in terms of r.

12 A skater of mass M is skating inattentively on a smooth frozen canal. She suddenly realises that she is heading perpendicularly towards the straight canal bank at speed V. She is at a distance d from the bank and can choose one of two methods of trying to avoid it; either she can apply a force of constant magnitude F, acting at right-angles to her velocity, so that she travels in a circle; or she can apply a force of magnitude $\frac{1}{2}F(V^2 + v^2)/V^2$ directly backwards, where v is her instantaneous speed. Treating the skater as a particle, find the set of values of d for which she can avoid hitting the bank. Comment **briefly** on the assumption that the skater is a particle.

13 A piece of circus apparatus consists of a rigid uniform plank of mass 1000 kg, suspended in a horizontal position by two equal light vertical ropes attached to the ends. The ropes each have natural length 10 m and modulus of elasticity 490 000 N. Initially the plank is hanging in equilibrium. Nellie, an elephant of mass 4000 kg, lands in the middle of the plank while travelling vertically downwards at speed 5 ms^{-1}. While carrying Nellie, the plank comes instantaneously to rest at a negligible height above the floor, and at this instant Nellie steps nimbly and gently off the plank onto the floor.

(i) Assuming that the plank remains horizontal, and the rope remain vertical, throughout the motion, find to three significant figures its initial height above the floor.

(ii) During the motion after Nellie alights, do the ropes ever become slack?

[Take g to be 9.8 ms^{-1}.]

Section C: Probability and Statistics

14 Let X be a standard normal random variable. If M is any real number, the random variable X_M is defined in terms of X by

$$X_M = \begin{cases} X & \text{if } X < M, \\ M & \text{if } X \geqslant M. \end{cases}$$

(i) Show that the expectation of X_M is given by

$$\mathrm{E}(X_M) = -\phi(M) + M(1 - \Phi(M)),$$

where ϕ is the probability density function, and Φ is the cumulative distribution function of X.

(ii) Fifty times a year, 1024 tourists disembark from a cruise liner at the port of Slaka. From there they must travel to the capital either by taxi or by bus. Officials of HOGPo are equally likely to direct a tourist to the bus station or to the taxi rank. Each bus of the bus coorperative holds 31 passengers, and the coorperative currently runs 16 buses. The bus coorperative makes a profit of 1 vloska for each passenger carried. It carries all the passengers it can, with any excess being (eventually) transported by taxi. What is the larges annual bribe the bus coorperative should consider paying to HOGPo in order to be allowed to run an extra bus?

15 **(i)** In Fridge football, each team scores two points for a goal and one point for a foul committed by the opposing team. In each game, for each team, the probability that the team scores n goals is $(3 - |2 - n|)/9$ for $0 \leqslant n \leqslant 4$ and zero otherwise, while the number of fouls committed against it will with equal probability be one of the numbers from 0 to 9 inclusive. The numbers of goals and fouls of each team are mutually independent. What is the probability that in some game a particular team gains more than half its points from fouls?

(ii) In response to criticisms that the game is boring and violent, the ruling body increases the number of penalty points awarded for a foul, in the hope that this will cause large numbers of fouls to be less probable. During the season following the rule change, 150 games are played and on 12 occasions (out of 300) a team committed 9 fouls. Is this good evidence of a change in the probability distribution of the number of fouls? Justify your answer.

16 Wondergoo is applied to all new cars. It protects them completely against rust for three years, but thereafter the probability density of the time of onset of rust is proportional to $t^2/(1+t^2)^2$ for a car of age $3+t$ years $(t \geqslant 0)$.

(i) Find the probability that a car becomes rusty before it is $3+t$ years old.

(ii) Every car is tested for rust annually on the anniversary of its manufacture. If a car is not rusty, it will certainly pass; if it is rusty, it will pass with probability $\frac{1}{2}$. Cars which do not pass are immediately taken off the road and destroyed. What is the probability that a randomly selected new car subsequently fails a test taken on the fifth anniversary of its manufacture?

(iii) Find also the probability that a car which was destroyed immediately after its fifth anniversary test was rusty when it passed its fourth anniversary test.

Section A: Pure Mathematics

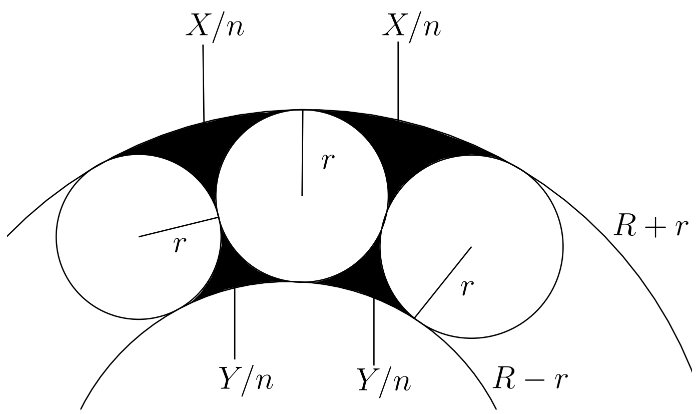
1 **(i)** Find the stationary points of the function f given by

$$\mathrm{f}(x) = \mathrm{e}^{ax} \cos bx, \ (a > 0, b > 0).$$

(ii) Show that the values of f at the stationary points with $x > 0$ form a geometric progression with common ratio $-\mathrm{e}^{a\pi/b}$.

(iii) Give a rough sketch of the graph of f.

2



The region A between concentric circles of radii $R + r$, $R - r$ contains n circles of radius r. Each circle of radius r touches both of the larger circles as well as its two neighbours of radius r, as shown in the figure.

(i) Find the relationship which must hold between n, R and r.

(ii) Show that Y, the total area of A outside the circle of radius r and adjacent to the circle of radius $R - r$, is given by

$$Y = nr\sqrt{R^2 - r^2} - \pi(R - r)^2 - n\pi r^2 \left(\frac{1}{2} - \frac{1}{n}\right).$$

(iii) Find similar expressions for X, the total area of A outside the circles of radius r and adjacent to the circle of radius $R + r$, and for Z, the total area inside the circle of radius r.

(iv) What value does $(X + Y)/Z$ approach when n becomes large?

3 **(i)** By substituting $y(x) = xv(x)$ in the differential equation

$$x^3 \frac{\mathrm{d}v}{\mathrm{d}x} + x^2 v = \frac{1 + x^2 v^2}{(1 + x^2)\, v},$$

or otherwise, find the solution $v(x)$ that satisfies $v = 1$ when $x = 1$.

(ii) What value does this solution approach when x becomes large?

4 Show that the sum of the infinite series

$$\log_2 \mathrm{e} - \log_4 \mathrm{e} + \log_{16} \mathrm{e} - \ldots + (-1)^n \log_{2^{2^n}} \mathrm{e} + \ldots$$

is

$$\frac{1}{\ln(2\sqrt{2})}.$$

[$\log_a b = c$ is equivalent to $a^c = b$.]

5 **(i)** Using the substitution $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$, show that, if $\alpha < \beta$,

$$\int_\alpha^\beta \frac{1}{\sqrt{(x - \alpha)(\beta - x)}} \, \mathrm{d}x = \pi.$$

(ii) What is the value of the above integral if $\alpha > \beta$?

(iii) Show also that, if $0 < \alpha < \beta$,

$$\int_\alpha^\beta \frac{1}{x\sqrt{(x - \alpha)(\beta - x)}} \, \mathrm{d}x = \frac{\pi}{\sqrt{\alpha\beta}}.$$

6 Let $y = \mathrm{f}(x)$, $(0 \leqslant x \leqslant a)$, be a continuous curve lying in the first quadrant and passing through the origin. Suppose that, for each non-negative value of y with $0 \leqslant y \leqslant \mathrm{f}(a)$, there is *exactly* one value of x such that $\mathrm{f}(x) = y$; thus we may write $x = \mathrm{g}(y)$, for a suitable function g.

For $0 \leqslant s \leqslant a,\ 0 \leqslant t \leqslant \mathrm{f}(a)$, define

$$\mathrm{F}(s) = \int_0^s \mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{G}(t) = \int_0^t \mathrm{g}(y)\,\mathrm{d}y.$$

(i) By a geometrical argument, show that

$$\mathrm{F}(s) + \mathrm{G}(t) \geqslant st. \qquad (*)$$

When does equality occur in $(*)$?

(ii) Suppose that $y = \sin x$ and that the ranges of x, y, s, t are restricted to $0 \leqslant x \leqslant s \leqslant \frac{1}{2}\pi$, $0 \leqslant y \leqslant t \leqslant 1$. By considering s such that the equality holds in $(*)$, show that

$$\int_0^t \sin^{-1} y\,\mathrm{d}y = t\sin^{-1} t - \left(1 - \cos(\sin^{-1} t)\right).$$

(iii) Check this result by differentiating both sides with respect to t.

7 Sum each of the series and give each answer in terms of the tangent of a single angle.

(i)

$$\sin\left(\frac{2\pi}{23}\right) + \sin\left(\frac{6\pi}{23}\right) + \sin\left(\frac{10\pi}{23}\right) + \cdots + \sin\left(\frac{38\pi}{23}\right) + \sin\left(\frac{42\pi}{23}\right)$$

(ii)

$$\sin\left(\frac{2\pi}{23}\right) - \sin\left(\frac{6\pi}{23}\right) + \sin\left(\frac{10\pi}{23}\right) - \cdots - \sin\left(\frac{38\pi}{23}\right) + \sin\left(\frac{42\pi}{23}\right)$$

8 **(i)** Explain why the use of the substitution $x = \dfrac{1}{t}$ does not demonstrate that the integrals

$$\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,\mathrm{d}x \quad \text{and} \quad \int_{-1}^{1} \frac{-t^2}{(1+t^2)^2}\,\mathrm{d}t$$

are equal.

(ii) Evaluate both integrals correctly.

9 ABC is a triangle whose vertices have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, relative to an origin in the plane ABC.

(i) Show that an arbitrary point P on the segment AB has position vector

$$\rho\mathbf{a} + \sigma\mathbf{b},$$

where $\rho \geqslant 0$, $\sigma \geqslant 0$ and $\rho + \sigma = 1$.

(ii) Give a similar expression for an arbitrary point on the segment PC, and deduce that any point inside ABC has position vector

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c},$$

where $\lambda \geqslant 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ and $\lambda + \mu + \nu = 1$.

(iii) Sketch the region of the plane in which the point $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ lies in each of the following cases:

(a) $\lambda + \mu + \nu = -1$, $\lambda \leqslant 0$, $\mu \leqslant 0$, $\nu \leqslant 0$;

(b) $\lambda + \mu + \nu = 1$, $\mu \leqslant 0$, $\nu \leqslant 0$.

Section B: Mechanics

10 A rubber band band of length 2π and modulus of elasticity λ encircles a smooth cylinder of unit radius, whose axis is horizontal. A particle of mass m is attached to the lowest point of the band, and hangs in equilibrium at a distance x below the axis of the cylinder.

(i) Obtain an expression in terms of x for the stretched length of the band in equilibrium.

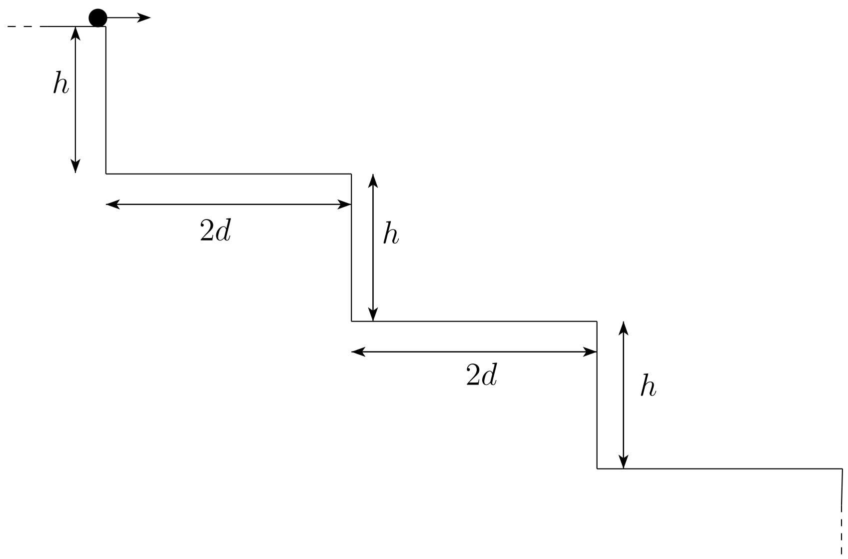
(ii) What is the value of λ if $x = 2$?

11 A smooth sphere of radius r stands fixed on a horizontal floor. A particle of mass m is displaced gently from equilibrium on top of the sphere.

(i) Find the angle its velocity makes with the horizontal when it loses contact with the sphere during the subsequent motion.

(ii) By energy considerations, or otherwise, find the vertical component of the momentum of the particle as it strikes the floor.

12



A particle is placed at the edge of the top step of a flight of steps. Each step is of width $2d$ and height h. The particle is kicked horizontally perpendicular to the edge of the top step. On its first and second bounces it lands exactly in the middle of each of the first and second steps from the top.

(i) Find the coefficient of restitution between the particle and the steps.

(ii) Determine whether it is possible for the particle to continue bouncing down the steps, hitting the middle of each successive step.

13 A particle of mass m moves along the x-axis. At time $t = 0$ it passes through $x = 0$ with velocity $v_0 > 0$. The particle is acted on by a force $\mathrm{F}(x)$, directed along the x-axis and measured in the direction of positive x, which is given by

$$\mathrm{F}(x) = \begin{cases} -m\mu^2 x & (x \geqslant 0), \\ -m\kappa \dfrac{\mathrm{d}x}{\mathrm{d}t} & (x < 0), \end{cases}$$

where μ and κ are positive constants. Obtain the particle's subsequent position as a function of time, and give a rough sketch of the x-t graph.

Section C: Probability and Statistics

14 A, B and C play a table tennis tournament. The winner of the tournament will be the first person to win two games in a row. In any game, whoever is not playing acts as a referee, and each player has equal chance of winning the game. The first game of the tournament is played between A and B, with C as referee. Thereafter, if the tournament is still undecided at the end of any game, the winner and referee of that game play the next game. The tournament is recorded by listing in order the winners of each game, so that, for example, ACC records a three-game tournament won by C, the first game having been won by A.

(i) Determine which of the following sequences of letters could be the record of a complete tournament, giving brief reasons for your answers:

(a) ACB,

(b) ABB,

(c) $ACBB$.

(ii) Find the probability that the tournament is still undecided after 5 games have been played. Find also the probabilities that each of A, B and C wins in 5 or fewer games.

(iii) Show that the probability that A wins eventually is $\frac{5}{14}$, and find the corresponding probabilities for B and C.

15 A point P is chosen at random (with uniform distribution) on the circle $x^2 + y^2 = 1$. The random variable X denotes the distance of P from $(1, 0)$.

(i) Find the mean and variance of X.

(ii) Find also the probability that X is greater than its mean.

16 The parliament of Laputa consists of 60 Preservatives and 40 Progressives. Preservatives never change their mind, always voting the same way on any given issue. Progressives vote at random on any given issue.

(i) A randomly selected member is known to have voted the same way twice on a given issue. Find the probability that the member will vote the same way a third time on that issue.

(ii) Following a policy change, a proportion α of Preservatives now consistently votes against Preservative policy. The Preservative leader decides that an election must be called if the value of α is such that, at any vote on an item of Preservative policy, the chance of a simple majority would be less than 80%. By making a suitable normal approximation, estimate the least value of α which will result in an election being called.

Section A: Pure Mathematics

1 **(i)** The real numbers x and y satisfy the equation $4x^2 + 16xy + y^2 + 24x = 0$. Prove that either $x \leqslant 0$ or $x \geqslant \frac{2}{5}$, and, similarly, find restrictions on the values of y.

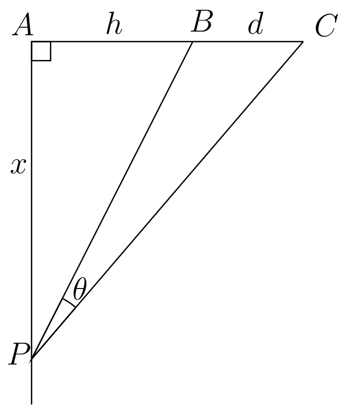
(ii) Find the coefficient of x^n in the expansion, in ascending powers of x, of

$$\frac{9}{(2-x)^2(1+x)}.$$

State the set of values of x for which this expansion is valid.

2 In the figure, the angle PAC is a right angle. $AB = h, BC = d, AP = x$ and the angle BPC is θ.

(i) Express $\tan\theta$ in terms of h, d and x.



(ii) Given that x may be varied, deduce, or find otherwise, the value of x (in terms of the constants d and h) for which θ has its maximum value.

3 Using the substitution $z = \frac{\mathrm{d}y}{\mathrm{d}x} - y$, or otherwise, solve the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\mathrm{e}^x,$$

given that

$$y = 1 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \text{when } x = 0.$$

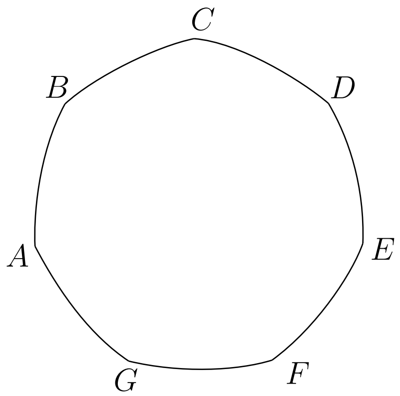
4 Evaluate

$$\int_1^\infty \frac{1}{(x+1)\sqrt{x^2+2x-2}}\,\mathrm{d}x.$$

5 **(i)** You are writing down all the positive integers in increasing order, starting from 1, until you have written 1000 digits, at which point you stop (even possibly in the middle of a number). How many times have you used the digit 7? Explain your reasoning briefly but clearly.

(ii) In how many zeros does the number $365!$ terminate, when written in base 10? Justify your answer.

6



The diagram shows a cross-section, parallel to its faces, of a British 50 pence coin. The seven arcs $AB, BC, \ldots, FG, GA$ are all of equal length and each arc is formed from the circle having its centre at the vertex diametrically opposite the mid-point of the arc. Given that the radius of each of these circles is a, show that the area of a face of the coin is

$$\frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right).$$

7 **(i)** Find the modulus and argument of $1 + \mathrm{e}^{2\mathrm{i}\alpha}$ where $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$.

(ii) By using de Moivre's theorem, or otherwise, sum the series

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\sin(2r+1)\alpha.$$

8 Let (a, b) be a fixed point, and (x, y) a variable point, on the curve $y = \mathrm{f}(x)$, $(x \geqslant a, \mathrm{f}'(x) \geqslant 0)$. The curve divides the rectangle with vertices $(a, b), (a, y), (x, y)$ and (x, b) into two portions, the lower of which has always half the area of the upper. Show that the curve is an arc of a parabola with its vertex at (a, b).

9 **(i)** Find the set of values of x which satisfy the inequality

$$\left| x + \frac{x-1}{x+1} \right| < 2,$$

expressing the set in terms of intervals whose end points are numbers of the form $a + b\sqrt{3}$, where a and b are integers.

(ii) Evaluate $\int_0^{\pi} |\cos x - \sin x| \, \mathrm{d}x$.

Section B: Mechanics

10 An engine of mass m moves along a straight level track against a resistance which at any instant has magnitude mkv, where v is the speed of the engine and k is a constant. The engine is working at a constant rate $9mku^2$.

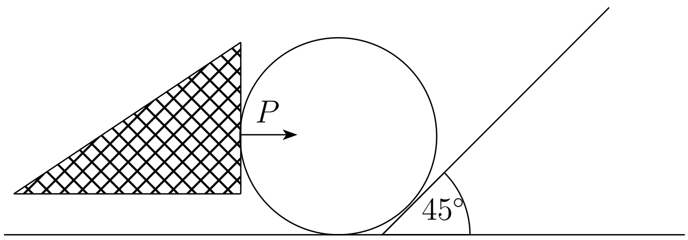
(i) Given that the engine starts from rest, state the maximum possible speed of the engine, and explain briefly whether this speed can be obtained in a finite time.

(ii) Show that the speed of the engine increases from u to $2u$ in time

$$\frac{1}{2k}\ln\left(\frac{8}{5}\right).$$

(iii) When the engine has attained a speed $2u$, the power is cut off, and an additional resistance to motion, of constant magnitude mkB, is applied so that the engine is brought to rest. Find the distance travelled by the engine with the power cut off.

11



A uniform cylinder of mass M rests on a horizontal floor touching a loading ramp at $45°$ to the horizontal, as shown in the diagram. The cylinder is pushed from the side with a force of magnitude P by the vertical face of a piece of moving equipment. The coefficient of friction between the cylinder and the vertical face is μ and the coefficient of friction between the cylinder and the ramp is v. The value of P is such that the cylinder is just about to roll up the ramp.

(i) State on which surface the friction must be limiting, and hence show that

$$P = \frac{Mg}{1-\mu(1+\sqrt{2})}.$$

(ii) Show further that $\mu < \sqrt{2}-1$ and $v \geqslant \mu/(\sqrt{2}-\mu)$.

12 A game consists of a player sliding a small uniform disc A, initially at a point O, across horizontal ice. The disc collides with an identical disc B at rest at a distance d away from O. After the collision, B comes to rest at a target C distant $2d$ away from O. All motion is along the line OC and the discs may be treated as point masses.

(i) Show that the speed of B immediately after the collision is $\sqrt{2\mu g d}$, where μ is the coefficient of friction between the discs and the ice.

(ii) Deduce that A is started with speed U given by

$$U = \frac{\sqrt{2\mu g d}}{1+e}(5 + 2e + e^2)^{\frac{1}{2}},$$

where e is the coefficient of restitution between the discs.

13 A particle of mass m is attached to one end of a light elastic string of natural length a, and the other end of the string is attached to a fixed point A. When at rest and hanging vertically the string has length $2a$. The particle is set in motion so that it moves in a horizontal circle below the level of A. The vertical plane through A containing the string rotates with constant angular speed ω. Show that for this motion to be possible, the string must be stretched to a length greater than $2a$ and ω must satisfy

$$\frac{g}{2a} < \omega^2 < \frac{g}{a}.$$

Section C: Probability and Statistics

14 A and B play the following game. A throws two unbiased four-faced dice onto a table (the four faces of each die are numbered $1, 2, 3, 4$ respectively). The total score is the sum of the numbers on the faces in contact with the table. B tries to guess this score, and guesses x. If his guess is right he wins $100x^2$ pence, and if his guess is wrong he loses $50x$ pence.

(i) Show that B's expected gain if he guesses 8 is 25 pence.

(ii) Which value of x would you advise B to choose, and what is his expected gain in this case?

15 The random variable C takes integral values in the interval -5 to 5, with probabilities

$$\mathrm{P}(C=-5)=\mathrm{P}(C=5)=\frac{1}{20},$$
$$\mathrm{P}(C=i)=\frac{1}{10}, \quad \text{for } -4 \leqslant i \leqslant 4.$$

(i) Calculate the expectation and variance of C.

(ii) A shopper buys 36 items at random in a supermarket and, instead of adding up her bill exactly, she first rounds the cost of each item to the nearest 10 pence, rounding up or down with equal probability if there is an odd amount of 5 pence. Should she suspect a mistake if the cashier asks her for 20 pence more than she estimated? Explain your reasoning briefly but clearly.

16 The length in minutes of a telephone call made by a man from a public call-box is a random variable denoted by T. The probability density function of T is given by:

$$\mathrm{f}(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & 0 \leqslant t < 1, \\ k\mathrm{e}^{-2t} & t \geqslant 1, \end{cases}$$

where k is constant.

(i) Show that the expected length of a call is one minute.

(ii) Find the cumulative distribution function of T.

(iii) To pay for a call, the man inserts into the coin box a 10 pence coin at the beginning of the call, and then another 10 pence coin after each half-minute of the call has elapsed. The random variable C is the cost in pence of the call. Prove that

$$\mathrm{E}(C) = 22\tfrac{1}{2} + \frac{5}{\mathrm{e}-1}.$$

Comment *briefly* on why this value differs from the expected length of a call multiplied by the charging rate of 20 pence per minute.

Section A: Pure Mathematics

1 Show that, if k is a root of the quartic equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0\,, \qquad (*)$$

then k^{-1} is a root.

You are now given that a and b in $(*)$ are both real and are such that the roots are all real.

(i) Write down all the values of a and b for which $(*)$ has only one distinct root.

(ii) Given that $(*)$ has exactly three distinct roots, show that either $b = 2a - 2$ or $b = -2a - 2$.

(iii) Solve $(*)$ in the case $b = 2a - 2$, giving your solutions in terms of a.

Given that a and b are both real and that the roots of $(*)$ are all real, find necessary and sufficient conditions, in terms of a and b, for $(*)$ to have exactly three distinct real roots.

2 A function $\mathrm{f}(x)$ is said to be *concave* for $a < x < b$ if

$$t\,\mathrm{f}(x_1) + (1-t)\,\mathrm{f}(x_2) \leqslant \mathrm{f}\big(tx_1 + (1-t)x_2\big)\,,$$

for $a < x_1 < b\,,\;\; a < x_2 < b$ and $0 \leqslant t \leqslant 1\,.$

Illustrate this definition by means of a sketch, showing the chord joining the points $\big(x_1, \mathrm{f}(x_1)\big)$ and $\big(x_2, \mathrm{f}(x_2)\big)$, in the case $x_1 < x_2$ and $\mathrm{f}(x_1) < \mathrm{f}(x_2)\,.$

Explain why a function $\mathrm{f}(x)$ satisfying $\mathrm{f}''(x) < 0$ for $a < x < b$ is concave for $a < x < b\,.$

(i) By choosing t, x_1 and x_2 suitably, show that, if $\mathrm{f}(x)$ is concave for $a < x < b$, then

$$\mathrm{f}\Big(\frac{u+v+w}{3}\Big) \geqslant \frac{\mathrm{f}(u)+\mathrm{f}(v)+\mathrm{f}(w)}{3}\,,$$

for $a < u < b$, $a < v < b$ and $a < w < b\,.$

(ii) Show that, if A, B and C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}\,.$$

(iii) By considering $\ln(\sin x)$, show that, if A, B and C are the angles of a triangle, then

$$\sin A \times \sin B \times \sin C \leqslant \frac{3\sqrt{3}}{8}\,.$$

3 **(i)** Let

$$f(x) = \frac{1}{1 + \tan x}$$

for $0 \leqslant x < \frac{1}{2}\pi$.

Show that $f'(x) = -\dfrac{1}{1 + \sin 2x}$ and hence find the range of $f'(x)$.

Sketch the curve $y = f(x)$.

(ii) The function $g(x)$ is continuous for $-1 \leqslant x \leqslant 1$.

Show that the curve $y = g(x)$ has rotational symmetry of order 2 about the point (a, b) on the curve if and only if

$$g(x) + g(2a - x) = 2b \, .$$

Given that the curve $y = g(x)$ passes through the origin and has rotational symmetry of order 2 about the origin, write down the value of

$$\int_{-1}^{1} g(x) \, dx \, .$$

(iii) Show that the curve $y = \dfrac{1}{1 + \tan^k x}$, where k is a positive constant and $0 < x < \frac{1}{2}\pi$,

has rotational symmetry of order 2 about a certain point (which you should specify) and evaluate

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \tan^k x} \, dx \, .$$

4 In this question, you may use the following identity without proof:

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\,\cos\tfrac{1}{2}(A-B)\,.$$

(i) Given that $0 \leqslant x \leqslant 2\pi$, find all the values of x that satisfy the equation

$$\cos x + 3\cos 2x + 3\cos 3x + \cos 4x = 0\,.$$

(ii) Given that $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$ and that

$$\cos(x+y) + \cos(x-y) - \cos 2x = 1\,,$$

show that either $x = y$ or x takes one specific value which you should find.

(iii) Given that $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$, find the values of x and y that satisfy the equation

$$\cos x + \cos y - \cos(x+y) = \tfrac{3}{2}\,.$$

5 In this question, you should ignore issues of convergence.

(i) Write down the binomial expansion, for $|x| < 1$, of $\dfrac{1}{1+x}$ and deduce that

$$\ln(1+x) = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$$

for $|x| < 1$.

(ii) Write down the series expansion in powers of x of e^{-ax}. Use this expansion to show that

$$\int_0^{\infty}\frac{(1-\mathrm{e}^{-ax})\,\mathrm{e}^{-x}}{x}\,\mathrm{d}x = \ln(1+a) \qquad (|a| < 1)\,.$$

(iii) Deduce the value of

$$\int_0^{1}\frac{x^p - x^q}{\ln x}\,\mathrm{d}x \qquad (|p| < 1,\ |q| < 1)\,.$$

6 **(i)** Find all pairs of positive integers (n, p), where p is a prime number, that satisfy

$$n! + 5 = p\,.$$

(ii) In this part of the question you may use the following two theorems:

1. For $n \geqslant 7$, $1! \times 3! \times \cdots \times (2n-1)! > (4n)!\,.$
2. For every positive integer n, there is a prime number between $2n$ and $4n$.

Find all pairs of positive integers (n, m) that satisfy

$$1! \times 3! \times \cdots \times (2n-1)! = m!\,.$$

7 The points O, A and B are the vertices of an acute-angled triangle. The points M and N lie on the sides OA and OB respectively, and the lines AN and BM intersect at Q. The position vector of A with respect to O is $\mathbf{a}$, and the position vectors of the other points are labelled similarly.

(i) Given that $|MQ| = \mu|QB|$, and that $|NQ| = \nu|QA|$, where μ and ν are positive and $\mu\nu < 1$, show that

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\,\mathbf{a}\,.$$

(ii) The point L lies on the side OB, and $|OL| = \lambda|OB|$. Given that ML is parallel to AN, express λ in terms of μ and ν.

What is the geometrical significance of the condition $\mu\nu < 1$?

8 **(i)** Use the substitution $v = \sqrt{y}$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{1}{2}} - \beta y \qquad (y \geqslant 0,\ \ t \geqslant 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_1(x)$ that satisfies $y_1(0) = 0\,.$

(ii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{2}{3}} - \beta y \qquad (y \geqslant 0,\ \ t \geqslant 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_2(x)$ that satisfies $y_2(0) = 0\,.$

(iii) In the case $\alpha = \beta$, sketch $y_1(x)$ and $y_2(x)$ on the same axes, indicating clearly which is $y_1(x)$ and which is $y_2(x)$. You should explain how you determined the positions of the curves relative to each other.

Section B: Mechanics

9 Two small beads, A and B, of the same mass, are threaded onto a vertical wire on which they slide without friction, and which is fixed to the ground at P. They are released simultaneously from rest, A from a height of $8h$ above P and B from a height of $17h$ above P.

(i) When A reaches the ground for the first time, it is moving with speed V. It then rebounds with coefficient of restitution $\frac{1}{2}$ and subsequently collides with B at height H above P.

Show that $H = \frac{15}{8}h$ and find, in terms of g and h, the speeds u_A and u_B of the two beads just before the collision.

(ii) When A reaches the ground for the second time, it is again moving with speed V. Determine the coefficient of restitution between the two beads.

10 **(i)** A uniform elastic string lies on a smooth horizontal table. One end of the string is attached to a fixed peg, and the other end is pulled at constant speed u. At time $t = 0$, the string is taut and its length is a. Obtain an expression for the speed, at time t, of the point on the string which is a distance x from the peg at time t.

(ii) An ant walks along the string starting at $t = 0$ at the peg. The ant walks at constant speed v along the string (so that its speed relative to the peg is the sum of v and the speed of the point on the string beneath the ant). At time t, the ant is a distance x from the peg. Write down a first order differential equation for x, and verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}\,.$$

(iii) Show that the time T taken for the ant to reach the end of the string is given by

$$uT = a(\mathrm{e}^k - 1)\,,$$

where $k = u/v$.

(iv) On reaching the end of the string, the ant turns round and walks back to the peg. Find in terms of T and k the time taken for the journey back.

11 The axles of the wheels of a motorbike of mass m are a distance b apart. Its centre of mass is a horizontal distance of d from the front axle, where $d < b$, and a vertical distance h above the road, which is horizontal and straight. The engine is connected to the rear wheel. The coefficient of friction between the ground and the rear wheel is μ, where $\mu < b/h$, and the front wheel is smooth.

(i) You may assume that the sum of the moments of the forces acting on the motorbike about the centre of mass is zero. By taking moments about the centre of mass show that, as the acceleration of the motorbike increases from zero, the rear wheel will slip before the front wheel loses contact with the road if

$$\mu < \frac{b-d}{h}\,. \qquad (*)$$

(ii) If the inequality $(*)$ holds and the rear wheel does not slip, show that the maximum acceleration is

$$\frac{\mu d g}{b - \mu h}\,.$$

(iii) If the inequality $(*)$ does not hold, find the maximum acceleration given that the front wheel remains in contact with the road.

Section C: Probability and Statistics

12 In a game, I toss a coin repeatedly. The probability, p, that the coin shows Heads on any given toss is given by

$$p = \frac{N}{N+1},$$

where N is a positive integer. The outcomes of any two tosses are independent.

The game has two versions. In each version, I can choose to stop playing after any number of tosses, in which case I win £H, where H is the number of Heads I have tossed. However, the game may end before that, in which case I win nothing.

(i) In version 1, the game ends when the coin first shows Tails (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Find, in terms of h and p, an expression for my expected winnings and show that I can maximise my expected winnings by choosing $h = N$.

(ii) In version 2, the game ends when the coin shows Tails on two *consecutive* tosses (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Show that my expected winnings are

$$\frac{hN^h(N+2)^h}{(N+1)^{2h}}.$$

In the case $N = 2$, use the approximation $\log_3 2 \approx 0.63$ to show that the maximum value of my expected winnings is approximately £3.

13 Four children, A, B, C and D, are playing a version of the game 'pass the parcel'. They stand in a circle, so that $ABCDA$ is the clockwise order. Each time a whistle is blown, the child holding the parcel is supposed to pass the parcel immediately exactly one place clockwise. In fact each child, independently of any other past event, passes the parcel clockwise with probability $\frac{1}{4}$, passes it anticlockwise with probability $\frac{1}{4}$ and fails to pass it at all with probability $\frac{1}{2}$. At the start of the game, child A is holding the parcel.

The probability that child A is holding the parcel just after the whistle has been blown for the nth time is A_n, and B_n, C_n and D_n are defined similarly.

(i) Find A_1, B_1, C_1 and D_1. Find also A_2, B_2, C_2 and D_2.

(ii) By first considering $B_{n+1} + D_{n+1}$, or otherwise, find B_n and D_n.

Find also expressions for A_n and C_n in terms of n.

Section A: Pure Mathematics

1 **Note:** In this question you may use without proof the result $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\arctan x\big) = \dfrac{1}{1+x^2}$.

Let

$$I_n = \int_0^1 x^n \arctan x \,\mathrm{d}x \ ,$$

where $n = 0$, 1, 2, 3, $\ldots$.

(i) Show that, for $n \geqslant 0$,

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x$$

and evaluate I_0.

(ii) Find an expression, in terms of n, for $(n+3)I_{n+2} + (n+1)I_n$.

Use this result to evaluate I_4.

(iii) Prove by induction that, for $n \geqslant 1$,

$$(4n+1)I_{4n} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r\frac{1}{r}\,,$$

where A is a constant to be determined.

2 The sequence of numbers x_0, x_1, x_2, $\ldots$ satisfies

$$x_{n+1} = \frac{ax_n - 1}{x_n + b}\,.$$

(You may assume that a, b and x_0 are such that $x_n + b \neq 0$.)

Find an expression for x_{n+2} in terms of a, b and x_n.

(i) Show that $a + b = 0$ is a necessary condition for the sequence to be periodic with period 2.

Note: The sequence is said to be periodic with period k if $x_{n+k} = x_n$ for all n, and there is no integer m with $0 < m < k$ such that $x_{n+m} = x_n$ for all n.

(ii) Find necessary and sufficient conditions for the sequence to have period 4.

3 **(i)** Sketch, on x-y axes, the set of all points satisfying $\sin y = \sin x$, for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$. You should give the equations of all the lines on your sketch.

(ii) Given that

$$\sin y = \tfrac{1}{2}\sin x$$

obtain an expression, in terms of x, for y' when $0 \leqslant x \leqslant \frac{1}{2}\pi$ and $0 \leqslant y \leqslant \frac{1}{2}\pi$, and show that

$$y'' = -\frac{3\sin x}{(4-\sin^2 x)^{\frac{3}{2}}} \,.$$

Use these results to sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and $0 \leqslant y \leqslant \frac{1}{2}\pi$.

Hence sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$.

(iii) Without further calculation, sketch the set of all points satisfying $\cos y = \frac{1}{2}\sin x$ for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$.

4 The Schwarz inequality is

$$\left(\int_a^b \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \leqslant \left(\int_a^b \big(\mathrm{f}(x)\big)^2 \mathrm{d}x\right)\left(\int_a^b \big(\mathrm{g}(x)\big)^2 \mathrm{d}x\right). \qquad (*)$$

(i) By setting $\mathrm{f}(x) = 1$ in $(*)$, and choosing $\mathrm{g}(x)$, a and b suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \leqslant \frac{t}{2}\,.$$

(ii) By setting $\mathrm{f}(x) = x$ in $(*)$, and choosing $\mathrm{g}(x)$ suitably, show that

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x \geqslant 12\big(1-\mathrm{e}^{-\frac{1}{4}}\big)^2\,.$$

(iii) Use $(*)$ to show that

$$\frac{64}{25\pi} \leqslant \int_0^{\frac{1}{2}\pi}\sqrt{\sin x}\,\mathrm{d}x \leqslant \sqrt{\frac{\pi}{2}}\,.$$

5 A curve C is determined by the parametric equations

$$x = at^2\,,\; y = 2at\,,$$

where $a > 0$.

(i) Show that the normal to C at a point P, with non-zero parameter p, meets C again at a point N, with parameter n, where

$$n = -\left(p + \frac{2}{p}\right).$$

(ii) Show that the distance $|PN|$ is given by

$$|PN|^2 = 16a^2\frac{(p^2+1)^3}{p^4}$$

and that this is minimised when $p^2 = 2$.

(iii) The point Q, with parameter q, is the point at which the circle with diameter PN cuts C again. By considering the gradients of QP and QN, show that

$$2 = p^2 - q^2 + \frac{2q}{p}\,.$$

Deduce that $|PN|$ is at its minimum when Q is at the origin.

6 Let

$$S_n = \sum_{r=1}^{n} \frac{1}{\sqrt{r}}\,,$$

where n is a positive integer.

(i) Prove by induction that

$$S_n \leqslant 2\sqrt{n} - 1\,.$$

(ii) Show that $(4k+1)\sqrt{k+1} > (4k+3)\sqrt{k}$ for $k \geqslant 0$.

Determine the smallest number C such that

$$S_n \geqslant 2\sqrt{n} + \frac{1}{2\sqrt{n}} - C\,.$$

7 The functions f and g are defined, for $x > 0$, by

$$\mathrm{f}(x) = x^x\,, \qquad \mathrm{g}(x) = x^{\mathrm{f}(x)}\,.$$

(i) By taking logarithms, or otherwise, show that $\mathrm{f}(x) > x$ for $0 < x < 1$. Show further that $x < \mathrm{g}(x) < \mathrm{f}(x)$ for $0 < x < 1$.

Write down the corresponding results for $x > 1$.

(ii) Find the value of x for which $\mathrm{f}'(x) = 0$.

(iii) Use the result $x \ln x \to 0$ as $x \to 0$ to find $\lim\limits_{x\to 0} \mathrm{f}(x)$, and write down $\lim\limits_{x\to 0} \mathrm{g}(x)$.

(iv) Show that $x^{-1} + \ln x \geqslant 1$ for $x > 0$.

Using this result, or otherwise, show that $\mathrm{g}'(x) > 0$.

Sketch the graphs, for $x > 0$, of $y = x$, $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ on the same axes.

8 All vectors in this question lie in the same plane.

The vertices of the non-right-angled triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to BC and CA, respectively.

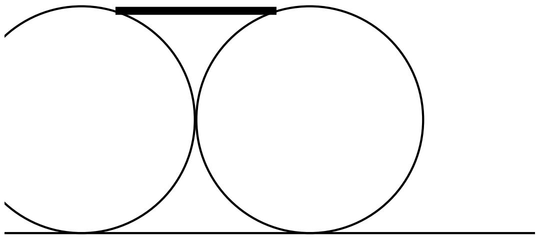
(i) Write down the vector equation of the line through A perpendicular to BC, in terms of $\mathbf{u}$, $\mathbf{a}$ and a parameter λ.

(ii) The line through A perpendicular to BC intersects the line through B perpendicular to CA at P. Find the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{u}$.

(iii) Hence show that the line CP is perpendicular to the line AB.

Section B: Mechanics

9 Two identical rough cylinders of radius r and weight W rest, not touching each other but a negligible distance apart, on a horizontal floor. A thin flat rough plank of width $2a$, where $a < r$, and weight kW rests symmetrically and horizontally on the cylinders, with its length parallel to the axes of the cylinders and its faces horizontal. A vertical cross-section is shown in the diagram below.



The coefficient of friction at all four contacts is $\frac{1}{2}$. The system is in equilibrium.

(i) Let F be the frictional force between one cylinder and the floor, and let R be the normal reaction between the plank and one cylinder. Show that

$$R\sin\theta = F(1+\cos\theta)\,,$$

where θ is the acute angle between the plank and the tangent to the cylinder at the point of contact.

Deduce that $2\sin\theta \leqslant 1+\cos\theta\,.$

(ii) Show that

$$N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F\,,$$

where N is the normal reaction between the floor and one cylinder.

Write down the condition that the cylinder does not slip on the floor and show that it is satisfied with no extra restrictions on θ.

(iii) Show that $\sin\theta \leqslant \frac{4}{5}$ and hence that $r \leqslant 5a\,.$

10 A car of mass m makes a journey of distance $2d$ in a straight line. It experiences air resistance and rolling resistance so that the total resistance to motion when it is moving with speed v is $Av^2 + R$, where A and R are constants.

The car starts from rest and moves with constant acceleration a for a distance d. Show that the work done by the engine for this half of the journey is

$$\int_0^d (ma + R + Av^2)\,\mathrm{d}x$$

and that it can be written in the form

$$\int_0^w \frac{(ma + R + Av^2)v}{a}\,\mathrm{d}v\,,$$

where $w = \sqrt{2ad}$.

For the second half of the journey, the acceleration of the car is $-a$.

(i) In the case $R > ma$, show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd\,.$$

(ii) In the case $ma - 2Aad < R < ma$, show that at a certain speed the driving force required to maintain the constant acceleration falls to zero.

Thereafter, the engine does no work (and the driver applies the brakes to maintain the constant acceleration). Show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd + \frac{(ma - R)^2}{4Aa}\,.$$

11 Two thin vertical parallel walls, each of height $2a$, stand a distance a apart on horizontal ground. The projectiles in this question move in a plane perpendicular to the walls.

(i) A particle is projected with speed $\sqrt{5ag}$ towards the two walls from a point A at ground level. It just clears the first wall. By considering the energy of the particle, find its speed when it passes over the first wall.

Given that it just clears the second wall, show that the angle its trajectory makes with the horizontal when it passes over the first wall is 45° .

Find the distance of A from the foot of the first wall.

(ii) A second particle is projected with speed $\sqrt{5ag}$ from a point B at ground level towards the two walls. It passes a distance h above the first wall, where $h > 0$. Show that it does not clear the second wall.

Section C: Probability and Statistics

12 Adam and Eve are catching fish. The number of fish, X, that Adam catches in any time interval is Poisson distributed with parameter λt, where λ is a constant and t is the length of the time interval. The number of fish, Y, that Eve catches in any time interval is Poisson distributed with parameter μt, where μ is a constant and t is the length of the time interval

The two Poisson variables are independent. You may assume that that expected time between Adam catching a fish and Adam catching his next fish is λ^{-1}, and similarly for Eve.

(i) By considering $\mathrm{P}(X+Y=r)$, show that the total number of fish caught by Adam and Eve in time T also has a Poisson distribution.

(ii) Given that Adam and Eve catch a total of k fish in time T, where k is fixed, show that the number caught by Adam has a binomial distribution.

(iii) Given that Adam and Eve start fishing at the same time, find the probability that the first fish is caught by Adam.

(iv) Find the expected time from the moment Adam and Eve start fishing until they have each caught at least one fish.

[**Note** This question has been redrafted to make the meaning clearer.]

13 In a television game show, a contestant has to open a door using a key. The contestant is given a bag containing n keys, where $n \geqslant 2$. Only one key in the bag will open the door. There are three versions of the game. In each version, the contestant starts by choosing a key at random from the bag.

(i) In version 1, after each failed attempt at opening the door the key that has been tried is put back into the bag and the contestant again selects a key at random from the bag. By considering the binomial expansion of $(1-q)^{-2}$, or otherwise, find the expected number of attempts required to open the door.

(ii) In version 2, after each failed attempt at opening the door the key that has been tried is put aside and the contestant selects another key at random from the bag. Find the expected number of attempts required to open the door.

(iii) In version 3, after each failed attempt at opening the door the key that has been tried is put back into the bag and another incorrect key is added to the bag. The contestant then selects a key at random from the bag. Show that the probability that the contestant draws the correct key at the kth attempt is

$$\frac{n-1}{(n+k-1)(n+k-2)} \,.$$

Show also, using partial fractions, that the expected number of attempts required to open the door is infinite.

You may use without proof the result that $\displaystyle\sum_{m=1}^{N} \frac{1}{m} \to \infty$ as $N \to \infty$.

Section A: Pure Mathematics

1 The curve C_1 has parametric equations $x = t^2$, $y = t^3$, where $-\infty < t < \infty$. Let O denote the point $(0, 0)$. The points P and Q on C_1 are such that $\angle POQ$ is a right angle.

(i) Show that the tangents to C_1 at P and Q intersect on the curve C_2 with equation $4y^2 = 3x - 1$.

(ii) Determine whether C_1 and C_2 meet, and sketch the two curves on the same axes.

2 Use the factor theorem to show that $a + b - c$ is a factor of

$$(a + b + c)^3 - 6(a + b + c)(a^2 + b^2 + c^2) + 8(a^3 + b^3 + c^3)\,. \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x + 1)^3 - 3(x + 1)(2x^2 + 5) + 2(4x^3 + 13) = 0\,.$$

(ii) By setting $d + e = c$, or otherwise, show that $(a + b - d - e)$ is a factor of

$$(a + b + d + e)^3 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x + 6)^3 - 6(x + 6)(x^2 + 14) + 8(x^3 + 36) = 0\,.$$

3 For each non-negative integer n, the polynomial f_n is defined by

$$\mathrm{f}_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\,.$$

(i) Show that $\mathrm{f}_n'(x) = \mathrm{f}_{n-1}(x)$ (for $n \geqslant 1$).

(ii) Show that, if a is a real root of the equation

$$\mathrm{f}_n(x) = 0\,, \qquad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geqslant 2$. Show that $\mathrm{f}_n'(a)\,\mathrm{f}_n'(b) > 0$ and use a sketch to deduce that $\mathrm{f}_n(c) = 0$ for some number c between a and b.

Deduce that $(*\ )$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?

4 Let

$$y = \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}\,.$$

(i) Given that x is real, show that

$$(y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2\,.$$

Deduce that

$$y^2 + 1 \geqslant 4(y-1)^2\,,$$

and hence that

$$\frac{4-\sqrt{7}}{3} \leqslant y \leqslant \frac{4+\sqrt{7}}{3}\,.$$

(ii) In the case $y = \dfrac{4+\sqrt{7}}{3}$, show that

$$\sqrt{y^2+1} = 2(y-1)$$

and find the corresponding values of x and $\tan\theta$.

5 In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!} & \text{if } p \geqslant q \geqslant 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n}.$$

(ii) Show that, for any positive integers m, n and r with $r \leqslant m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}.$$

(iii) Show that, for any positive integers m and N,

$$\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}.$$

6 This question concerns solutions of the differential equation

$$(1-x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + k^2y^2 = k^2 \qquad (*)$$

where k is a positive integer.

For each value of k, let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k.

(i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.

(ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2\big(y_n(x)\big)^2 - 1$. Show that

$$(1-x^2)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + 4n^2z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

(iv) Let $v(x) = y_n\big(y_m(x)\big)$. Show that $v(x) = y_{mn}(x)$.

7 Show that

$$\int_0^a \mathrm{f}(x)\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\mathrm{d}x\,, \qquad (*)$$

where f is any function for which the integrals exist.

(i) Use $(*)$ to evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x\,.$$

(ii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,\mathrm{d}x\,.$$

(iv) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x\,(\cos x + \sin x)}\,\mathrm{d}x\,.$$

8 Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad (m > \tfrac{1}{2})\,.$$

Show by means of a sketch that

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x\,, \qquad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E. Use $(*)$ to obtain the following approximations for E:

$$E \approx 2\,; \qquad E \approx \frac{5}{3}\,; \qquad E \approx \frac{33}{20}\,.$$

(ii) Show that, when r is large, the error in approximating $\dfrac{1}{r^2}$ by $\displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x$ is approximately $\dfrac{1}{4r^4}$.

Given that $E \approx 1.645$, show that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08\,.$

Section B: Mechanics

9 A small bullet of mass m is fired into a block of wood of mass M which is at rest. The speed of the bullet on entering the block is u. Its trajectory within the block is a horizontal straight line and the resistance to the bullet's motion is R, which is constant.

(i) The block is fixed. The bullet travels a distance a inside the block before coming to rest. Find an expression for a in terms of m, u and R.

(ii) Instead, the block is free to move on a smooth horizontal table. The bullet travels a distance b inside the block before coming to rest relative to the block, at which time the block has moved a distance c on the table. Find expressions for b and c in terms of M, m and a.

10 A thin uniform wire is bent into the shape of an isosceles triangle ABC, where AB and AC are of equal length and the angle at A is 2θ. The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ. The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geqslant 2\tan\theta(1+\sin\theta)\,.$$

11 **(i)** Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a + ut\cos\alpha\,,\ ut\sin\alpha)$ and $(vt\cos\beta\,,\ b + vt\sin\beta)$, where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geqslant 0$.

Given that the two particles collide, show that

$$u\sin(\theta + \alpha) = v\sin(\theta + \beta)\,,$$

where θ is the acute angle satisfying $\tan\theta = \dfrac{b}{a}$.

(ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg\,.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Section C: Probability and Statistics

12 Starting with the result $\mathrm{P}(A \cup B) = \mathrm{P}(A) + P(B) - \mathrm{P}(A \cap B)$, prove that

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C)\,.$$

Write down, without proof, the corresponding result for four events A, B, C and D.

A pack of n cards, numbered $1, 2, \ldots, n$, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the ith position in the row. Write down the following probabilities:

(i) $\mathrm{P}(E_i)\,;$

(ii) $\mathrm{P}(E_i \cap E_j)$, where $i \neq j$;

(iii) $\mathrm{P}(E_i \cap E_j \cap E_k)$, where $i \neq j$, $j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}\,.$$

Find the probability that exactly one card is in the same position as the number it bears.

13 **(i)** The random variable X has a binomial distribution with parameters n and p, where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}$, that

$$\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}\,.$$

(ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}\,.$$

(iii) By considering a Poisson distribution with parameter n, show that

$$n! \approx \sqrt{2\pi n}\,\mathrm{e}^{-n}\,n^n\,.$$

Section A: Pure Mathematics

1 **(i)** By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)\,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2\,.$$

2 **(i)** In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

(ii) The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

(iii) The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

3 Three rods have lengths a, b and c, where $a < b < c$. The three rods can be made into a triangle (possibly of zero area) if $a + b \geqslant c$.

(i) Let T_n be the number of triangles that can be made with three rods chosen from n rods of lengths $1, 2, 3, \ldots, n$ (where $n \geqslant 3$).

(ii) Show that $T_8 - T_7 = 2+4+6$ and evaluate $T_8 - T_6$. Write down expressions for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

(iii) Prove by induction that $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$, and find the corresponding result for an odd number of rods.

4 **(i)** The continuous function f is defined by

$$\tan \mathrm{f}(x) = x \qquad (-\infty < x < \infty)$$

and $\mathrm{f}(0) = \pi$. Sketch the curve $y = \mathrm{f}(x)$.

(ii) The continuous function g is defined by

$$\tan \mathrm{g}(x) = \frac{x}{1+x^2} \qquad (-\infty < x < \infty)$$

and $\mathrm{g}(0) = \pi$. Sketch the curves $y = \dfrac{x}{1+x^2}$ and $y = \mathrm{g}(x)$.

(iii) The continuous function h is defined by $\mathrm{h}(0) = \pi$ and

$$\tan \mathrm{h}(x) = \frac{x}{1-x^2} \qquad (x \neq \pm 1).$$

(The values of $\mathrm{h}(x)$ at $x = \pm 1$ are such that $\mathrm{h}(x)$ is continuous at these points.) Sketch the curves $y = \dfrac{x}{1-x^2}$ and $y = \mathrm{h}(x)$.

5 In this question, the arctan function satisfies $0 \leqslant \arctan x < \frac{1}{2}\pi$ for $x \geqslant 0$.

(i) Let

$$S_n = \sum_{m=1}^{n} \arctan\left(\frac{1}{2m^2}\right),$$

for $n = 1, 2, 3, \ldots$. Prove by induction that

$$\tan S_n = \frac{n}{n+1}.$$

Prove also that

$$S_n = \arctan \frac{n}{n+1}.$$

(ii) In a triangle ABC, the lengths of the sides AB and BC are $4n^2$ and $4n^4 - 1$, respectively, and the angle at B is a right angle. Let $\angle BCA = 2\alpha_n$. Show that

$$\sum_{n=1}^{\infty} \alpha_n = \tfrac{1}{4}\pi.$$

6 **(i)** Show that

$$\sec^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \frac{2}{1+\sin x}\,.$$

Hence integrate $\dfrac{1}{1+\sin x}$ with respect to x.

(ii) By means of the substitution $y = \pi - x$, show that

$$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x,$$

where f is any function for which these integrals exist. Hence evaluate

$$\int_0^\pi \frac{x}{1+\sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x.$$

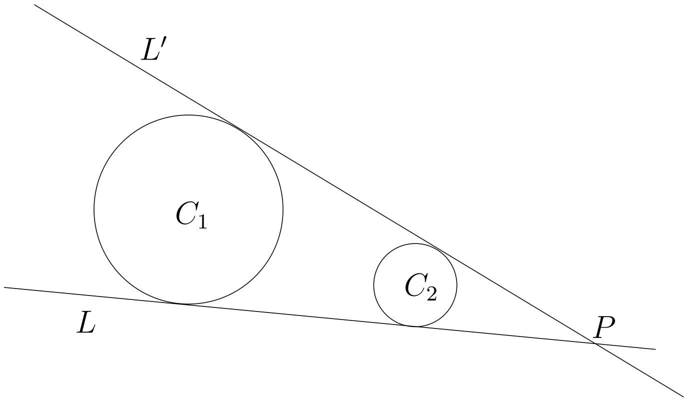
7 A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

(i) Let C be the circle of radius a in the x-y plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \leqslant a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\dfrac{a_2^2 - a_1^2}{4d}$. Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \geqslant \left(4d^2 + (a_2 - a_1)^2\right)\left(4d^2 + (a_2 + a_1)^2\right).$$

8



The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

Section B: Mechanics

9 An equilateral triangle ABC is made of three light rods each of length a. It is free to rotate in a vertical plane about a horizontal axis through A. Particles of mass $3m$ and $5m$ are attached to B and C respectively. Initially, the system hangs in equilibrium with BC below A.

(i) Show that, initially, the angle θ that BC makes with the horizontal is given by $\sin\theta = \frac{1}{7}$.

(ii) The triangle receives an impulse that imparts a speed v to the particle B. Find the minimum speed v_0 such that the system will perform complete rotations if $v > v_0$.

10 A particle of mass m is pulled along the floor of a room in a straight line by a light string which is pulled at constant speed V through a hole in the ceiling. The floor is smooth and horizontal, and the height of the room is h.

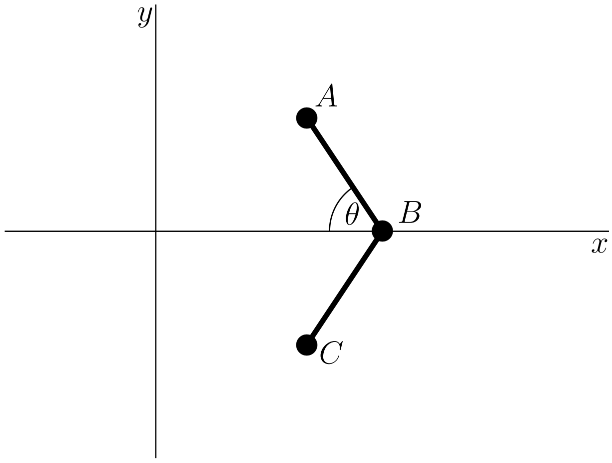
(i) Find, in terms of V and θ, the speed of the particle when the string makes an angle of θ with the vertical (and the particle is still in contact with the floor).

(ii) Find also the acceleration, in terms of V, h and θ.

(iii) Find the tension in the string and hence show that the particle will leave the floor when

$$\tan^4\theta = \frac{V^2}{gh}\,.$$

11 Three particles, A, B and C, each of mass m, lie on a smooth horizontal table. Particles A and C are attached to the two ends of a light inextensible string of length $2a$ and particle B is attached to the midpoint of the string. Initially, A, B and C are at rest at points $(0, a)$, $(0, 0)$ and $(0, -a)$, respectively. An impulse is delivered to B, imparting to it a speed u in the positive x direction. The string remains taut throughout the subsequent motion.



(i) At time t, the angle between the x-axis and the string joining A and B is θ, as shown in the diagram, and B is at $(x, 0)$. Write down the coordinates of A in terms of x, a and θ. Given that the velocity of B is $(v, 0)$, show that the velocity of A is $(\dot{x} + a\sin\theta\,\dot{\theta}\,,\ a\cos\theta\,\dot{\theta})$, where the dot denotes differentiation with respect to time.

(ii) Show that, before particles A and C first collide,

$$3\dot{x} + 2a\dot{\theta}\sin\theta = v \qquad \text{and} \qquad \dot{\theta}^2 = \frac{v^2}{a^2(3 - 2\sin^2\theta)}\,.$$

(iii) When A and C collide, the collision is elastic (no energy is lost). At what value of θ does the second collision between particles A and C occur? (You should justify your answer.)

(iv) When $v = 0$, what are the possible values of θ? Is $v = 0$ whenever θ takes these values?

Section C: Probability and Statistics

12 Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows: Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT. The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT? Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$. Find the probability that C wins.

13 The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{for } x \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant. It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geqslant 0$. If $X \leqslant y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$\mathrm{E}(C) = ky + \frac{a}{\lambda}\mathrm{e}^{-\lambda y}\,.$$

How should y be chosen in order to minimise $\mathrm{E}(C)$, in the different cases that arise according to the value of a/k?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.

Section A: Pure Mathematics

1 In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leqslant \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$\big(1 + 2\cos(\alpha+\beta)\big)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0\,. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leqslant \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45^\circ$ and (b) $\alpha = 30^\circ$, $\beta = 90^\circ$, and illustrate each case with a sketch.

2 This question concerns the inequality

$$\int_0^\pi \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x \leqslant \int_0^\pi \big(\mathrm{f}'(x)\big)^2\,\mathrm{d}x\,. \qquad (*)$$

(i) Show that $(*)$ is satisfied in the case $\mathrm{f}(x) = \sin nx$, where n is a positive integer. Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $\mathrm{f}(0) \neq 0$ or $\mathrm{f}(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $\mathrm{f}(0) = \mathrm{f}(\pi) = 0$. By setting $\mathrm{f}(x) = ax^2 + bx + c$, where a, b and c are suitably chosen, show that $\pi^2 \leqslant 10$. By setting $\mathrm{f}(x) = p\sin\frac{1}{2}x + q\cos\frac{1}{2}x + r$, where p, q and r are suitably chosen, obtain another inequality for π. Which of these inequalities leads to a better estimate for π^2?

3 **(i)** Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geqslant 0$, is $c(m^2+1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C. Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2\big(1 + (y')^2\big)\,. \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1+m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

4 **(i)** By using the substitution $u = 1/x$, show that for $b > 0$

$$\int_{1/b}^{b} \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)}\,\mathrm{d}x = 0\,.$$

(ii) By using the substitution $u = 1/x$, show that for $b > 0$,

$$\int_{1/b}^{b} \frac{\arctan x}{x}\,\mathrm{d}x = \frac{\pi \ln b}{2}\,.$$

(iii) By using the result $\displaystyle\int_0^\infty \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = k/x$, for suitable k, show that

$$\int_0^\infty \frac{1}{(a^2 + x^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3} \qquad (a > 0).$$

5 Given that $y = xu$, where u is a function of x, write down an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$. Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

6 By simplifying $\sin(r + \frac{1}{2})x - \sin(r - \frac{1}{2})x$ or otherwise show that, for $\sin \frac{1}{2}x \neq 0$,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}.$$

The functions S_n, for $n = 1, 2, \ldots$, are defined by

$$\mathrm{S}_n(x) = \sum_{r=1}^{n} \frac{1}{r} \sin rx (0 \leqslant x \leqslant \pi).$$

(i) Find the stationary points of $\mathrm{S}_2(x)$ for $0 \leqslant x \leqslant \pi$, and sketch this function.

(ii) Show that if $\mathrm{S}_n(x)$ has a stationary point at $x = x_0$, where $0 < x_0 < \pi$, then

$$\sin nx_0 = (1 - \cos nx_0) \tan \tfrac{1}{2}x_0$$

and hence that $\mathrm{S}_n(x_0) \geqslant \mathrm{S}_{n-1}(x_0)$. Deduce that if $\mathrm{S}_{n-1}(x) > 0$ for all x in the interval $0 < x < \pi$, then $\mathrm{S}_n(x) > 0$ for all x in this interval.

(iii) Prove that $\mathrm{S}_n(x) \geqslant 0$ for $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$.

7 **(i)** The function f is defined by $\mathrm{f}(x) = |x-a| + |x-b|$, where $a < b$. Sketch the graph of $\mathrm{f}(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $\mathrm{g}(x)$, where $\mathrm{g}(x) = |2x - a - b|$. What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $(b, \mathrm{f}(b))$ and $(a, \mathrm{f}(a))$?

(ii) Show graphically that the equation

$$|x-a| + |x-b| = |x-c|\,,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x-a| + |x-b| = |x-c| + |x-d|\,,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

8 For positive integers n, a and b, the integer c_r $(0 \leqslant r \leqslant n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r, n, a and b. For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \geqslant c_r$ for $0 \leqslant r \leqslant n$). Show that

$$\frac{b(n+1)}{a+b} - 1 \leqslant m \leqslant \frac{b(n+1)}{a+b}\,.$$

Deduce that m is either a unique integer or one of two consecutive integers. Let $\mathrm{G}(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

(i) Evaluate $\mathrm{G}(9, 1, 3)$ and $\mathrm{G}(9, 2, 3)$.

(ii) For any positive integer k, find $\mathrm{G}(2k, a, a)$ and $\mathrm{G}(2k-1, a, a)$ in terms of k.

(iii) For fixed n and b, determine a value of a for which $\mathrm{G}(n, a, b)$ is greatest.

(iv) For fixed n, find the greatest possible value of $\mathrm{G}(n, 1, b)$. For which values of b is this greatest value achieved?

Section B: Mechanics

9 A uniform rectangular lamina $ABCD$ rests in equilibrium in a vertical plane with the corner A in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side AB at a distance d from A. The other end of the string is attached to a point on the wall above A where it makes an acute angle θ with the downwards vertical. The side AB makes an acute angle ϕ with the upwards vertical at A. The sides BC and AB have lengths $2a$ and $2b$ respectively. The coefficient of friction between the lamina and the wall is μ.

(i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d\sin(\theta+\phi) = (\cos\theta + \mu\sin\theta)(a\cos\phi + b\sin\phi)\,. \qquad (*)$$

(ii) How should $(*)$ be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

(iii) Find a condition on d, in terms of a, b, $\tan\theta$ and $\tan\phi$, which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if $b(2\tan\theta + \tan\phi) < a$.

10 A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O.

(i) Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

(ii) Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g.

(iii) Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u.

(iv) Hence find the furthest distance from O that can be achieved by such a projectile.

11 A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O, of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion. Let x be the distance between O and the ring. Taking the y-axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x, L and α.

(i) By considering the vertical component of the equation of motion of P, show that

$$km\ddot{x}\cos\alpha = T\cos\alpha - kmg\,,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R.

(ii) Show that $\dfrac{\sin\alpha}{(1-\sin\alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k.

(iii) Show that $\ddot{x} = -g\tan\alpha\,.$

Section C: Probability and Statistics

12 The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $\mathrm{f}(t)$ and cumulative distribution function $\mathrm{F}(t)$. The *hazard function*, $\mathrm{h}(t)$, is defined, for $\mathrm{F}(t) < 1$, by

$$\mathrm{h}(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)}\,.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $\mathrm{h}(t)\,\delta t$ for small values of δt.

(ii) Find the hazard function in the case $\mathrm{F}(t) = t/a$ for $0 < t < a$. Sketch $\mathrm{f}(t)$ and $\mathrm{h}(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $\mathrm{h}(t)$ is constant for $t > b$ and zero otherwise if and only if $\mathrm{f}(t) = k\mathrm{e}^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$\mathrm{h}(t) = \left(\frac{\lambda}{\theta^{\lambda}}\right) t^{\lambda - 1}\,,$$

where λ and θ are positive constants. Find the probability density function for T.

13 A random number generator prints out a sequence of integers I_1, I_2, I_3, $\ldots$. Each integer is independently equally likely to be any one of 1, 2, $\ldots$, n, where n is fixed. The random variable X takes the value r, where I_r is the first integer which is a repeat of some earlier integer. Write down an expression for $\mathrm{P}(X = 4)$.

(i) Find an expression for $\mathrm{P}(X = r)$, where $2 \leqslant r \leqslant n + 1$. Hence show that, for any positive integer n,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots \ = \ 1\,.$$

(ii) Write down an expression for $\mathrm{E}(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $\mathrm{P}(X \geqslant k)$.

(iv) Show that, for any discrete random variable Y taking the values 1, 2, $\ldots$, N,

$$\mathrm{E}(Y) = \sum_{k=1}^{N} \mathrm{P}(Y \geqslant k)\,.$$

Hence show that, for any positive integer n,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right) + \cdots \ = \ 0.$$

Section A: Pure Mathematics

1 **(i)** Find the value of m for which the line $y = mx$ touches the curve $y = \ln x$. If instead the line intersects the curve when $x = a$ and $x = b$, where $a < b$, show that $a^b = b^a$. Show by means of a sketch that $a < \mathrm{e} < b$.

(ii) The line $y = mx + c$, where $c > 0$, intersects the curve $y = \ln x$ when $x = p$ and $x = q$, where $p < q$. Show by means of a sketch, or otherwise, that $p^q > q^p$.

(iii) Show by means of a sketch that the straight line through the points $(p, \ln p)$ and $(q, \ln q)$, where $\mathrm{e} \leqslant p < q$, intersects the y-axis at a positive value of y. Which is greater, π^{e} or e^{π}?

(iv) Show, using a sketch or otherwise, that if $0 < p < q$ and $\dfrac{\ln q - \ln p}{q - p} = \mathrm{e}^{-1}$, then $q^p > p^q$.

2 For $n \geqslant 0$, let

$$I_n = \int_0^1 x^n (1 - x)^n \mathrm{d}x .$$

(i) For $n \geqslant 1$, show by means of a substitution that

$$\int_0^1 x^{n-1}(1 - x)^n \mathrm{d}x = \int_0^1 x^n (1 - x)^{n-1} \mathrm{d}x$$

and deduce that

$$2 \int_0^1 x^{n-1}(1 - x)^n \mathrm{d}x = I_{n-1} .$$

Show also, for $n \geqslant 1$, that

$$I_n = \frac{n}{n+1} \int_0^1 x^{n-1}(1 - x)^{n+1} \mathrm{d}x$$

and hence that $I_n = \dfrac{n}{2(2n+1)} I_{n-1}$.

(ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!} .$$

(iii) Use the substitution $x = \sin^2\theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.

3 **(i)** Given that the cubic equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and $c < 0$, show with the help of sketches that either exactly one of the roots is positive or all three of the roots are positive.

(ii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real positive roots show that

$$a^2 > b > 0, \quad a < 0, \quad c < 0\,. \qquad (*)$$

[**Hint**: Consider the turning points.]

(iii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and that

$$ab < 0, \quad c > 0\,,$$

determine, with the help of sketches, the signs of the roots.

(iv) Show by means of an explicit example (giving values for a, b and c) that it is possible for the conditions $(*)$ to be satisfied even though the corresponding cubic equation has only one real root.

4 The line passing through the point $(a, 0)$ with gradient b intersects the circle of unit radius centred at the origin at P and Q, and M is the midpoint of the chord PQ. Find the coordinates of M in terms of a and b.

(i) Suppose b is fixed and positive. As a varies, M traces out a curve (the *locus* of M). Show that $x = -by$ on this curve. Given that a varies with $-1 \leqslant a \leqslant 1$, show that the locus is a line segment of length $2b/(1+b^2)^{\frac{1}{2}}$. Give a sketch showing the locus and the unit circle.

(ii) Find the locus of M in the following cases, giving in each case its cartesian equation, describing it geometrically and sketching it in relation to the unit circle:

(**a**) a is fixed with $0 < a < 1$, and b varies with $-\infty < b < \infty$;

(**b**) $ab = 1$, and b varies with $0 < b \leqslant 1$.

5 **(i)** A function $\mathrm{f}(x)$ satisfies $\mathrm{f}(x) = \mathrm{f}(1-x)$ for all x. Show, by differentiating with respect to x, that $\mathrm{f}'(\frac{1}{2}) = 0$. If, in addition, $\mathrm{f}(x) = \mathrm{f}(\frac{1}{x})$ for all (non-zero) x, show that $\mathrm{f}'(-1) = 0$ and that $\mathrm{f}'(2) = 0$.

(ii) The function f is defined, for $x \neq 0$ and $x \neq 1$, by

$$\mathrm{f}(x) = \frac{(x^2 - x + 1)^3}{(x^2 - x)^2}\,.$$

Show that $\mathrm{f}(x) = \mathrm{f}(\frac{1}{x})$ and $\mathrm{f}(x) = \mathrm{f}(1-x)$. Given that it has exactly three stationary points, sketch the curve $y = \mathrm{f}(x)$.

(iii) Hence, or otherwise, find all the roots of the equation $\mathrm{f}(x) = \dfrac{27}{4}$ and state the ranges of values of x for which $\mathrm{f}(x) > \dfrac{27}{4}$. Find also all the roots of the equation $\mathrm{f}(x) = \dfrac{343}{36}$ and state the ranges of values of x for which $\mathrm{f}(x) > \dfrac{343}{36}$.

6 In this question, the following theorem may be used.
Let u_1, u_2, ... be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leqslant b$ for all n, where b is some fixed number) and increasing (that is, $u_n \geqslant u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges). The sequence u_1, u_2, ... is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \qquad (n \geqslant 1)\,. \qquad (*)$$

(i) Show that, for $n \geqslant 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leqslant u_n \leqslant 2$ for all n.

(iii) Show that the sequence u_1, u_3, u_5, ... tends to a limit, and that the sequence u_2, u_4, u_6, ... tends to a limit. Find these limits and deduce that the sequence u_1, u_2, u_3, ... tends to a limit. Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?

7 **(i)** Write down a solution of the equation

$$x^2 - 2y^2 = 1\,, \qquad (*)$$

for which x and y are non-negative integers. Show that, if $x = p$, $y = q$ is a solution of $(*)$, then so also is $x = 3p + 4q$, $y = 2p + 3q$. Hence find two solutions of $(*)$ for which x is a positive odd integer and y is a positive even integer.

(ii) Show that, if x is an odd integer and y is an even integer, $(*)$ can be written in the form

$$n^2 = \tfrac{1}{2}m(m+1)\,,$$

where m and n are integers.

(iii) The positive integers a, b and c satisfy

$$b^3 = c^4 - a^2\,,$$

where b is a prime number. Express a and c^2 in terms of b in the two cases that arise. Find a solution of $a^2 + b^3 = c^4$, where a, b and c are positive integers but b is not prime.

8 The function f satisfies $\mathrm{f}(x) > 0$ for $x \geqslant 0$ and is strictly decreasing (which means that $\mathrm{f}(b) < \mathrm{f}(a)$ for $b > a$).

(i) For $t \geqslant 0$, let $A_0(t)$ be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve $y = \mathrm{f}(x)$, the y-axis and the line $y = \mathrm{f}(t)$. Show that $A_0(t)$ can be written in the form

$$A_0(t) = x_0\left(\mathrm{f}(x_0) - \mathrm{f}(t)\right),$$

where x_0 satisfies $x_0\mathrm{f}'(x_0) + \mathrm{f}(x_0) = \mathrm{f}(t)$.

(ii) The function g is defined, for $t > 0$, by

$$\mathrm{g}(t) = \frac{1}{t}\int_0^t \mathrm{f}(x)\mathrm{d}x\,.$$

Show that $t\mathrm{g}'(t) = \mathrm{f}(t) - \mathrm{g}(t)$. Making use of a sketch show that, for $t > 0$,

$$\int_0^t \left(\mathrm{f}(x) - \mathrm{f}(t)\right)\mathrm{d}x > A_0(t)$$

and deduce that $-t^2\mathrm{g}'(t) > A_0(t)$.

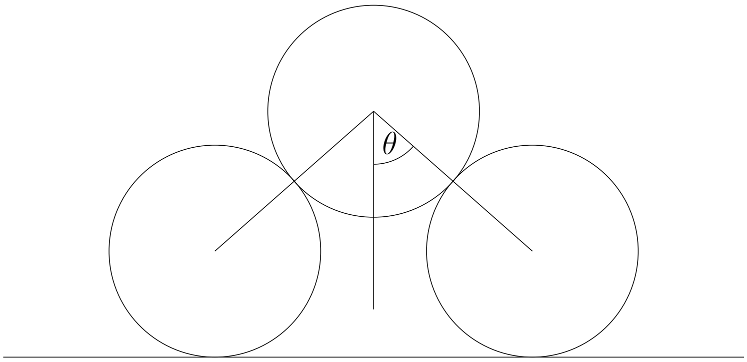
(iii) In the case $\mathrm{f}(x) = \dfrac{1}{1+x}$, use the above to establish the inequality

$$\ln\sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}}\,,$$

for $t > 0$.

Section B: Mechanics

9 The diagram shows three identical discs in equilibrium in a vertical plane. Two discs rest, not in contact with each other, on a horizontal surface and the third disc rests on the other two. The angle at the upper vertex of the triangle joining the centres of the discs is 2θ.



The weight of each disc is W. The coefficient of friction between a disc and the horizontal surface is μ and the coefficient of friction between the discs is also μ.

(i) Show that the normal reaction between the horizontal surface and a disc in contact with the surface is $\frac{3}{2}W$.

(ii) Find the normal reaction between two discs in contact and show that the magnitude of the frictional force between two discs in contact is $\dfrac{W\sin\theta}{2(1+\cos\theta)}$.

(iii) Show that if $\mu < 2-\sqrt{3}$ there is no value of θ for which equilibrium is possible.

10 A particle is projected at an angle of elevation α (where $\alpha > 0$) from a point A on horizontal ground. At a general point in its trajectory the angle of elevation of the particle from A is θ and its direction of motion is at an angle ϕ above the horizontal (with $\phi \geqslant 0$ for the first half of the trajectory and $\phi \leqslant 0$ for the second half). Let B denote the point on the trajectory at which $\theta = \frac{1}{2}\alpha$ and let C denote the point on the trajectory at which $\phi = -\frac{1}{2}\alpha$.

(i) Show that, at a general point on the trajectory, $2\tan\theta = \tan\alpha + \tan\phi$.

(ii) Show that, if B and C are the same point, then $\alpha = 60°$.

(iii) Given that $\alpha < 60°$, determine whether the particle reaches the point B first or the point C first.

11 Three identical particles lie, not touching one another, in a straight line on a smooth horizontal surface. One particle is projected with speed u directly towards the other two which are at rest. The coefficient of restitution in all collisions is e, where $0 < e < 1$.

(i) Show that, after the second collision, the speeds of the particles are $\frac{1}{2}u(1-e)$, $\frac{1}{4}u(1-e^2)$ and $\frac{1}{4}u(1+e)^2$. Deduce that there will be a third collision whatever the value of e.

(ii) Show that there will be a fourth collision if and only if e is less than a particular value which you should determine.

Section C: Probability and Statistics

12 The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1,\ 3,\ 5,\ 7,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2,\ 4,\ 6,\ 8,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots .$$

(ii) Show that

$$\mathrm{Var}\,(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\mathrm{Var}\,(Y)$. Are there any non-zero values of λ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X+Y)$?

13 A biased coin has probability p of showing a head and probability q of showing a tail, where $p \neq 0$, $q \neq 0$ and $p \neq q$. When the coin is tossed repeatedly, runs occur. A *straight run* of length n is a sequence of n consecutive heads or n consecutive tails. An *alternating run* of length n is a sequence of length n alternating between heads and tails. An alternating run can start with either a head or a tail. Let S be the length of the longest straight run beginning with the first toss and let A be the length of the longest alternating run beginning with the first toss.

(i) Explain why $\mathrm{P}(A=1) = p^2 + q^2$ and find $\mathrm{P}(S=1)$. Show that $\mathrm{P}(S=1) < \mathrm{P}(A=1)$.

(ii) Show that $\mathrm{P}(S=2) = \mathrm{P}(A=2)$ and determine the relationship between $\mathrm{P}(S=3)$ and $\mathrm{P}(A=3)$.

(iii) Show that, for $n > 1$, $\mathrm{P}(S=2n) > \mathrm{P}(A=2n)$ and determine the corresponding relationship between $\mathrm{P}(S=2n+1)$ and $\mathrm{P}(A=2n+1)$. [You are advised *not* to use $p+q=1$ in this part.]

Section A: Pure Mathematics

1 Write down the general term in the expansion in powers of x of $(1-x^6)^{-2}$.

(i) Find the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}.$$

Obtain also, and simplify, formulae for the coefficient of x^n in the different cases that arise.

(ii) Show that the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}(1-x)^{-1}$$

is 55, and find the coefficients of x^{25} and x^{66}.

2 If $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are polynomials of degree m and n, respectively, what is the degree of $\mathrm{p}(\mathrm{q}(x))$?

(i) The polynomial $\mathrm{p}(x)$ satisfies

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) - 3\mathrm{p}(x) = -2x$$

for all x. Explain carefully why $\mathrm{p}(x)$ must be of degree 1, and find all polynomials that satisfy this equation.

(ii) Find all polynomials that satisfy

$$2\mathrm{p}(\mathrm{p}(x)) + 3[\mathrm{p}(x)]^2 - 4\mathrm{p}(x) = x^4$$

for all x.

3 **(i)** Show that, for any function f (for which the integrals exist),

$$\int_0^{\infty} \mathrm{f}\big(x+\sqrt{1+x^2}\big)\,\mathrm{d}x = \frac{1}{2}\int_1^{\infty}\left(1+\frac{1}{t^2}\right)\mathrm{f}(t)\,\mathrm{d}t\,.$$

(ii) Hence evaluate

$$\int_0^{\infty} \frac{1}{2x^2+1+2x\sqrt{x^2+1}}\,\mathrm{d}x\,,$$

and, using the substitution $x=\tan\theta$,

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta\,.$$

4 In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x|<1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}\,.$$

Hence show that $\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}\,.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$. Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\,.$$

(iii) Use parts (i) and (ii) to show that as $n\to\infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}\,.$$

5 **(i)** Sketch the curve $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{(x-a)^2 - 1} \qquad (x \neq a \pm 1),$$

and a is a constant.

(ii) The function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = \frac{1}{\big((x-a)^2 - 1\big)\big((x-b)^2 - 1\big)} \qquad (x \neq a \pm 1,\ x \neq b \pm 1),$$

where a and b are constants, and $b > a$. Sketch the curves $y = \mathrm{g}(x)$ in the two cases $b > a + 2$ and $b = a + 2$, finding the values of x at the stationary points.

6 A cyclic quadrilateral $ABCD$ has sides AB, BC, CD and DA of lengths a, b, c and d, respectively. The area of the quadrilateral is Q, and angle DAB is θ.

(i) Find an expression for $\cos\theta$ in terms of a, b, c and d, and an expression for $\sin\theta$ in terms of a, b, c, d and Q.

(ii) Hence show that

$$16Q^2 = 4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2\,,$$

and deduce that

$$Q^2 = (s-a)(s-b)(s-c)(s-d)\,,$$

where $s = \frac{1}{2}(a + b + c + d)$.

(iii) Deduce a formula for the area of a triangle with sides of length a, b and c.

7 Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly.

(i) Given that $\overrightarrow{GY_1} = -\lambda_1 \overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1} = \tfrac{1}{3}\big((1 - 2\lambda_1)\mathbf{x}_1 + (1 + \lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\big)$$

and hence that

$$\lambda_1 = \frac{3 - \alpha - \beta - \gamma}{3 + \alpha - 2\beta - 2\gamma}\,,$$

where $\alpha = \mathbf{x}_2 \,.\, \mathbf{x}_3$, $\beta = \mathbf{x}_3 \,.\, \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \,.\, \mathbf{x}_2$.

(ii) Deduce that $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = 3\,.$

8 **(i)** The positive numbers α, β and q satisfy $\beta - \alpha > q$. Show that

$$\frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0\,.$$

(ii) The sequence u_0, u_1, ... is defined by $u_0 = \alpha$, $u_1 = \beta$ and

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \qquad (n \geqslant 1),$$

where α, β and q are given positive numbers (and α and β are such that no term in the sequence is zero). Prove that $u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$. Prove also that

$$u_{n+1} - pu_n + u_{n-1} = 0$$

for some number p which you should express in terms of α, β and q.

(iii) Hence, or otherwise, show that if $\beta > \alpha + q$, the sequence is strictly increasing (that is, $u_{n+1} - u_n > 0$ for all n). Comment on the case $\beta = \alpha + q$.

Section B: Mechanics

9 **(i)** A tennis ball is projected from a height of $2h$ above horizontal ground with speed u and at an angle of α below the horizontal. It travels in a plane perpendicular to a vertical net of height h which is a horizontal distance of a from the point of projection. Given that the ball passes over the net, show that

$$\frac{1}{u^2} < \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}\,.$$

(ii) The ball lands before it has travelled a horizontal distance of b from the point of projection. Show that

$$\sqrt{u^2\sin^2\alpha + 4gh}\; < \frac{bg}{u\cos\alpha} + u\sin\alpha\,.$$

(iii) Hence show that

$$\tan\alpha < \frac{h(b^2 - 2a^2)}{ab(b-a)}\,.$$

10 A hollow circular cylinder of internal radius r is held fixed with its axis horizontal. A uniform rod of length $2a$ (where $a < r$) rests in equilibrium inside the cylinder inclined at an angle of θ to the horizontal, where $\theta \neq 0$. The vertical plane containing the rod is perpendicular to the axis of the cylinder. The coefficient of friction between the cylinder and each end of the rod is μ, where $\mu > 0$.

(i) Show that, if the rod is on the point of slipping, then the normal reactions R_1 and R_2 of the lower and higher ends of the rod, respectively, on the cylinder are related by

$$\mu(R_1 + R_2) = (R_1 - R_2)\tan\phi$$

where ϕ is the angle between the rod and the radius to an end of the rod.

(ii) Show further that

$$\tan\theta = \frac{\mu r^2}{r^2 - a^2(1+\mu^2)}\,.$$

(iii) Deduce that $\lambda < \phi$, where $\tan\lambda = \mu$.

11 A small block of mass km is initially at rest on a smooth horizontal surface. Particles P_1, P_2, P_3, $\ldots$ are fired, in order, along the surface from a fixed point towards the block. The mass of the ith particle is im ($i = 1, 2, \ldots$) and the speed at which it is fired is u/i. Each particle that collides with the block is embedded in it.

(i) Show that, if the nth particle collides with the block, the speed of the block after the collision is

$$\frac{2nu}{2k + n(n+1)}\,.$$

In the case $2k = N(N+1)$, where N is a positive integer, determine the number of collisions that occur.

(ii) Show that the total kinetic energy lost in all the collisions is

$$\tfrac{1}{2}mu^2\left(\sum_{n=2}^{N+1}\frac{1}{n}\right).$$

Section C: Probability and Statistics

12 A modern villa has complicated lighting controls. In order for the light in the swimming pool to be on, a particular switch in the hallway must be on and a particular switch in the kitchen must be on. There are four identical switches in the hallway and four identical switches in the kitchen. Guests cannot tell whether the switches are on or off, or what they control. Each Monday morning a guest arrives, and the switches in the hallway are either all on or all off. The probability that they are all on is p and the probability that they are all off is $1-p$. The switches in the kitchen are each on or off, independently, with probability $\frac{1}{2}$.

(i) On the first Monday, a guest presses one switch in the hallway at random and one switch in the kitchen at random. Find the probability that the swimming pool light is on at the end of this process. Show that the probability that the guest has pressed the swimming pool light switch in the hallway, given that the light is on at the end of the process, is $\dfrac{1-p}{1+2p}$.

(ii) On each of seven Mondays, guests go through the above process independently of each other, and each time the swimming pool light is found to be on at the end of the process. Given that the most likely number of days on which the swimming pool light switch in the hallway was pressed is 3, show that $\frac{1}{4} < p < \frac{5}{14}$.

13 In this question, you may assume that $\displaystyle\int_0^\infty \mathrm{e}^{-x^2/2}\mathrm{d}x = \sqrt{\tfrac{1}{2}\pi}$.

(i) The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y.

(ii) The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $\mathrm{P}(Y < y)$ and hence show that the probability density function of Y is $2\pi yk\mathrm{e}^{-\pi ky^2}$ for $y \geqslant 0$.

(iii) Find $\mathrm{E}(Y)$ and show that $\mathrm{Var}\,(Y) = \dfrac{4-\pi}{4\pi k}$.

Section A: Pure Mathematics

1 **(i)** Sketch the curve $y = \sqrt{1-x} + \sqrt{3+x}$. Use your sketch to show that only one real value of x satisfies

$$\sqrt{1-x} + \sqrt{3+x} = x + 1\,,$$

and give this value.

(ii) Determine graphically the number of real values of x that satisfy

$$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}\ .$$

Solve this equation.

2 Write down the cubes of the integers 1, 2, . . . , 10 . The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3\,, \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2\ .$$

Deduce that $(4z^3 - k^2)/3$ is a perfect square and that $\frac{1}{4}k^2 \leqslant z^3 < k^2$. Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

3 In this question, you may assume without proof that any function f for which $\mathrm{f}'(x) \geqslant 0$ is *increasing*; that is, $\mathrm{f}(x_2) \geqslant \mathrm{f}(x_1)$ if $x_2 \geqslant x_1$.

(i) **(a)** Let $\mathrm{f}(x) = \sin x - x\cos x$. Show that $\mathrm{f}(x)$ is increasing for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and deduce that $\mathrm{f}(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(b) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \geqslant 1$ for $0 \leqslant x < 1$, show that

$$\arcsin x \geqslant x \qquad (0 \leqslant x < 1).$$

(c) Let $\mathrm{g}(x) = x\operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)\, x^{-1} \geqslant x\operatorname{cosec} x \qquad (0 < x < 1).$$

(ii) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arctan x) \leqslant 1$ for $x \geqslant 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geqslant x^2 \qquad (0 < x < \tfrac{1}{2}\pi).$$

4 **(i)** Find all the values of θ, in the range $0^\circ < \theta < 180^\circ$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18^\circ = \frac{1}{4}\left(\sqrt{5} - 1\right).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x\,,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0^\circ < \alpha < 90^\circ$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha\,.$$

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A.

(i) Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \dfrac{2\,\mathbf{a.b}}{\mathbf{a.a}}$.

(ii) The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

(iii) Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

6 For any given function f, let

$$I = \int [\mathrm{f}'(x)]^2\,[\mathrm{f}(x)]^n \mathrm{d}x\,, \qquad (*)$$

where n is a positive integer. Show that, if $\mathrm{f}(x)$ satisfies $\mathrm{f}''(x) = k\mathrm{f}(x)\mathrm{f}'(x)$ for some constant k, then $(*)$ can be integrated to obtain an expression for I in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, k and n.

(i) Verify your result in the case $\mathrm{f}(x) = \tan x$. Hence find

$$\int \frac{\sin^4 x}{\cos^8 x}\,\mathrm{d}x\,.$$

(ii) Find

$$\int \sec^2 x\,(\sec x + \tan x)^6\,\mathrm{d}x\,.$$

7 The two sequences a_0, a_1, a_2, ... and b_0, b_1, b_2, ... have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n\,,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\displaystyle\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}}\, a_{n+1}$, and give a corresponding result for $\displaystyle\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \tfrac{1}{2} b_{n+1}^2\,,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2\,,$$

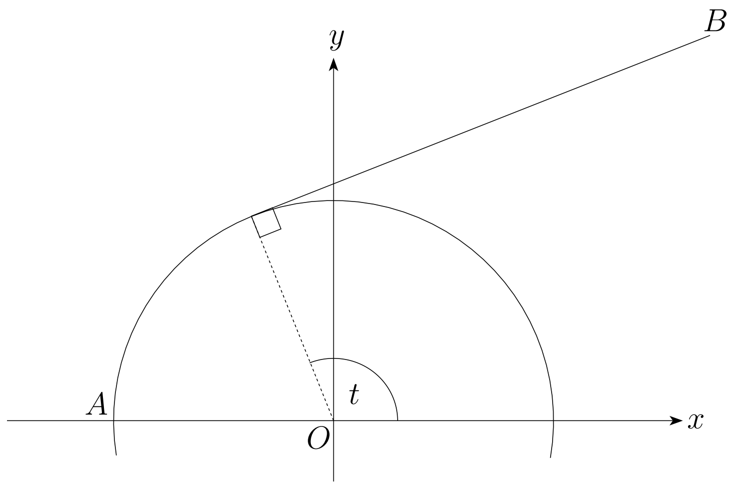
and give a corresponding result when n is odd.

8 The end A of an inextensible string AB of length π is attached to a point on the circumference of a fixed circle of unit radius and centre O. Initially the string is straight and tangent to the circle. The string is then wrapped round the circle until the end B comes into contact with the circle. The string remains taut during the motion, so that a section of the string is in contact with the circumference and the remaining section is straight.

(i) Taking O to be the origin of cartesian coordinates with A at $(-1, 0)$ and B initially at $(-1, \pi)$, show that the curve described by B is given parametrically by

$$x = \cos t + t \sin t\,, \qquad y = \sin t - t \cos t\,,$$

where t is the angle shown in the diagram.



Find the value, t_0, of t for which x takes its maximum value on the curve, and sketch the curve.

(ii) Use the area integral $\displaystyle\int y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ to find the area between the curve and the x axis for $\pi \geqslant t \geqslant t_0$.

(iii) Find the area swept out by the string (that is, the area between the curve described by B and the semicircle shown in the diagram).

Section B: Mechanics

9 Two particles, A of mass $2m$ and B of mass m, are moving towards each other in a straight line on a smooth horizontal plane, with speeds $2u$ and u respectively. They collide directly.

(i) Given that the coefficient of restitution between the particles is e, where $0 < e \leqslant 1$, determine the speeds of the particles after the collision.

(ii) After the collision, B collides directly with a smooth vertical wall, rebounding and then colliding directly with A for a second time. The coefficient of restitution between B and the wall is f, where $0 < f \leqslant 1$. Show that the velocity of B after its second collision with A is

$$\tfrac{2}{3}(1-e^2)u - \tfrac{1}{3}(1-4e^2)fu$$

towards the wall and that B moves towards (not away from) the wall for all values of e and f.

10 **(i)** A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ.

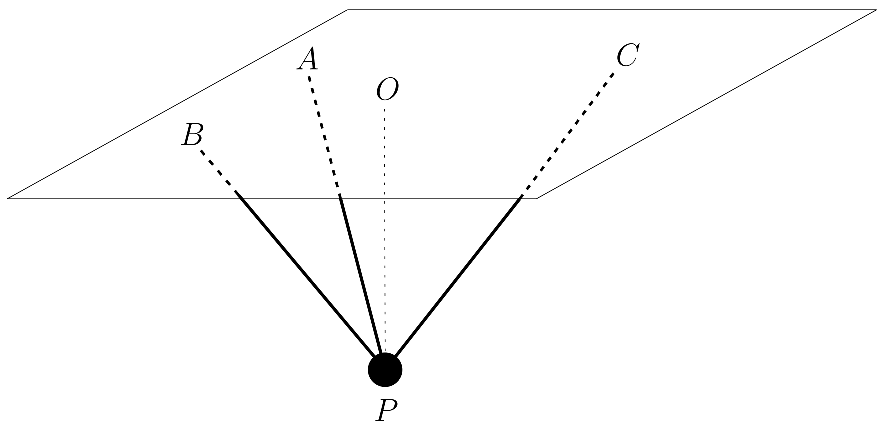
(ii) A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\frac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

$$(\sqrt{2}-1)\sqrt{H/g}\,.$$

(iii) When the string becomes taut, R leaves the plane, the string remaining taut.

(iv) Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ. Given that $D = H$, find the value of $\tan\theta$.

11 Three non-collinear points A, B and C lie in a horizontal ceiling. A particle P of weight W is suspended from this ceiling by means of three light inextensible strings AP, BP and CP, as shown in the diagram. The point O lies vertically above P in the ceiling.



The angles AOB and AOC are $90° + \theta$ and $90° + \phi$, respectively, where θ and ϕ are acute angles such that $\tan\theta = \sqrt{2}$ and $\tan\phi = \frac{1}{4}\sqrt{2}$. The strings AP, BP and CP make angles $30°$, $90° - \theta$ and $60°$, respectively, with the vertical, and the tensions in these strings have magnitudes T, U and V respectively.

(i) Show that the unit vector in the direction PB can be written in the form

$$-\frac{1}{3}\,\mathbf{i} - \frac{\sqrt{2}}{3}\,\mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}}\,\mathbf{k}\,,$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual mutually perpendicular unit vectors with $\mathbf{j}$ parallel to OA and $\mathbf{k}$ vertically upwards.

(ii) Find expressions in vector form for the forces acting on P.

(iii) Show that $U = \sqrt{6}V$ and find T, U and V in terms of W.

Section C: Probability and Statistics

12 Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points. Xavier has probability p and Younis has probability $1-p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1-p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1-p^2}{2-p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him $£1$; if Younis wins the match, Xavier gives him $£k$. Find the value of k for which the game is 'fair' in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

13 What property of a distribution is measured by its *skewness*?

(i) One measure of skewness, γ, is given by

$$\gamma = \frac{\mathrm{E}\big((X-\mu)^3\big)}{\sigma^3}\,,$$

where μ and σ^2 are the mean and variance of the random variable X. Show that

$$\gamma = \frac{\mathrm{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}\,.$$

The continuous random variable X has probability density function f where

$$\mathrm{f}(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that for this distribution $\gamma = -\dfrac{2\sqrt{2}}{5}$.

(ii) The *decile skewness*, D, of a distribution is defined by

$$D = \frac{\mathrm{F}^{-1}(\frac{9}{10}) - 2\mathrm{F}^{-1}(\frac{1}{2}) + \mathrm{F}^{-1}(\frac{1}{10})}{\mathrm{F}^{-1}(\frac{9}{10}) - \mathrm{F}^{-1}(\frac{1}{10})}\,,$$

where F^{-1} is the inverse of the cumulative distribution function. Show that, for the above distribution, $D = 2 - \sqrt{5}$. The *Pearson skewness*, P, of a distribution is defined by

$$P = \frac{3(\mu - M)}{\sigma}\,,$$

where M is the median. Find P for the above distribution and show that $D > P > \gamma$.

Section A: Pure Mathematics

1 Let P be a given point on a given curve C. The *osculating circle* to C at P is defined to be the circle that satisfies the following two conditions at P: it touches C; and the rate of change of its gradient is equal to the rate of change of the gradient of C. Find the centre and radius of the osculating circle to the curve $y = 1 - x + \tan x$ at the point on the curve with x-coordinate $\frac{1}{4}\pi$.

2 Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x\,.$$

Find and prove a similar result for $\sin 3x$ in terms of $\sin x$.

(i) Let

$$\mathrm{I}(\alpha) = \int_0^\alpha \big(7\sin x - 8\sin^3 x\big)\mathrm{d}x\,.$$

Show that

$$\mathrm{I}(\alpha) = -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}\,,$$

where $c = \cos\alpha$. Write down one value of c for which $\mathrm{I}(\alpha) = 0$.

(ii) Useless Eustace believes that

$$\int \sin^n x\,\mathrm{d}x = \frac{\sin^{n+1} x}{n+1}$$

for $n = 1,\ 2,\ 3, \ldots$. Show that Eustace would obtain the correct value of $\mathrm{I}(\beta)$, where $\cos\beta = -\frac{1}{6}$. Find all values of α for which he would obtain the correct value of $\mathrm{I}(\alpha)$.

3 The first four terms of a sequence are given by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_3 = 2$. The general term is given by

$$F_n = a\lambda^n + b\mu^n\,, \qquad (*)$$

where a, b, λ and μ are independent of n, and a is positive.

(i) Show that $\lambda^2 + \lambda\mu + \mu^2 = 2$, and find the values of λ, μ, a and b.

(ii) Use $(*)$ to evaluate F_6.

(iii) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}}$.

4 **(i)** Let

$$I = \int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x\,.$$

Use a substitution to show that

$$I = \int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x$$

and hence evaluate I in terms of a. Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x+\frac{\pi}{4})}\,\mathrm{d}x\,.$$

(ii) Evaluate

$$\int_{\frac{1}{2}}^2 \frac{\sin x}{x\big(\sin x + \sin\frac{1}{x}\big)}\,\mathrm{d}x\,.$$

5 The points A and B have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $5\mathbf{i}-\mathbf{j}-\mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{i}+n\mathbf{j}+p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w. Hence describe the surface with Cartesian equation $x^2+y^2+z^2 = 2(yz+zx+xy)$.

6 Each edge of the tetrahedron $ABCD$ has unit length. The face ABC is horizontal, and P is the point in ABC that is vertically below D.

(i) Find the length of PD.

(ii) Show that the cosine of the angle between adjacent faces of the tetrahedron is $1/3$.

(iii) Find the radius of the largest sphere that can fit inside the tetrahedron.

7 **(i)** By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1+q)\,,$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1+q) = 0\,,$$

and that

$$x = u + q/u\,,$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp\,.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.

8 The curves C_1 and C_2 are defined by

$$y = \mathrm{e}^{-x} \quad (x > 0) \qquad \text{and} \qquad y = \mathrm{e}^{-x}\sin x \quad (x > 0),$$

respectively.

(i) Sketch roughly C_1 and C_2 on the same diagram.

(ii) Let x_n denote the x-coordinate of the nth point of contact between the two curves, where $0 < x_1 < x_2 < \cdots$, and let A_n denote the area of the region enclosed by the two curves between x_n and x_{n+1}. Show that

$$A_n = \tfrac{1}{2}(\mathrm{e}^{2\pi} - 1)\mathrm{e}^{-(4n+1)\pi/2}$$

and hence find $\displaystyle\sum_{n=1}^{\infty} A_n$.

Section B: Mechanics

9 Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h\cot\beta < b < 4h\cot\beta$$

and

$$2h\cot\alpha < a < 4h\cot\alpha\,,$$

where a is the horizontal distance from A to the point of collision.

10 **(i)** In an experiment, a particle A of mass m is at rest on a smooth horizontal table. A particle B of mass bm, where $b > 1$, is projected along the table directly towards A with speed u. The collision is perfectly elastic. Find an expression for the speed of A after the collision in terms of b and u, and show that, irrespective of the relative masses of the particles, A cannot be made to move at twice the initial speed of B.

(ii) In a second experiment, a particle B_1 is projected along the table directly towards A with speed u. This time, particles B_2, B_3, $\ldots$, B_n are at rest in order on the line between B_1 and A. The mass of B_i ($i = 1,\, 2,\, \ldots,\, n$) is $\lambda^{n+1-i}m$, where $\lambda > 1$. All collisions are perfectly elastic. Show that, by choosing n sufficiently large, there is no upper limit on the speed at which A can be made to move. In the case $\lambda = 4$, determine the least value of n for which A moves at more than $20u$. You may use the approximation $\log_{10} 2 \approx 0.30103$.

11 A uniform rod AB of length $4L$ and weight W is inclined at an angle θ to the horizontal. Its lower end A rests on a fixed support and the rod is held in equilibrium by a string attached to the rod at a point C which is $3L$ from A. The reaction of the support on the rod acts in a direction α to AC and the string is inclined at an angle β to CA.

(i) Show that

$$\cot\alpha = 3\tan\theta + 2\cot\beta\,.$$

(ii) Given that $\theta = 30^\circ$ and $\beta = 45^\circ$, show that $\alpha = 15^\circ$.

Section C: Probability and Statistics

12 The continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a-b)}\,.$$

(ii) Show that the median, M, of X is given by $M = \dfrac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

13 Rosalind wants to join the Stepney Chess Club. In order to be accepted, she must play a challenge match consisting of several games against Pardeep (the Club champion) and Quentin (the Club secretary), in which she must win at least one game against each of Pardeep and Quentin. From past experience, she knows that the probability of her winning a single game against Pardeep is p and the probability of her winning a single game against Quentin is q, where $0 < p < q < 1$.

(i) The challenge match consists of three games. Before the match begins, Rosalind must choose either to play Pardeep twice and Quentin once or to play Quentin twice and Pardeep once. Show that she should choose to play Pardeep twice.

(ii) In order to ease the entry requirements, it is decided instead that the challenge match will consist of four games. Now, before the match begins, Rosalind must choose whether to play Pardeep three times and Quentin once (strategy 1), or to play Pardeep twice and Quentin twice (strategy 2) or to play Pardeep once and Quentin three times (strategy 3). Show that, if $q - p > \frac{1}{2}$, Rosalind should choose strategy 1. If $q - p < \frac{1}{2}$, give examples of values of p and q to show that strategy 2 can be better or worse than strategy 1.

Section A: Pure Mathematics

1 **(i)** Two curves have equations $x^4 + y^4 = u$ and $xy = v$, where u and v are positive constants. State the equations of the lines of symmetry of each curve.

(ii) The curves intersect at the distinct points A, B, C and D (taken anticlockwise from A). The coordinates of A are (α, β), where $\alpha > \beta > 0$. Write down, in terms of α and β, the coordinates of B, C and D.

(iii) Show that the quadrilateral $ABCD$ is a rectangle and find its area in terms of u and v only. Verify that, for the case $u = 81$ and $v = 4$, the area is 14.

2 The curve C has equation

$$y = a^{\sin(\pi e^x)},$$

where $a > 1$.

(i) Find the coordinates of the stationary points on C.

(ii) Use the approximations $e^t \approx 1 + t$ and $\sin t \approx t$ (both valid for small values of t) to show that

$$y \approx 1 - \pi x \ln a$$

for small values of x.

(iii) Sketch C.

(iv) By approximating C by means of straight lines joining consecutive stationary points, show that the area between C and the x-axis between the kth and $(k+1)$th maxima is approximately

$$\left(\frac{a^2+1}{2a}\right) \ln\left(1 + (k - \tfrac{3}{4})^{-1}\right).$$

3 Prove that

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) \equiv \sec x - \tan x\,. \qquad (*)$$

(i) Use $(*)$ to find the value of $\tan\frac{1}{8}\pi$. Hence show that

$$\tan\tfrac{11}{24}\pi = \frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1}\,.$$

(ii) Show that

$$\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1} = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}\,.$$

(iii) Use $(*)$ to show that

$$\tan\tfrac{1}{48}\pi = \sqrt{16+10\sqrt{2}+8\sqrt{3}+6\sqrt{6}}\, - 2 - \sqrt{2} - \sqrt{3} - \sqrt{6}\,.$$

4 The polynomial $\mathrm{p}(x)$ is of degree 9 and $\mathrm{p}(x)-1$ is exactly divisible by $(x-1)^5$.

(i) Find the value of $\mathrm{p}(1)$.

(ii) Show that $\mathrm{p}'(x)$ is exactly divisible by $(x-1)^4$.

(iii) Given also that $\mathrm{p}(x)+1$ is exactly divisible by $(x+1)^5$, find $\mathrm{p}(x)$.

5 Expand and simplify $(\sqrt{x-1}+1)^2$.

(i) Evaluate

$$\int_5^{10} \frac{\sqrt{x+2\sqrt{x-1}}\,+\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}\,\mathrm{d}x\,.$$

(ii) Find the total area between the curve

$$y = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}$$

and the x-axis between the points $x=\frac{5}{4}$ and $x=10$.

(iii) Evaluate

$$\int_{\frac{5}{4}}^{10} \frac{\sqrt{x+2\sqrt{x-1}}\,+\sqrt{x-2\sqrt{x+1}+2}}{\sqrt{x^2-1}}\,\mathrm{d}x\,.$$

6 The Fibonacci sequence F_1, F_2, F_3, ... is defined by $F_1 = 1$, $F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} (n \geqslant 2).$$

Write down the values of F_3, F_4, ..., F_{10}. Let $S = \sum_{i=1}^{\infty} \frac{1}{F_i}$.

(i) Show that $\frac{1}{F_i} > \frac{1}{2F_{i-1}}$ for $i \geqslant 4$ and deduce that $S > 3$. Show also that $S < 3\frac{2}{3}$.

(ii) Show further that $3.2 < S < 3.5$.

7 Let $y = (x-a)^n \mathrm{e}^{bx} \sqrt{1+x^2}$, where n and a are constants and b is a non-zero constant. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-a)^{n-1} \mathrm{e}^{bx} \mathrm{q}(x)}{\sqrt{1+x^2}},$$

where $\mathrm{q}(x)$ is a cubic polynomial. Using this result, determine:

(i) $\displaystyle\int \frac{(x-4)^{14} \mathrm{e}^{4x} (4x^3 - 1)}{\sqrt{1+x^2}} \, \mathrm{d}x$;

(ii) $\displaystyle\int \frac{(x-1)^{21} \mathrm{e}^{12x} (12x^4 - x^2 - 11)}{\sqrt{1+x^2}} \, \mathrm{d}x$;

(iii) $\displaystyle\int \frac{(x-2)^{6} \mathrm{e}^{4x} (4x^4 + x^3 - 2)}{\sqrt{1+x^2}} \, \mathrm{d}x$.

8 The non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, given by

$$\mathbf{p} = \lambda \mathbf{a} + (1-\lambda)\mathbf{b} \quad \text{and} \quad \mathbf{q} = \mu \mathbf{a} + (1-\mu)\mathbf{c}$$

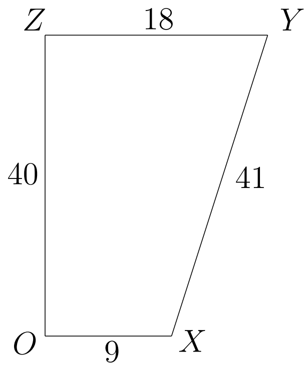
where $0 < \lambda < 1$ and $\mu > 1$.

(i) Draw a diagram showing A, B, C, P and Q.

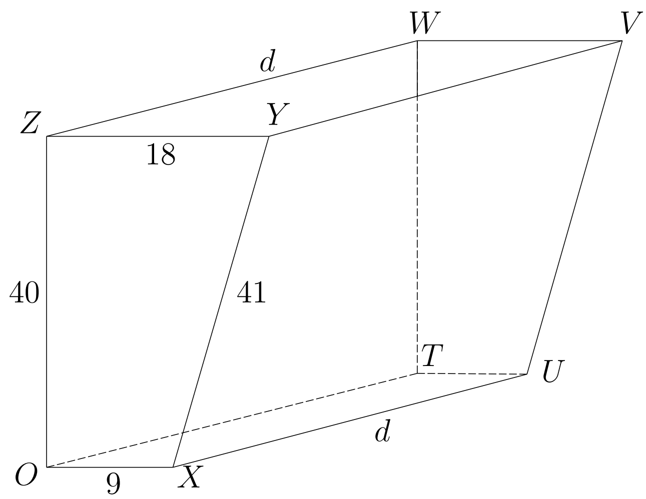
(ii) Given that $CQ \times BP = AB \times AC$, find μ in terms of λ, and show that, for all values of λ, the the line PQ passes through the fixed point D, with position vector $\mathbf{d}$ given by $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$. What can be said about the quadrilateral $ABDC$?

Section B: Mechanics

9 **(i)** A uniform lamina $OXYZ$ is in the shape of the trapezium shown in the diagram. It is right-angled at O and Z, and OX is parallel to YZ. The lengths of the sides are given by $OX = 9$ cm, $XY = 41$ cm, $YZ = 18$ cm and $ZO = 40$ cm. Show that its centre of mass is a distance 7 cm from the edge OZ.



(ii) The diagram shows a tank with no lid made of thin sheet metal. The base $OXUT$, the back $OTWZ$ and the front $XUVY$ are rectangular, and each end is a trapezium as in part **(i)** . The width of the tank is d cm.

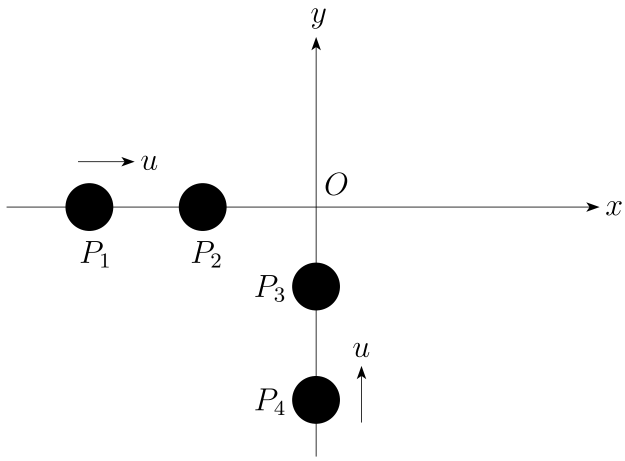


Show that the centre of mass of the tank, when empty, is a distance

$$\frac{3(140 + 11d)}{5(12 + d)} \text{ cm}$$

from the back of the tank. The tank is then filled with a liquid. The mass per unit volume of this liquid is k times the mass per unit area of the sheet metal. In the case $d = 20$, find an expression for the distance of the centre of mass of the filled tank from the back of the tank.

10



Four particles P_1, P_2, P_3 and P_4, of masses m_1, m_2, m_3 and m_4, respectively, are arranged on smooth horizontal axes as shown in the diagram. Initially, P_2 and P_3 are stationary, and both P_1 and P_4 are moving towards O with speed u. Then P_1 and P_2 collide, at the same moment as P_4 and P_3 collide. Subsequently, P_2 and P_3 collide at O, as do P_1 and P_4 some time later. The coefficient of restitution between each pair of particles is e, and $e > 0$. Show that initially P_2 and P_3 are equidistant from O.

11 A train consists of an engine and n trucks. It is travelling along a straight horizontal section of track. The mass of the engine and of each truck is M. The resistance to motion of the engine and of each truck is R, which is constant. The maximum power at which the engine can work is P. Obtain an expression for the acceleration of the train when its speed is v and the engine is working at maximum power. The train starts from rest with the engine working at maximum power. Obtain an expression for the time T taken to reach a given speed V, and show that this speed is only achievable if

$$P > (n+1)RV\,.$$

(i) In the case when $(n+1)RV/P$ is small, use the approximation $\ln(1-x) \approx -x - \frac{1}{2}x^2$ (valid for small x) to obtain the approximation

$$PT \approx \tfrac{1}{2}(n+1)MV^2$$

and interpret this result.

(ii) In the general case, the distance moved from rest in time T is X. *Write down*, with explanation, an equation relating P, T, X, M, V, R and n and hence show that

$$X = \frac{2PT - (n+1)MV^2}{2(n+1)R}\,.$$

Section C: Probability and Statistics

12 A continuous random variable X has probability density function given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x < 0 \\ k\mathrm{e}^{-2x^2} & \text{for } 0 \leqslant x < \infty\,, \end{cases}$$

where k is a constant.

(i) Sketch the graph of $\mathrm{f}(x)$.

(ii) Find the value of k.

(iii) Determine $\mathrm{E}(X)$ and $\mathrm{Var}\,(X)$.

(iv) Use statistical tables to find, to three significant figures, the median value of X.

13 Satellites are launched using two different types of rocket: the Andover and the Basingstoke. The Andover has four engines and the Basingstoke has six. Each engine has a probability p of failing during any given launch. After the launch, the rockets are retrieved and repaired by replacing some or all of the engines. The cost of replacing each engine is K.

(i) For the Andover, if more than one engine fails, all four engines are replaced. Otherwise, only the failed engine (if there is one) is replaced. Show that the expected repair cost for a single launch using the Andover is

$$4Kp(1 + q + q^2 - 2q^3) \qquad (q = 1 - p) \qquad (*)$$

(ii) For the Basingstoke, if more than two engines fail, all six engines are replaced. Otherwise only the failed engines (if there are any) are replaced. Find, in a form similar to $(*)$, the expected repair cost for a single launch using the Basingstoke. Find the values of p for which the expected repair cost for the Andover is $\frac{2}{3}$ of the expected repair cost for the Basingstoke.

Section A: Pure Mathematics

1 A sequence of points (x_1, y_1), (x_2, y_2), ... in the cartesian plane is generated by first choosing (x_1, y_1) then applying the rule, for $n = 1, 2, \ldots$,

$$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + a,\ 2x_n y_n + b + 2),$$

where a and b are given real constants.

(i) In the case $a = 1$ and $b = -1$, find the values of (x_1, y_1) for which the sequence is constant.

(ii) Given that $(x_1, y_1) = (-1, 1)$, find the values of a and b for which the sequence has period 2.

2 Let a_n be the coefficient of x^n in the series expansion, in ascending powers of x, of

$$\frac{1+x}{(1-x)^2(1+x^2)},$$

where $|x| < 1$. Show, using partial fractions, that either $a_n = n + 1$ or $a_n = n + 2$ according to the value of n.

Hence find a decimal approximation, to nine significant figures, for the fraction $\dfrac{11\,000}{8181}$.

[You are not required to justify the accuracy of your approximation.]

3 **(i)** Find the coordinates of the turning points of the curve $y = 27x^3 - 27x^2 + 4$. Sketch the curve and deduce that $x^2(1 - x) \leqslant 4/27$ for all $x \geqslant 0$.

Given that each of the numbers a, b and c lies between 0 and 1, prove by contradiction that at least one of the numbers $bc(1-a)$, $ca(1-b)$ and $ab(1-c)$ is less than or equal to $4/27$.

(ii) Given that each of the numbers p and q lies between 0 and 1, prove that at least one of the numbers $p(1-q)$ and $q(1-p)$ is less than or equal to $1/4$.

4 A curve is given by

$$x^2 + y^2 + 2axy = 1,$$

where a is a constant satisfying $0 < a < 1$. Show that the gradient of the curve at the point P with coordinates (x, y) is

$$-\frac{x + ay}{ax + y},$$

provided $ax + y \neq 0$. Show that θ, the acute angle between OP and the normal to the curve at P, satisfies

$$\tan\theta = a|y^2 - x^2| \, .$$

Show further that, if $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$ at P, then:

(i) $a(x^2 + y^2) + 2xy = 0\,;$

(ii) $(1 + a)(x^2 + y^2 + 2xy) = 1\,;$

(iii) $\tan\theta = \dfrac{a}{\sqrt{1 - a^2}}\,.$

5 **(i)** Evaluate the integrals

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2x}{1 + \sin^2 x}\mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \frac{\sin x}{1 + \sin^2 x}\mathrm{d}x\,.$$

(ii) Show, using the binomial expansion, that $(1+\sqrt{2}\,)^5 < 99$. Show also that $\sqrt{2} > 1.4$. Deduce that $2^{\sqrt{2}} > 1 + \sqrt{2}$. Use this result to determine which of the above integrals is greater.

6 A curve has the equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of $\mathrm{f}(x)$.

(ii) Determine all values of x in the interval $-\pi \leqslant x \leqslant \pi$ for which $\mathrm{f}(x) = 0$. Find a value of x in this interval at which the curve touches the x-axis without crossing it.

(iii) Find the value or values of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $\mathrm{f}(x) = 2\,.$

7 **(i)** By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1}y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

8 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1-\lambda) : \lambda\,.$$

(i) Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.

(ii) Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

(iii) The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b-a)^2\,.$$

Section B: Mechanics

9 In this question, use $g = 10\,\text{m s}^{-2}$.

In cricket, a fast bowler projects a ball at $40\,\text{m s}^{-1}$ from a point h m above the ground, which is horizontal, and at an angle α above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of $20\,\text{m}$ from the point of projection.

(i) Determine, in terms of h, the two possible values of $\tan\alpha$.

Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.

(ii) State the range of values of h for which the bowler projects the ball below the horizontal.

(iii) In the case $h = 2.5$, give an approximate value in degrees, correct to two significant figures, for α. You need not justify the accuracy of your approximation.

[You may use the small-angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.]

10 The lengths of the sides of a rectangular billiards table $ABCD$ are given by $AB = DC = a$ and $AD = BC = 2b$. There are small pockets at the midpoints M and N of the sides AD and BC, respectively. The sides of the table may be taken as smooth vertical walls.

A small ball is projected along the table from the corner A. It strikes the side BC at X, then the side DC at Y and then goes directly into the pocket at M. The angles BAX, CXY and DYM are α, β and γ respectively. On each stage of its path, the ball moves with constant speed in a straight line, the speeds being u, v and w respectively. The coefficient of restitution between the ball and the sides is e, where $e > 0$.

(i) Show that $\tan\alpha\tan\beta = e$ and find γ in terms of α.

(ii) Show that $\tan\alpha = \dfrac{(1+2e)b}{(1+e)a}$ and deduce that the shot is possible whatever the value of e.

(iii) Find an expression in terms of e for the fraction of the kinetic energy of the ball that is lost during the motion.

11 A wedge of mass km has the shape (in cross-section) of a right-angled triangle. It stands on a smooth horizontal surface with one face vertical. The inclined face makes an angle θ with the horizontal surface. A particle P, of mass m, is placed on the inclined face and released from rest. The horizontal face of the wedge is smooth, but the inclined face is rough and the coefficient of friction between P and this face is μ.

(i) When P is released, it slides down the inclined plane at an acceleration a relative to the wedge. Show that the acceleration of the wedge is

$$\frac{a\cos\theta}{k+1}\,.$$

To a stationary observer, P appears to descend along a straight line inclined at an angle 45° to the horizontal. Show that

$$\tan\theta = \frac{k}{k+1}\,.$$

In the case $k = 3$, find an expression for a in terms of g and μ.

(ii) What happens when P is released if $\tan\theta \leqslant \mu$?

Section C: Probability and Statistics

12 In the High Court of Farnia, the outcome of each case is determined by three judges: the ass, the beaver and the centaur. Each judge decides its verdict independently. Being simple creatures, they make their decisions entirely at random. Past verdicts show that the ass gives a guilty verdict with probability p, the beaver gives a guilty verdict with probability $p/3$ and the centaur gives a guilty verdict with probability p^2.

(i) Let X be the number of guilty verdicts given by the three judges in a case. Given that $\mathrm{E}(X) = 4/3$, find the value of p.

(ii) The probability that a defendant brought to trial is guilty is t. The King pronounces that the defendant is guilty if at least two of the judges give a guilty verdict; otherwise, he pronounces the defendant not guilty. Find the value of t such that the probability that the King pronounces correctly is $1/2$.

13 Bag P and bag Q each contain n counters, where $n \geqslant 2$. The counters are identical in shape and size, but coloured either black or white. First, k counters ($0 \leqslant k \leqslant n$) are drawn at random from bag P and placed in bag Q. Then, k counters are drawn at random from bag Q and placed in bag P.

(i) If initially $n-1$ counters in bag P are white and one is black, and all n counters in bag Q are white, find the probability in terms of n and k that the black counter ends up in bag P.

Find the value or values of k for which this probability is maximised.

(ii) If initially $n-1$ counters in bag P are white and one is black, and $n-1$ counters in bag Q are white and one is black, find the probability in terms of n and k that the black counters end up in the same bag.

Find the value or values of k for which this probability is maximised.

Section A: Pure Mathematics

1 *In this question, you are not required to justify the accuracy of the approximations.*

(i) Write down the binomial expansion of $\left(1+\frac{k}{100}\right)^{\frac{1}{2}}$ in ascending powers of k, up to and including the k^3 term.

(a) Use the value $k=8$ to find an approximation to five decimal places for $\sqrt{3}$.

(b) By choosing a suitable integer value of k, find an approximation to five decimal places for $\sqrt{6}$.

(i) By considering the first two terms of the binomial expansion of $\left(1+\frac{k}{1000}\right)^{\frac{1}{3}}$, show that $\frac{3029}{2100}$ is an approximation to $\sqrt[3]{3}$.

2 A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q-p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q-p)^4$.

3 By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}$.

(i) Let $I = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x$.

(a) Evaluate I.

(b) Use the substitution $t = \tan\frac{1}{2}x$ to show that $\displaystyle\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t = \tfrac{1}{2}I$.

(ii) Evaluate $\displaystyle\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,\mathrm{d}t$.

4 Given that $\cos A$, $\cos B$ and β are non-zero, show that the equation

$$\alpha\sin(A-B) + \beta\cos(A+B) = \gamma\sin(A+B)$$

reduces to the form

$$(\tan A - m)(\tan B - n) = 0\,,$$

where m and n are independent of A and B, if and only if $\alpha^2 = \beta^2 + \gamma^2$.

Determine all values of x, in the range $0 \leqslant x < 2\pi$, for which:

(i) $2\sin(x - \frac{1}{4}\pi) + \sqrt{3}\cos(x + \frac{1}{4}\pi) = \sin(x + \frac{1}{4}\pi)$;

(ii) $2\sin(x - \frac{1}{6}\pi) + \sqrt{3}\cos(x + \frac{1}{6}\pi) = \sin(x + \frac{1}{6}\pi)$;

(iii) $2\sin(x + \frac{1}{3}\pi) + \sqrt{3}\cos(3x) = \sin(3x)$.

5 In this question, $\mathrm{f}^2(x)$ denotes $\mathrm{f}(\mathrm{f}(x))$, $\mathrm{f}^3(x)$ denotes $\mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$, and so on.

(i) The function f is defined, for $x \neq \pm 1/\sqrt{3}$, by

$$\mathrm{f}(x) = \frac{x+\sqrt{3}}{1-\sqrt{3}\,x}\,.$$

Find by direct calculation $\mathrm{f}^2(x)$ and $\mathrm{f}^3(x)$, and determine $\mathrm{f}^{2007}(x)$.

(ii) Show that $\mathrm{f}^n(x) = \tan(\theta + \frac{1}{3}n\pi)$, where $x = \tan\theta$ and n is any positive integer.

(iii) The function $\mathrm{g}(t)$ is defined, for $|t| \leqslant 1$ by $\mathrm{g}(t) = \frac{\sqrt{3}}{2}t + \frac{1}{2}\sqrt{1-t^2}$. Find an expression for $\mathrm{g}^n(t)$ for any positive integer n.

6 **(i)** Differentiate $\ln\left(x+\sqrt{3+x^2}\right)$ and $x\sqrt{3+x^2}$ and simplify your answers.

Hence find $\int \sqrt{3+x^2}\,\mathrm{d}x$.

(ii) Find the two solutions of the differential equation

$$3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

that satisfy $y = 0$ when $x = 1$.

7 A function $\mathrm{f}(x)$ is said to be concave on some interval if $\mathrm{f}''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $\mathrm{f}(x)$ be concave on a given interval and let x_1, x_2, …, x_n lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{f}(x_k) \leqslant \mathrm{f}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$. You may use this result without proving it.

(i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}\,.$$

(ii) By choosing a suitable function f, prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \;\leqslant\; \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers t_1, t_2, …, t_n.

Hence:

(a) show that $x^4 + y^4 + z^4 + 16 \geqslant 8xyz$, where x, y and z are any positive numbers;

(b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.

8 The points B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$, respectively, relative to the origin A, and A, B and C are not collinear.

(i) The point X has position vector $s\mathbf{b} + t\mathbf{c}$. Describe the locus of X when $s + t = 1$.

(ii) The point P has position vector $\beta\mathbf{b} + \gamma\mathbf{c}$, where β and γ are non-zero, and $\beta + \gamma \neq 1$. The line AP cuts the line BC at D. Show that $BD : DC = \gamma : \beta$.

(iii) The line BP cuts the line CA at E, and the line CP cuts the line AB at F. Show that

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1\,.$$

Section B: Mechanics

9 A solid right circular cone, of mass M, has semi-vertical angle α and smooth surfaces. It stands with its base on a smooth horizontal table. A particle of mass m is projected so that it strikes the curved surface of the cone at speed u. The coefficient of restitution between the particle and the cone is e. The impact has no rotational effect on the cone and the cone has no vertical velocity after the impact.

(i) The particle strikes the cone in the direction of the normal at the point of impact. Explain why the trajectory of the particle immediately after the impact is parallel to the normal to the surface of the cone. Find an expression, in terms of M, m, α, e and u, for the speed at which the cone slides along the table immediately after impact.

(ii) If instead the particle falls vertically onto the cone, show that the speed w at which the cone slides along the table immediately after impact is given by

$$w = \frac{mu(1+e)\sin\alpha\cos\alpha}{M + m\cos^2\alpha}\,.$$

Show also that the value of α for which w is greatest is given by

$$\cos\alpha = \sqrt{\frac{M}{2M+m}}\,.$$

10 A solid figure is composed of a uniform solid cylinder of density ρ and a uniform solid hemisphere of density 3ρ. The cylinder has circular cross-section, with radius r, and height $3r$, and the hemisphere has radius r. The flat face of the hemisphere is joined to one end of the cylinder, so that their centres coincide.

The figure is held in equilibrium by a force P so that one point of its flat base is in contact with a rough horizontal plane and its base is inclined at an angle α to the horizontal. The force P is horizontal and acts through the highest point of the base. The coefficient of friction between the solid and the plane is μ. Show that

$$\mu \geqslant \left|\tfrac{9}{8} - \tfrac{1}{2}\cot\alpha\right|\,.$$

11 *In this question take the acceleration due to gravity to be $10\,\mathrm{m\,s^{-2}}$ and neglect air resistance.*

The point O lies in a horizontal field. The point B lies 50 m east of O. A particle is projected from B at speed $25\,\mathrm{m\,s^{-1}}$ at an angle $\arctan\frac{1}{2}$ above the horizontal and in a direction that makes an angle $60°$ with OB; it passes to the north of O.

(i) Taking unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to O at time t seconds after the particle was projected, and show that its distance from O is

$$5(t^2 - \sqrt{5}t + 10)\,\mathrm{m}.$$

When this distance is shortest, the particle is at point P. Find the position vector of P and its horizontal bearing from O.

(ii) Show that the particle reaches its maximum height at P.

(iii) When the particle is at P, a marksman fires a bullet from O directly at P. The initial speed of the bullet is $350\,\mathrm{m\,s^{-1}}$. Ignoring the effect of gravity on the bullet show that, when it passes through P, the distance between P and the particle is approximately 3 m.

Section C: Probability and Statistics

12 I have two identical dice. When I throw either one of them, the probability of it showing a 6 is p and the probability of it not showing a 6 is q, where $p+q=1$. As an experiment to determine p, I throw the dice simultaneously until at least one die shows a 6. If both dice show a six on this throw, I stop. If just one die shows a six, I throw the other die until it shows a 6 and then stop.

(i) Show that the probability that I stop after r throws is $pq^{r-1}(2-q^{r-1}-q^r)$, and find an expression for the expected number of throws.

[**Note:** You may use the result $\displaystyle\sum_{r=0}^{\infty} rx^r = x(1-x)^{-2}$.]

(ii) In a large number of such experiments, the mean number of throws was m. Find an estimate for p in terms of m.

13 **(i)** Given that $0<r<n$ and r is much smaller than n, show that $\dfrac{n-r}{n} \approx \mathrm{e}^{-r/n}$.

(ii) There are k guests at a party. Assuming that there are exactly 365 days in the year, and that the birthday of any guest is equally likely to fall on any of these days, show that the probability that there are at least two guests with the same birthday is approximately $1-\mathrm{e}^{-k(k-1)/730}$.

(iii) Using the approximation $\frac{253}{365} \approx \ln 2$, find the smallest value of k such that the probability that at least two guests share the same birthday is at least $\frac{1}{2}$.

(iv) How many guests must there be at the party for the probability that at least one guest has the same birthday as the host to be at least $\frac{1}{2}$?

14 The random variable X has a continuous probability density function $\mathrm{f}(x)$ given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ \ln x & \text{for } 1 \leqslant x \leqslant k \\ \ln k & \text{for } k \leqslant x \leqslant 2k \\ a - bx & \text{for } 2k \leqslant x \leqslant 4k \\ 0 & \text{for } x \geqslant 4k \end{cases}$$

where k, a and b are constants.

(i) Sketch the graph of $y = \mathrm{f}(x)$.

(ii) Determine a and b in terms of k and find the numerical values of k, a and b.

(iii) Find the median value of X.

Section A: Pure Mathematics

1 The sequence of real numbers u_1, u_2, u_3, ... is defined by

$$u_1 = 2\,, \qquad \text{and} \qquad u_{n+1} = k - \frac{36}{u_n} \quad \text{for } n \geqslant 1, \qquad (*)$$

where k is a constant.

(i) Determine the values of k for which the sequence $(*)$ is:

(a) constant;

(b) periodic with period 2;

(c) periodic with period 4.

(ii) In the case $k = 37$, show that $u_n \geqslant 2$ for all n. Given that in this case the sequence $(*)$ converges to a limit ℓ, find the value of ℓ.

2 **(i)** Using the series

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\,,$$

show that $\mathrm{e} > \frac{8}{3}$.

(ii) Show that $n! > 2^n$ for $n \geqslant 4$ and hence show that $\mathrm{e} < \frac{67}{24}$.

(iii) Show that the curve with equation

$$y = 3\mathrm{e}^{2x} + 14\ln(\tfrac{4}{3} - x)\,, \qquad x < \tfrac{4}{3}$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and give a sketch to show the shape of the curve.

3 **(i)** Show that $\left(5+\sqrt{24}\,\right)^4 + \dfrac{1}{\left(5+\sqrt{24}\,\right)^4}$ is an integer.

Show also that

$$0.1 < \frac{1}{5+\sqrt{24}} < \frac{2}{19} < 0.11\,.$$

Hence determine, with clear reasoning, the value of $\left(5+\sqrt{24}\right)^4$ correct to four decimal places.

(ii) If N is an integer greater than 1, show that $(N+\sqrt{N^2-1}\,)^k$, where k is a positive integer, differs from the integer nearest to it by less than $\left(2N-\frac{1}{2}\right)^{-k}$.

4 By making the substitution $x = \pi - t$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \tfrac{1}{2}\pi\int_0^{\pi}\mathrm{f}(\sin x)\,\mathrm{d}x\,,$$

where $\mathrm{f}(\sin x)$ is a given function of $\sin x$.

Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi}\frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_0^{2\pi}\frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

(iii) $\displaystyle\int_0^{\pi}\frac{x\left|\sin 2x\right|}{3+\sin^2 x}\,\mathrm{d}x\,.$

5 The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x. Thus, for example, $\lfloor \pi \rfloor = 3$, $\lfloor 18 \rfloor = 18$ and $\lfloor -4.2 \rfloor = -5$.

(i) Two curves are given by $y = x^2 + 3x - 1$ and $y = x^2 + 3\lfloor x \rfloor - 1$. Sketch the curves, for $1 \leqslant x \leqslant 3$, on the same axes.

Find the area between the two curves for $1 \leqslant x \leqslant n$, where n is a positive integer.

(ii) Two curves are given by $y = x^2 + 3x - 1$ and $y = \lfloor x \rfloor^2 + 3\lfloor x \rfloor - 1$. Sketch the curves, for $1 \leqslant x \leqslant 3$, on the same axes.

Show that the area between the two curves for $1 \leqslant x \leqslant n$, where n is a positive integer, is

$$\tfrac{1}{6}(n-1)(3n+11)\,.$$

6 By considering a suitable scalar product, prove that

$$(ax + by + cz)^2 \leqslant (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$$

for any real numbers a, b, c, x, y and z. Deduce a necessary and sufficient condition on a, b, c, x, y and z for the following equation to hold:

$$(ax + by + cz)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\,.$$

(i) Show that $(x + 2y + 2z)^2 \leqslant 9(x^2 + y^2 + z^2)$ for all real numbers x, y and z.

(ii) Find real numbers p, q and r that satisfy both

$$p^2 + 4q^2 + 9r^2 = 729 \quad \text{and} \quad 8p + 8q + 3r = 243\,.$$

7 An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(i) Show that the equation of the tangent at the point $(a\cos\alpha, b\sin\alpha)$ is

$$y = -\frac{b\cot\alpha}{a}\,x + b\,\mathrm{cosec}\,\alpha\,.$$

(ii) The point A has coordinates $(-a,-b)$, where a and b are positive. The point E has coordinates $(-a,0)$ and the point P has coordinates (a,kb), where $0<k<1$. The line through E parallel to AP meets the line $y=b$ at the point Q. Show that the line PQ is tangent to the above ellipse at the point given by $\tan(\alpha/2)=k$.

(iii) Determine by means of sketches, or otherwise, whether this result holds also for $k=0$ and $k=1$.

8 Show that the line through the points with position vectors $\mathbf{x}$ and $\mathbf{y}$ has equation

$$\mathbf{r} = (1-\alpha)\mathbf{x} + \alpha\mathbf{y}\,,$$

where α is a scalar parameter.

The sides OA and CB of a trapezium $OABC$ are parallel, and $OA > CB$. The point E on OA is such that $OE : EA = 1:2$, and F is the midpoint of CB. The point D is the intersection of OC produced and AB produced; the point G is the intersection of OB and EF; and the point H is the intersection of DG produced and OA. Let $\mathbf{a}$ and $\mathbf{c}$ be the position vectors of the points A and C, respectively, with respect to the origin O.

(i) Show that B has position vector $\lambda\mathbf{a} + \mathbf{c}$ for some scalar parameter λ.

(ii) Find, in terms of $\mathbf{a}$, $\mathbf{c}$ and λ only, the position vectors of D, E, F, G and H. Determine the ratio $OH : HA$.

Section B: Mechanics

9 A painter of weight kW uses a ladder to reach the guttering on the outside wall of a house. The wall is vertical and the ground is horizontal. The ladder is modelled as a uniform rod of weight W and length $6a$.

The ladder is not long enough, so the painter stands the ladder on a uniform table. The table has weight $2W$ and a square top of side $\frac{1}{2}a$ with a leg of length a at each corner. The foot of the ladder is at the centre of the table top and the ladder is inclined at an angle $\arctan 2$ to the horizontal. The edge of the table nearest the wall is parallel to the wall.

The coefficient of friction between the foot of the ladder and the table top is $\frac{1}{2}$. The contact between the ladder and the wall is sufficiently smooth for the effects of friction to be ignored.

(i) Show that, if the legs of the table are fixed to the ground, the ladder does not slip on the table however high the painter stands on the ladder.

(ii) It is given that $k = 9$ and that the coefficient of friction between each table leg and the ground is $\frac{1}{3}$. If the legs of the table are not fixed to the ground, so that the table can tilt or slip, determine which occurs first when the painter slowly climbs the ladder.

[Note: $\arctan 2$ is another notation for $\tan^{-1} 2$.]

10 Three particles, A, B and C, of masses m, km and $3m$ respectively, are initially at rest lying in a straight line on a smooth horizontal surface. Then A is projected towards B at speed u. After the collision, B collides with C. The coefficient of restitution between A and B is $\frac{1}{2}$ and the coefficient of restitution between B and C is $\frac{1}{4}$.

(i) Find the range of values of k for which A and B collide for a second time.

(ii) Given that $k = 1$ and that B and C are initially a distance d apart, show that the time that elapses between the two collisions of A and B is $\dfrac{60d}{13u}$.

11 A projectile of unit mass is fired in a northerly direction from a point on a horizontal plain at speed u and an angle θ above the horizontal. It lands at a point A on the plain. In flight, the projectile experiences two forces: gravity, of magnitude g; and a horizontal force of constant magnitude f due to a wind blowing from North to South. Derive an expression, in terms of u, g, f and θ for the distance OA.

(i) Determine the angle α such that, for all $\theta > \alpha$, the wind starts to blow the projectile back towards O before it lands at A.

(ii) An identical projectile, which experiences the same forces, is fired from O in a northerly direction at speed u and angle 45° above the horizontal and lands at a point B on the plain. Given that θ is chosen to maximise OA, show that

$$\frac{OB}{OA} = \frac{g-f}{\sqrt{g^2+f^2}-f} \ .$$

Describe carefully the motion of the second projectile when $f = g$.

Section C: Probability and Statistics

12 A cricket team has only three bowlers, Arthur, Betty and Cuba, each of whom bowls 30 balls in any match. Past performance reveals that, on average, Arthur takes one wicket for every 36 balls bowled, Betty takes one wicket for every 25 balls bowled, and Cuba takes one wicket for every 41 balls bowled.

(i) In one match, the team took exactly one wicket, but the name of the bowler was not recorded. Using a binomial model, find the probability that Arthur was the bowler.

(ii) Show that the average number of wickets taken by the team in a match is approximately 3. Give with brief justification a suitable model for the number of wickets taken by the team in a match and show that the probability of the team taking at least five wickets in a given match is approximately $\frac{1}{5}$.

[You may use the approximation $e^3 = 20$.]

13 I know that ice-creams come in n different sizes, but I don't know what the sizes are. I am offered one of each in succession, in random order. I am certainly going to choose one - the bigger the better - but I am not allowed more than one. My strategy is to reject the first ice-cream I am offered and choose the first one thereafter that is bigger than the first one I was offered; if the first ice-cream offered is in fact the biggest one, then I have to put up with the last one, however small.

Let $\mathrm{P}_n(k)$ be the probability that I choose the kth biggest ice-cream, where $k = 1$ is the biggest and $k = n$ is the smallest.

(i) Show that $\mathrm{P}_4(1) = \frac{11}{24}$ and find $\mathrm{P}_4(2)$, $\mathrm{P}_4(3)$ and $\mathrm{P}_4(4)$.

(ii) Find an expression for $\mathrm{P}_n(1)$.

14 Sketch the graph of $y = \dfrac{1}{x \ln x}$ for $x > 0$, $x \neq 1$. You may assume that $x \ln x \to 0$ as $x \to 0$.

The continuous random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{\lambda}{x \ln x} & \text{for } a \leqslant x \leqslant b \,, \\[2ex] 0 & \text{otherwise} \,, \end{cases}$$

where a, b and λ are suitably chosen constants.

(i) In the case $a = 1/4$ and $b = 1/2$, find λ.

(ii) In the case $\lambda = 1$ and $a > 1$, show that $b = a^{\mathrm{e}}$.

(iii) In the case $\lambda = 1$ and $a = \mathrm{e}$, show that $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2) \approx \frac{31}{108}$.

(iv) In the case $\lambda = 1$ and $a = \mathrm{e}^{1/2}$, find $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2)$.

Section A: Pure Mathematics

1 **(i)** Find the three values of x for which the derivative of $x^2 \mathrm{e}^{-x^2}$ is zero.

(ii) Given that a and b are distinct positive numbers, find a polynomial $\mathrm{P}(x)$ such that the derivative of $\mathrm{P}(x)\mathrm{e}^{-x^2}$ is zero for $x = 0$, $x = \pm a$ and $x = \pm b$, but for no other values of x.

2 For any positive integer N, the function $\mathrm{f}(N)$ is defined by

$$\mathrm{f}(N) = N\Big(1 - \frac{1}{p_1}\Big)\Big(1 - \frac{1}{p_2}\Big)\cdots\Big(1 - \frac{1}{p_k}\Big)$$

where p_1, p_2, … , p_k are the only prime numbers that are factors of N.
Thus $\mathrm{f}(80) = 80(1 - \frac{1}{2})(1 - \frac{1}{5})$.

(i) **(a)** Evaluate $\mathrm{f}(12)$ and $\mathrm{f}(180)$. **(b)** Show that $\mathrm{f}(N)$ is an integer for all N.

(ii) Prove, or disprove by means of a counterexample, each of the following: **(a)** $\mathrm{f}(m)\mathrm{f}(n) = \mathrm{f}(mn)$; **(b)** $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ if p and q are distinct prime numbers; **(c)** $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ only if p and q are distinct prime numbers.

(iii) Find a positive integer m and a prime number p such that $\mathrm{f}(p^m) = 146410$.

3 Give a sketch, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, of the curve

$$y = (\sin x - x\cos x)\ ,$$

and show that $0 \leqslant y \leqslant 1$. Show that:

(i) $$\int_0^{\frac{1}{2}\pi} y \,\mathrm{d}x = 2 - \frac{\pi}{2}\ ;$$

(ii) $$\int_0^{\frac{1}{2}\pi} y^2 \,\mathrm{d}x = \frac{\pi^3}{48} - \frac{\pi}{8}\ .$$

Deduce that $\pi^3 + 18\pi < 96$.

4 **(i)** The positive numbers a, b and c satisfy $bc = a^2 + 1$. Prove that

$$\arctan\left(\frac{1}{a+b}\right) + \arctan\left(\frac{1}{a+c}\right) = \arctan\left(\frac{1}{a}\right).$$

The positive numbers p, q, r, s, t, u and v satisfy

$$st = (p+q)^2 + 1\,, \qquad uv = (p+r)^2 + 1\,, \qquad qr = p^2 + 1\,.$$

(ii) Prove that

$$\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan\left(\frac{1}{p}\right).$$

(iii) Hence show that

$$\arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{82}\right) + \arctan\left(\frac{1}{187}\right) = \arctan\left(\frac{1}{7}\right).$$

[Note that $\arctan x$ is another notation for $\tan^{-1} x$.]

5 **(i)** The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

(ii) Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1)\,,$$

where $q = \dfrac{b+c}{a}$. Deduce that

$$R \leqslant \frac{1}{\pi(\pi - 1)}\,.$$

6 **(i)** Write down the general term in the expansion in powers of x of $(1-x)^{-1}$, $(1-x)^{-2}$ and $(1-x)^{-3}$, where $|x|<1$. Evaluate $\displaystyle\sum_{n=1}^{\infty} n2^{-n}$ and $\displaystyle\sum_{n=1}^{\infty} n^2 2^{-n}$.

(ii) Show that $(1-x)^{-\frac{1}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\frac{x^n}{2^{2n}}$, for $|x|<1$. Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 2^{2n} 3^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{n(2n)!}{(n!)^2 2^{2n} 3^n}$.

7 The position vectors, relative to an origin O, at time t of the particles P and Q are

$$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k} \qquad \text{and} \qquad \cos(t+\tfrac{1}{4}\pi)\left[\tfrac{3}{2}\mathbf{i} + \tfrac{3\sqrt{3}}{2}\mathbf{k}\right] + 3\sin(t+\tfrac{1}{4}\pi)\,\mathbf{j}\,,$$

respectively, where $0 \leqslant t \leqslant 2\pi$.

(i) Give a geometrical description of the motion of P and Q.

(ii) Let θ be the angle POQ at time t that satisfies $0 \leqslant \theta \leqslant \pi$. Show that

$$\cos\theta = \tfrac{3\sqrt{2}}{8} - \tfrac{1}{4}\cos(2t+\tfrac{1}{4}\pi)\,.$$

(iii) Show that the total time for which $\theta \geqslant \frac{1}{4}\pi$ is $\frac{3}{2}\pi$.

8 For $x \geqslant 0$ the curve C is defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3y^2}{(1+x^2)^{5/2}}$$

with $y=1$ when $x=0$.

(i) Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}\,.$$

(ii) Draw a sketch of C.

(iii) On a separate diagram draw a sketch of the two curves defined for $x \geqslant 0$ by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x^3z^3}{2(1+x^2)^{5/2}}$$

with $z=1$ at $x=0$ on one curve, and $z=-1$ at $x=0$ on the other.

Section B: Mechanics

9 Two particles, A and B, of masses m and $2m$, respectively, are placed on a line of greatest slope, ℓ, of a rough inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between A and the plane is $\frac{1}{6}\sqrt{3}$ and the coefficient of friction between B and the plane is $\frac{1}{3}\sqrt{3}$. The particles are at rest with B higher up ℓ than A and are connected by a light inextensible string which is taut. A force P is applied to B.

(i) Show that the least magnitude of P for which the two particles move upwards along ℓ is $\frac{11}{8}\sqrt{3}\,mg$ and give, in this case, the direction in which P acts.

(ii) Find the least magnitude of P for which the particles do not slip downwards along ℓ.

10 The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of $100\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line AB of $\arcsin\frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B, at speed of $200\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line BA of $\arcsin\frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B, and the missiles collide. Taking $g = 10\,\mathrm{m\,s^{-2}}$ and ignoring air resistance, find T. [Note that $\arcsin\frac{3}{5}$ is another notation for $\sin^{-1}\frac{3}{5}$.]

11 A plane is inclined at an angle $\arctan\frac{3}{4}$ to the horizontal and a small, smooth, light pulley P is fixed to the top of the plane. A string, APB, passes over the pulley. A particle of mass m_1 is attached to the string at A and rests on the inclined plane with AP parallel to a line of greatest slope in the plane. A particle of mass m_2, where $m_2 > m_1$, is attached to the string at B and hangs freely with BP vertical. The coefficient of friction between the particle at A and the plane is $\frac{1}{2}$.

(i) The system is released from rest with the string taut. Show that the acceleration of the particles is $\dfrac{m_2 - m_1}{m_2 + m_1}g$.

(ii) At a time T after release, the string breaks. Given that the particle at A does not reach the pulley at any point in its motion, find an expression in terms of T for the time after release at which the particle at A reaches its maximum height. It is found that, regardless of when the string broke, this time is equal to the time taken by the particle at A to descend from its point of maximum height to the point at which it was released. Find the ratio $m_1 : m_2$.

[Note that $\arctan\frac{3}{4}$ is another notation for $\tan^{-1}\frac{3}{4}$.]

Section C: Probability and Statistics

12 The twins Anna and Bella share a computer and never sign their e-mails. When I e-mail them, only the twin currently online responds. The probability that it is Anna who is online is p and she answers each question I ask her truthfully with probability a, independently of all her other answers, even if a question is repeated. The probability that it is Bella who is online is q, where $q = 1 - p$, and she answers each question truthfully with probability b, independently of all her other answers, even if a question is repeated.

(i) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. Show that the probability that the coin did come down heads is $\frac{1}{2}$ if and only if $2(ap + bq) = 1$.

(ii) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'no'. Show that the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

(iii) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'yes'. Show that, if $2(ap + bq) = 1$, the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

13 The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geqslant 2$) which contain fewer than two errors is denoted by Y. Show that $\mathrm{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1 + \lambda)e^{-\lambda}$ and $q = 1 - p$. Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$;

(ii) the largest value of n for which $\mathrm{P}(Y = n) \geqslant 1 - \lambda$ is approximately $2/\lambda$;

(iii) $\mathrm{P}(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

14 The probability density function $\mathrm{f}(x)$ of the random variable X is given by

$$\mathrm{f}(x) = k\left[\phi(x) + \lambda \mathrm{g}(x)\right],$$

where $\phi(x)$ is the probability density function of a normal variate with mean 0 and variance 1, λ is a positive constant, and $\mathrm{g}(x)$ is a probability density function defined by

$$\mathrm{g}(x) = \begin{cases} 1/\lambda & \text{for } 0 \leqslant x \leqslant \lambda\,; \\ 0 & \text{otherwise.} \end{cases}$$

Find μ, the mean of X, in terms of λ, and prove that σ, the standard deviation of X, satisfies.

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1+\lambda)^2}\,.$$

In the case $\lambda = 2$:

(i) draw a sketch of the curve $y = \mathrm{f}(x)$;

(ii) express the cumulative distribution function of X in terms of $\Phi(x)$, the cumulative distribution function corresponding to $\phi(x)$;

(iii) evaluate $\mathrm{P}(0 < X < \mu + 2\sigma)$, given that $\Phi(\frac{2}{3} + \frac{2}{3}\sqrt{7}) = 0.9921$.

Section A: Pure Mathematics

1 Find all real values of x that satisfy:

(i) $\sqrt{3x^2+1}+\sqrt{x}-2x-1=0$;

(ii) $\sqrt{3x^2+1}-2\sqrt{x}+x-1=0$;

(iii) $\sqrt{3x^2+1}-2\sqrt{x}-x+1=0$.

2 **(i)** Prove that, if $|\alpha|<2\sqrt{2}$, then there is no value of x for which

$$x^2-\alpha|x|+2<0\,. \qquad (*)$$

(ii) Find the solution set of $(*)$ for $\alpha=3$.

(iii) For $\alpha>2\sqrt{2}$, the sum of the lengths of the intervals in which x satisfies $(*)$ is denoted by S .

(iv) Find S in terms of α and deduce that $S<2\alpha$.

(v) Sketch the graph of S against α .

3 The curve C has equation

$$y=x(x+1)(x-2)^4.$$

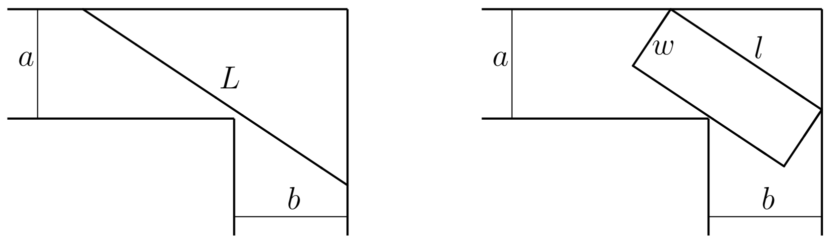
Determine the coordinates of all the stationary points of C and the nature of each. Sketch C. In separate diagrams draw sketches of the curves whose equations are:

(i) $y^2=x(x+1)(x-2)^4$;

(ii) $y=x^2(x^2+1)(x^2-2)^4$.

4

Figure 1 Figure 2



(i) An attempt is made to move a rod of length L from a corridor of width a into a corridor of width b, where $a \neq b$. The corridors meet at right angles, as shown in Figure 1 and the rod remains horizontal. Show that if the attempt is to be successful then

$$L \leqslant a \operatorname{cosec} \alpha + b \sec \alpha \ ,$$

where α satisfies

$$\tan^3 \alpha = \frac{a}{b} \ .$$

(ii) An attempt is made to move a rectangular table-top, of width w and length l, from one corridor to the other, as shown in the Figure 2. The table-top remains horizontal. Show that if the attempt is to be successful then

$$l \leqslant a \operatorname{cosec} \beta + b \sec \beta - 2w \operatorname{cosec} 2\beta,$$

where β satisfies

$$w = \left(\frac{a - b\tan^3 \beta}{1 - \tan^2 \beta} \right) \cos \beta \ .$$

5 **(i)** Evaluate $\int_0^\pi x \sin x \, \mathrm{d}x$ and $\int_0^\pi x \cos x \, \mathrm{d}x$.

(ii) The function f satisfies the equation

$$\mathrm{f}(t) = t + \int_0^\pi \mathrm{f}(x) \sin(x+t) \, \mathrm{d}x \ . \qquad (*)$$

Show that

$$\mathrm{f}(t) = t + A \sin t + B \cos t \ ,$$

where $A = \int_0^\pi \mathrm{f}(x) \cos x \, \mathrm{d}x$ and $B = \int_0^\pi \mathrm{f}(x) \sin x \, \mathrm{d}x$.

(iii) Find A and B by substituting for $\mathrm{f}(t)$ and $\mathrm{f}(x)$ in $(*)$ and equating coefficients of $\sin t$ and $\cos t$.

6 **(i)** The vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the plane Π. Given that $|\mathbf{a}| = 1$ and $\mathbf{a}.\mathbf{b} = 3$, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, a vector $\mathbf{p}$ parallel to $\mathbf{a}$ and a vector $\mathbf{q}$ perpendicular to $\mathbf{a}$, both lying in the plane Π, such that

$$\mathbf{p} + \mathbf{q} = \mathbf{a} + \mathbf{b} .$$

(ii) The vector $\mathbf{c}$ is not parallel to the plane Π and is such that $\mathbf{a}.\mathbf{c} = -2$ and $\mathbf{b}.\mathbf{c} = 2$. Given that $|\mathbf{b}| = 5$, find, in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are parallel to $\mathbf{p}$ and $\mathbf{q}$, respectively, $\mathbf{R}$ is perpendicular to the plane Π and

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} .$$

7 The function f is defined by

$$\mathrm{f}(x) = 2\sin x - x .$$

(i) Show graphically that the equation $\mathrm{f}(x) = 0$ has exactly one root in the interval $[\frac{1}{2}\pi,\ \pi]$. This interval is denoted I_0.

(ii) In order to determine the root, a sequence of intervals $I_1,\ I_2,\ \ldots$ is generated in the following way. If the interval $I_n = [a_n, b_n]$, and $c_n = (a_n + b_n)/2$, then

$$I_{n+1} = \begin{cases} [a_n, c_n] & \text{if } \mathrm{f}(a_n)\mathrm{f}(c_n) < 0\,; \\ [c_n, b_n] & \text{if } \mathrm{f}(c_n)\mathrm{f}(b_n) < 0\,. \end{cases}$$

By using the approximations $\dfrac{1}{\sqrt{2}} \approx 0.7$ and $\pi \approx \sqrt{10}$, show that $I_2 = [\frac{1}{2}\pi,\ \frac{5}{8}\pi]$ and find I_3.

8 Let x satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 - x^n\right)^{1/n}$$

and the condition $x = 0$ when $t = 0$.

(i) Solve the equation in the case $n = 1$ and sketch the graph of the solution for $t > 0$.

(ii) Prove that $1 - x < (1 - x^2)^{1/2}$ for $0 < x < 1$. Use this result to sketch the graph of the solution in the case $n = 2$ for $0 < t < \frac{1}{2}\pi$, using the same axes as your previous sketch. By setting $x = \sin y$, solve the equation in this case.

(iii) Use the result (which you need not prove)

$$(1 - x^2)^{1/2} < (1 - x^3)^{1/3} \quad \text{for} \quad 0 < x < 1\,,$$

to sketch, without solving the equation, the graph of the solution of the equation in the case $n = 3$ using the same axes as your previous sketches. Use your sketch to show that $x = 1$ at a value of t less than $\frac{1}{2}\pi$.

Section B: Mechanics

9 The base of a non-uniform solid hemisphere, of mass M, has radius r. The distance of the centre of gravity, G, of the hemisphere from the base is p and from the centre of the base is $\sqrt{p^2+q^2}$. The hemisphere rests in equilibrium with its curved surface on a horizontal plane.

(i) A particle of mass $m,$ where m is small, is attached to A, the lowest point of the circumference of the base. In the new position of equilibrium, find the angle, α, that the base makes with the horizontal.

(ii) The particle is removed and attached to the point B of the base which is at the other end of the diameter through A. In the new position of equilibrium the base makes an angle β with the horizontal. Show that

$$\tan(\alpha-\beta)=\frac{2mMrp}{M^2\left(p^2+q^2\right)-m^2r^2}\,.$$

10 In this question take $g=10ms^{-2}$. The point A lies on a fixed rough plane inclined at 30° to the horizontal and ℓ is the line of greatest slope through A. A particle P is projected up ℓ from A with initial speed 6ms^{-1}. A time T seconds later, a particle Q is projected from A up ℓ, also with speed 6ms^{-1}. The coefficient of friction between each particle and the plane is $1/(5\sqrt{3})$ and the mass of each particle is 4kg.

(i) Given that $T<1+\sqrt{3/2}$, show that the particles collide at a time $(3-\sqrt{6})b+1$ seconds after P is projected.

(ii) In the case $T=1+\sqrt{2/3}$, determine the energy lost due to friction from the instant at which P is projected to the time of the collision.

11 The maximum power that can be developed by the engine of train A, of mass m, when travelling at speed v is $Pv^{3/2}$, where P is a constant. The maximum power that can be developed by the engine of train B, of mass $2m$, when travelling at speed v is $2Pv^{3/2}$. For both A and B resistance to motion is equal to kv, where k is a constant. For $t \leqslant 0$, the engines are crawling along at very low equal speeds. At $t = 0$, both drivers switch on full power and at time t the speeds of A and B are v_A and v_B, respectively.

(i) Show that

$$v_A = \frac{P^2\left(1 - \mathrm{e}^{-kt/2m}\right)^2}{k^2}$$

and write down the corresponding result for v_B.

(ii) Find v_A and v_B when $9v_A = 4v_B$.

(iii) Both engines are switched off when $9v_A = 4v_B$. Show that thereafter $k^2 v_B^2 = 4P^2 v_A$.

Section C: Probability and Statistics

12 **(i)** Sketch the graph, for $x \geqslant 0$, of

$$y = kx\mathrm{e}^{-ax^2} ,$$

where a and k are positive constants.

(ii) The random variable X has probability density function $\mathrm{f}(x)$ given by

$$\mathrm{f}(x) = \begin{cases} kx\mathrm{e}^{-ax^2} & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $k = \dfrac{2a}{1-\mathrm{e}^{-a}}$ and find the mode m in terms of a, distinguishing between the cases $a < \frac{1}{2}$ and $a > \frac{1}{2}$.

(iii) Find the median h in terms of a and show that $h > m$ if $a > -\ln\left(2\mathrm{e}^{-1/2} - 1\right)$. Show that, $-\ln\left(2\mathrm{e}^{-1/2} - 1\right) > \frac{1}{2}$. Show also that, if $a > -\ln\left(2\mathrm{e}^{-1/2} - 1\right)$, then

$$P(X > m \mid X < h) = \frac{2\mathrm{e}^{-1/2} - \mathrm{e}^{-a} - 1}{1 - \mathrm{e}^{-a}} .$$

13 A bag contains b balls, r of them red and the rest white. In a game the player must remove balls one at a time from the bag (without replacement). She may remove as many balls as she wishes, but if she removes any red ball, she loses and gets no reward at all. If she does not remove a red ball, she is rewarded with £1 for each white ball she has removed.

(i) If she removes n white balls on her first n draws, calculate her expected gain on the next draw and show that it is zero if $n = \dfrac{b-r}{r+1}$.

(ii) Hence, or otherwise, show that she will maximise her expected total reward if she aims to remove n balls, where

$$n = \text{ the integer part of } \frac{b+1}{r+1} .$$

(iii) With this value of n, show that in the case $r = 1$ and b even, her expected total reward is $\pounds\frac{1}{4}b$, and find her expected total reward in the case $r = 1$ and b odd.

14 **(i)** Explain why, if A, B and C are three events,

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) + \mathrm{P}(\mathrm{C}) - \mathrm{P}(\mathrm{A} \cap \mathrm{B}) - \mathrm{P}(\mathrm{B} \cap \mathrm{C}) - \mathrm{P}(\mathrm{C} \cap \mathrm{A}) + \mathrm{P}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}),$$

where $\mathrm{P}(\mathrm{X})$ denotes the probability of event X.

(ii) A cook makes three plum puddings for Christmas. He stirs r silver sixpences thoroughly into the pudding mixture before dividing it into three equal portions. Find an expression for the probability that each pudding contains at least one sixpence. Show that the cook must stir 6 or more sixpences into the mixture if there is to be less than $\frac{1}{3}$ chance that at least one of the puddings contains no sixpence.

(iii) Given that the cook stirs 6 sixpences into the mixture and that each pudding contains at least one sixpence, find the probability that there are two sixpences in each pudding.

Section A: Pure Mathematics

1 Consider the equations

$$\begin{aligned} ax - y - z &= 3\,, \\ 2ax - y - 3z &= 7\,, \\ 3ax - y - 5z &= b\,, \end{aligned}$$

where a and b are given constants.

(i) In the case $a = 0$, show that the equations have a solution if and only if $b = 11$.

(ii) In the case $a \neq 0$ and $b = 11$ show that the equations have a solution with $z = \lambda$ for any given number λ.

(iii) In the case $a = 2$ and $b = 11$ find the solution for which $x^2 + y^2 + z^2$ is least.

(iv) Find a value for a for which there is a solution such that $x > 10^6$ and $y^2 + z^2 < 1$.

2 **(i)** Write down a value of θ in the interval $\frac{1}{4}\pi < \theta < \frac{1}{2}\pi$ that satisfies the equation

$$4\cos\theta + 2\sqrt{3}\,\sin\theta = 5\,.$$

(ii) Hence, or otherwise, show that

$$\pi = 3\arccos(5/\sqrt{28}) + 3\arctan(\sqrt{3}/2)\,.$$

(iii) Show that

$$\pi = 4\arcsin(7\sqrt{2}/10) - 4\arctan(3/4)\,.$$

3 **(i)** Prove that the cube root of any irrational number is an irrational number.

(ii) Let $u_n = 5^{1/(3^n)}$. Given that $\sqrt[3]{5}$ is an irrational number, prove by induction that u_n is an irrational number for every positive integer n.

(iii) Hence, or otherwise, give an example of an infinite sequence of irrational numbers which converges to a given integer m.

[An irrational number is a number that cannot be expressed as the ratio of two integers.]

4 The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H. Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2}\,. \qquad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression $(*)$ should be modified to give the area of the minor segment.

(i) Line: $y = c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

(ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.

(iii) Line: $y = mx + c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

5 **(i)** The position vectors of the points A, B and P with respect to an origin O are $a\mathbf{i}$, $b\mathbf{j}$ and $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, respectively, where a, b, and n are all non-zero. The points E, F, G and H are the midpoints of OA, BP, OB and AP, respectively. Show that the lines EF and GH intersect.

(ii) Let D be the point with position vector $d\mathbf{k}$, where d is non-zero, and let S be the point of intersection of EF and GH. The point T is such that the mid-point of DT is S. Find the position vector of T and hence find d in terms of n if T lies in the plane OAB.

6 The function f is defined by

$$\mathrm{f}(x) = |x-1|\,,$$

where the domain is $\mathbf{R}$, the set of all real numbers. The function $\mathrm{g}_n = \mathrm{f}^n$, with domain $\mathbf{R}$, so for example $\mathrm{g}_3(x) = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$. In separate diagrams, sketch graphs of g_1, g_2, g_3 and g_4. The function h is defined by

$$\mathrm{h}(x) = \left|\sin\frac{\pi x}{2}\right|\,,$$

where the domain is $\mathbf{R}$. Show that if n is even,

$$\int_0^n \big(\mathrm{h}(x) - \mathrm{g}_n(x)\big)\,\mathrm{d}x = \frac{2n}{\pi} - \frac{n}{2}\,.$$

7 **(i)** Show that, if $n > 0$, then

$$\int_{\mathrm{e}^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x = \frac{2}{n^2\mathrm{e}}\,.$$

You may assume that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

(ii) Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x < \int_a^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x\,.$$

(iii) Deduce that

$$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{\mathrm{e}}{2}\int_{\mathrm{e}^{1/N}}^{\infty} \left(\frac{1-x^{-N}}{x^2-x}\right)\ln x\,\mathrm{d}x\,,$$

where N is any integer greater than 1.

8 **(i)** It is given that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} + k\left(\frac{t^2-3t+2}{t+1}\right)y = 0\,,$$

where k is a constant, and $y = A$ when $t = 0$, where A is a positive constant. Find y in terms of t, k and A.

(ii) Show that y has two stationary values whose ratio is $(3/2)^{6k}\mathrm{e}^{-5k/2}$.

(iii) Describe the behaviour of y as $t \to +\infty$ for the case where $k > 0$ and for the case where $k < 0$.

(iv) In separate diagrams, sketch the graph of y for $t > 0$ for each of these cases.

Section B: Mechanics

9 AB is a uniform rod of weight W. The point C on AB is such that $AC > CB$. The rod is in contact with a rough horizontal floor at A and with a cylinder at C. The cylinder is fixed to the floor with its axis horizontal. The rod makes an angle α with the horizontal and lies in a vertical plane perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the floor is $\tan\lambda_1$ and the coefficient of friction between the rod and the cylinder is $\tan\lambda_2$. Show that if friction is limiting both at A and at C, and $\alpha \neq \lambda_2 - \lambda_1$, then the frictional force acting on the rod at A has magnitude

$$\frac{W\sin\lambda_1\,\sin(\alpha-\lambda_2)}{\sin(\alpha+\lambda_1-\lambda_2)}\,.$$

10 A bead B of mass m can slide along a rough horizontal wire. A light inextensible string of length 2ℓ has one end attached to a fixed point A of the wire and the other to B. A particle P of mass $3m$ is attached to the mid-point of the string and B is held at a distance ℓ from A. The bead is released from rest.

(i) Let a_1 and a_2 be the magnitudes of the horizontal and vertical components of the initial acceleration of P. Show by considering the motion of P relative to A, or otherwise, that $a_1 = \sqrt{3}a_2$. Show also that the magnitude of the initial acceleration of B is $2a_1$.

(ii) Given that the frictional force opposing the motion of B is equal to $(\sqrt{3}/6)R$, where R is the normal reaction between B and the wire, show that the magnitude of the initial acceleration of P is $g/18$.

11 A particle P_1 is projected with speed V at an angle of elevation $\alpha\ (> 45^\circ)$, from a point in a horizontal plane.

(i) Find T_1, the flight time of P_1, in terms of α, V and g.

(ii) Show that the time after projection at which the direction of motion of P_1 first makes an angle of 45° with the horizontal is $\frac{1}{2}(1-\cot\alpha)T_1$.

(iii) A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of 45° with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2, show that

$$\frac{2T_2}{T_1} = 1 + \cot\alpha + \sqrt{1+3\cot^2\alpha}\,.$$

Section C: Probability and Statistics

12 The life of a certain species of elementary particles can be described as follows. Each particle has a life time of T seconds, after which it disintegrates into X particles of the same species, where X is a random variable with binomial distribution $\mathrm{B}(2,p)$. A population of these particles starts with the creation of a single such particle at $t=0$. Let X_n be the number of particles in existence in the time interval $nT < t < (n+1)T$, where $n = 1$, 2 , $\ldots$.

(i) Show that $\mathrm{P}(X_1 = 2 \text{ and } X_2 = 2) = 6p^4q^2$, where $q = 1-p$.

(ii) Find the possible values of p if it is known that $\mathrm{P}(X_1 = 2 | X_2 = 2) = 9/25$.

(iii) Explain briefly why $\mathrm{E}(X_n) = 2p\mathrm{E}(X_{n-1})$ and hence determine $\mathrm{E}(X_n)$ in terms of p.

(iv) Show that for one of the values of p found above $\lim_{n\to\infty} \mathrm{E}(X_n) = 0$ and that for the other $\lim_{n\to\infty} \mathrm{E}(X_n) = +\infty$.

13 The random variable X takes the values $k = 1, 2, 3, \ldots$, and has probability distribution

$$\mathrm{P}(X = k) = A\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} \,,$$

where λ is a positive constant.

(i) Show that $A = (1 - \mathrm{e}^{-\lambda})^{-1}$.

(ii) Find the mean μ in terms of λ and show that

$$\mathrm{Var}\,(X) = \mu(1 - \mu + \lambda) \;.$$

(iii) Deduce that $\lambda < \mu < 1 + \lambda$.

(iv) Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

14 The probability of throwing a 6 with a biased die is p. It is known that p is equal to one or other of the numbers A and B where $0 < A < B < 1$. Accordingly the following statistical test of the hypothesis $H_0 : p = B$ against the alternative hypothesis $H_1 : p = A$ is performed. The die is thrown repeatedly until a 6 is obtained. Then if X is the total number of throws, H_0 is accepted if $X \leqslant M$, where M is a given positive integer; otherwise H_1 is accepted. Let α be the probability that H_1 is accepted if H_0 is true, and let β be the probability that H_0 is accepted if H_1 is true. Show that $\beta = 1 - \alpha^K$, where K is independent of M and is to be determined in terms of A and B. Sketch the graph of β against α.

Section A: Pure Mathematics

1 **(i)** Show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1-\cos 2\theta}\,\mathrm{d}\theta = \frac{\sqrt{3}}{2} - \frac{1}{2}\,.$$

(ii) By using the substitution $x = \sin 2\theta$, or otherwise, show that

$$\int_{\sqrt{3}/2}^{1} \frac{1}{1-\sqrt{1-x^2}}\,\mathrm{d}x = \sqrt{3} - 1 - \frac{\pi}{6}\,.$$

(iii) Hence evaluate the integral

$$\int_{1}^{2/\sqrt{3}} \frac{1}{y(y-\sqrt{y^2-1^2})}\,\mathrm{d}y\,.$$

2 **(i)** Show that setting $z - z^{-1} = w$ in the quartic equation

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$

results in the quadratic equation $w^2 + 5w + 6 = 0$.

(ii) Hence solve the above quartic equation.

(iii) Solve similarly the equation

$$2z^8 - 3z^7 - 12z^6 + 12z^5 + 22z^4 - 12z^3 - 12z^2 + 3z + 2 = 0\,.$$

3 The nth Fermat number, F_n, is defined by

$$F_n = 2^{2^n} + 1\,, \quad n = 0,\ 1,\ 2,\ \ldots\ ,$$

where 2^{2^n} means 2 raised to the power 2^n .

(i) Calculate F_0, F_1, F_2 and F_3 .

(ii) Show that, for $k = 1$, $k = 2$ and $k = 3$,

$$F_0 F_1 \ldots F_{k-1} = F_k - 2\ . \qquad (*)$$

(iii) Prove, by induction, or otherwise, that $(*)$ holds for all $k \geqslant 1$.

(iv) Deduce that no two Fermat numbers have a common factor greater than 1.

(v) Hence show that there are infinitely many prime numbers.

4 Give a sketch to show that, if $\mathrm{f}(x) > 0$ for $p < x < q$, then $\int_p^q \mathrm{f}(x)\mathrm{d}x > 0$.

(i) By considering $\mathrm{f}(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $\mathrm{f}(x) = a\sin^2 x - b\sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b\ln(q/p) < a\left(\frac{1}{p} - \frac{1}{q}\right) + c(q-p)\ .$$

5 The numbers x_n, where $n = 0, 1, 2, \ldots$, satisfy

$$x_{n+1} = kx_n(1 - x_n) .$$

(i) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n.

(ii) Given that $x_0 = x_1 = x_2 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a.

(iii) Given instead that $x_0 = x_2 = x_4 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a.

6 The lines l_1, l_2 and l_3 lie in an inclined plane P and pass through a common point A. The line l_2 is a line of greatest slope in P. The line l_1 is perpendicular to l_3 and makes an acute angle α with l_2. The angles between the horizontal and l_1, l_2 and l_3 are $\pi/6$, β and $\pi/4$, respectively.

(i) Show that $\cos\alpha \sin\beta = \frac{1}{2}$ and find the value of $\sin\alpha \sin\beta$.

(ii) Deduce that $\beta = \pi/3$.

(iii) The lines l_1 and l_3 are rotated in P about A so that l_1 and l_3 remain perpendicular to each other. The new acute angle between l_1 and l_2 is θ. The new angles which l_1 and l_3 make with the horizontal are ϕ and 2ϕ, respectively. Show that

$$\tan^2\theta = \frac{3 + \sqrt{13}}{2} .$$

7 In 3-dimensional space, the lines m_1 and m_2 pass through the origin and have directions $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$, respectively. Find the directions of the two lines m_3 and m_4 that pass through the origin and make angles of $\pi/4$ with both m_1 and m_2. Find also the cosine of the acute angle between m_3 and m_4. The points A and B lie on m_1 and m_2 respectively, and are each at distance $\lambda\sqrt{2}$ units from O. The points P and Q lie on m_3 and m_4 respectively, and are each at distance 1 unit from O. If all the coordinates (with respect to axes $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) of A, B, P and Q are non-negative, prove that:

(i) there are only two values of λ for which AQ is perpendicular to BP;

(ii) there are no non-zero values of λ for which AQ and BP intersect.

8 Find y in terms of x, given that:

$$\text{for } x<0\,,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=-y \quad\text{and}\quad y=a \text{ when } x=-1\,;$$
$$\text{for } x>0\,,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=y \quad\text{and}\quad y=b \text{ when } x=1\,.$$

(i) Sketch a solution curve.

(ii) Determine the condition on a and b for the solution curve to be continuous (that is, for there to be no 'jump' in the value of y) at $x=0$.

(iii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=|\mathrm{e}^x-1|\,y$$

given that $y=\mathrm{e}^{\mathrm{e}}$ when $x=1$ and that y is continuous at $x=0$. Write down the following limits:

(i) $\displaystyle\lim_{x\to+\infty} y\exp(-\mathrm{e}^x)\,;$ **(ii)** $\displaystyle\lim_{x\to-\infty} y\mathrm{e}^{-x}\,.$

Section B: Mechanics

9 A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h, at distances a and b from O.

(i) Given that the particle just passes over the walls, find $\tan\alpha$ in terms of a, b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab} \, .$$

(ii) The heights of the walls are now increased by the same small positive amount δh. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively.

Show that

$$\sec^2\alpha \, \delta\alpha \approx \frac{a+b}{ab} \, \delta h \, ,$$

and deduce that $\delta\alpha > 0$.

(iii) Show also that δV is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

10 **(i)** A competitor in a Marathon of $42\frac{3}{8}$ km runs the first t hours of the race at a constant speed of 13 km h^{-1} and the remainder at a constant speed of $14 + 2t/T$ km h^{-1}, where T hours is her time for the race. Show that the minimum possible value of T over all possible values of t is 3.

(ii) The speed of another competitor decreases linearly with respect to time from 16 km h^{-1} at the start of the race. If both of these competitors have a run time of 3 hours, find the maximum distance between them at any stage of the race.

11 **(i)** A rigid straight beam AB has length l and weight W. Its weight per unit length at a distance x from B is $\alpha W l^{-1}(x/l)^{\alpha-1}$, where α is a positive constant. Show that the centre of mass of the beam is at a distance $\alpha l/(\alpha+1)$ from B.

(ii) The beam is placed with the end A on a rough horizontal floor and the end B resting against a rough vertical wall. The beam is in a vertical plane at right angles to the plane of the wall and makes an angle of θ with the floor. The coefficient of friction between the floor and the beam is μ and the coefficient of friction between the wall and the beam is also μ. Show that, if the equilibrium is limiting at both A and B, then

$$\tan\theta = \frac{1-\alpha\mu^2}{(1+\alpha)\mu}\,.$$

(iii) Given that $\alpha = 3/2$ and given also that the beam slides for any $\theta < \pi/4$ find the greatest possible value of μ.

Section C: Probability and Statistics

12 On K consecutive days each of L identical coins is thrown M times. For each coin, the probability of throwing a head in any one throw is p (where $0 < p < 1$).

(i) Show that the probability that on exactly k of these days more than l of the coins will each produce fewer than m heads can be approximated by

$$\binom{K}{k} q^k (1-q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h-2l-1}{2\sqrt{h}}\right), \ h = L\Phi\left(\frac{2m-1-2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(.)$ is the cumulative distribution function of a standard normal variate.

(ii) Would you expect this approximation to be accurate in the case $K = 7$, $k = 2$, $L = 500$, $l = 4$, $M = 100$, $m = 48$ and $p = 0.6$?

13 Let $\mathrm{F}(x)$ be the cumulative distribution function of a random variable X, which satisfies $\mathrm{F}(a) = 0$ and $\mathrm{F}(b) = 1$, where $a > 0$. Let

$$\mathrm{G}(y) = \frac{\mathrm{F}(y)}{2 - \mathrm{F}(y)} \ .$$

(i) Show that $\mathrm{G}(a) = 0$, $\mathrm{G}(b) = 1$ and that $\mathrm{G}'(y) \geqslant 0$.

(ii) Show also that

$$\frac{1}{2} \leqslant \frac{2}{(2-\mathrm{F}(y))^2} \leqslant 2 \ .$$

(iii) The random variable Y has cumulative distribution function $\mathrm{G}(y)$. Show that

$$\tfrac{1}{2}\,\mathrm{E}(X) \leqslant \mathrm{E}(Y) \leqslant 2\mathrm{E}(X) \ ,$$

and that

$$\mathrm{Var}\,(Y) \leqslant 2\mathrm{Var}\,(X) + \tfrac{7}{4}\big(\mathrm{E}(X)\big)^2 \ .$$

14 A densely populated circular island is divided into N concentric regions R_1, R_2, ..., R_N, such that the inner and outer radii of R_n are $n-1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2-n/N$, independently of the number of accidents in any other region. Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X, that occur in it.

(i) Show that, in the long term, the average number of recorded accidents per day will be

$$2-\frac{1}{6}\left(1+\frac{1}{N}\right)\left(4-\frac{1}{N}\right).$$

[Note: $\sum\limits_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$.]

(ii) Show also that

$$\mathrm{P}(X=k)=\frac{\mathrm{e}^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k\mathrm{e}^{n/N}.$$

(iii) Suppose now that $N=3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48+45\mathrm{e}^{1/3}+25\mathrm{e}^{-1/3}}.$$

Section A: Pure Mathematics

1 Use the binomial expansion to obtain a polynomial of degree 2 which is a good approximation to $\sqrt{1-x}$ when x is small.

(i) By taking $x = 1/100$, show that $\sqrt{11} \approx 79599/24000$, and estimate, correct to 1 significant figure, the error in this approximation. (You may assume that the error is given approximately by the first neglected term in the binomial expansion.)

(ii) Find a rational number which approximates $\sqrt{1111}$ with an error of about 2×10^{-12}.

2 Sketch the graph of the function $[x/N]$, for $0 < x < 2N$, where the notation $[y]$ means the integer part of y. (Thus $[2.9] = 2,\ \ [4] = 4$.)

(i) Prove that

$$\sum_{k=1}^{2N} (-1)^{[k/N]} k = 2N - N^2.$$

(ii) Let

$$S_N = \sum_{k=1}^{2N} (-1)^{[k/N]} 2^{-k}.$$

Find S_N in terms of N and determine the limit of S_N as $N \to \infty$.

3 The cuboid $ABCDEFGH$ is such AE, BF, CG, DH are perpendicular to the opposite faces $ABCD$ and $EFGH$, and $AB = 2, BC = 1, AE = \lambda$.

(i) Show that if α is the acute angle between the diagonals AG and BH then

$$\cos\alpha = \left|\frac{3-\lambda^2}{5+\lambda^2}\right|$$

(ii) Let R be the ratio of the volume of the cuboid to its surface area. Show that $R < \frac{1}{3}$ for all possible values of λ.

(iii) Prove that, if $R \geqslant \frac{1}{4}$, then $\alpha \leqslant \arccos\frac{1}{9}$.

4 **(i)** Let

$$f(x) = P\sin x + Q\sin 2x + R\sin 3x \ .$$

Show that if $Q^2 < 4R(P-R)$, then the only values of x for which $f(x) = 0$ are given by $x = m\pi$, where m is an integer.
[You may assume that $\sin(3x) = \sin(x)(4\cos^2(x) - 1)$.]

(ii) Now let

$$g(x) = \sin(2nx) + \sin(4nx) - \sin(6nx),$$

where n is a positive integer and $0 < x < \frac{1}{2}\pi$. Find an expression for the largest root of the equation $g(x) = 0$, distinguishing between the cases where n is even and n is odd.

5 **(i)** The curve C_1 passes through the origin in the x–y plane and its gradient is given by

$$\frac{dy}{dx} = x(1-x^2)e^{-x^2}.$$

Show that C_1 has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}e^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of C_1.

(ii) The curve C_2 passes through the origin and its gradient is given by

$$\frac{dy}{dx} = x(1-x^2)e^{-x^3}.$$

Show that C_2 has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}e^{-1}$. (You need not find k.)

6 Show that

$$\int_0^1 \frac{x^4}{1+x^2}\,dx = \frac{\pi}{4} - \frac{2}{3} \ .$$

Determine the values of

(i) $\displaystyle\int_0^1 x^3 \ \tan^{-1}\left(\frac{1-x}{1+x}\right) dx$,

(ii) $\displaystyle\int_0^1 \frac{(1-y)^3}{(1+y)^5} \tan^{-1}y \, dy$.

7 In an Argand diagram, O is the origin and P is the point $2+0\mathrm{i}$. The points Q, R and S are such that the lengths OP, PQ, QR and RS are all equal, and the angles OPQ, PQR and QRS are all equal to $5\pi/6$, so that the points O, P, Q, R and S are five vertices of a regular 12-sided polygon lying in the upper half of the Argand diagram.

(i) Show that Q is the point $2+\sqrt{3}+\mathrm{i}$ and find S.

(ii) The point C is the centre of the circle that passes through the points O, P and Q. Show that, if the polygon is rotated anticlockwise about O until C first lies on the real axis, the new position of S is

$$-\tfrac{1}{2}(3\sqrt{2}+\sqrt{6})(\sqrt{3}-\mathrm{i})\ .$$

8 The function f satisfies $\mathrm{f}(x+1)=\mathrm{f}(x)$ and $\mathrm{f}(x)>0$ for all x.

(i) Give an example of such a function.

(ii) The function F satisfies

$$\frac{\mathrm{dF}}{\mathrm{d}x}=\mathrm{f}(x)$$

and $\mathrm{F}(0)=0$. Show that $\mathrm{F}(n)=n\mathrm{F}(1)$, for any positive integer n.

(iii) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{f}(x)y=0$$

that satisfies $y=1$ when $x=0$. Show that $y(n)\to 0$ as $n\to\infty$, where $n=1,\,2,\,3,\,\ldots$

Section B: Mechanics

9 A particle of unit mass is projected vertically upwards with speed u. At height x, while the particle is moving upwards, it is found to experience a total force F, due to gravity and air resistance, given by $F = \alpha e^{-\beta x}$, where α and β are positive constants.

(i) Calculate the energy expended in reaching this height.

(ii) Show that

$$F = \tfrac{1}{2}\beta v^2 + \alpha - \tfrac{1}{2}\beta u^2 \ ,$$

where v is the speed of the particle, and explain why $\alpha = \frac{1}{2}\beta u^2 + g$, where g is the acceleration due to gravity.

(iii) Determine an expression, in terms of y, g and β, for the air resistance experienced by the particle on its downward journey when it is at a distance y below its highest point.

10 Two particles A and B of masses m and km, respectively, are at rest on a smooth horizontal surface. The direction of the line passing through A and B is perpendicular to a vertical wall which is on the other side of B from A. The particle A is now set in motion towards B with speed u. The coefficient of restitution between A and B is e_1 and between B and the wall is e_2.

(i) Show that there will be a second collision between A and B provided

$$k < \frac{1 + e_2(1 + e_1)}{e_1} \ .$$

(ii) Show that, if $e_1 = \frac{1}{3}$, $e_2 = \frac{1}{2}$ and $k < 5$, then the kinetic energy of A and B immediately after B rebounds from the wall is greater than $mu^2/27$.

11 A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity.

(i) Show that P will hit the ground at a distance R from A given by

$$\frac{gR}{u} = v + v_e + \sqrt{v_e^2 + v^2}\,.$$

(ii) It is required that the range R should be greater than a certain distance D (where $D > 2uv/g$). Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[*The effect of air resistance is to be neglected.*]

Section C: Probability and Statistics

12 The national lottery of Ruritania is based on the positive integers from 1 to N, where N is very large and fixed. Tickets cost $\pounds 1$ each. For each ticket purchased, the punter (i.e. the purchaser) chooses a number from 1 to N. The winning number is chosen at random, and the jackpot is shared equally amongst those punters who chose the winning number. A syndicate decides to buy N tickets, choosing every number once to be sure of winning a share of the jackpot. The total number of tickets purchased in this draw is $3.8N$ and the jackpot is $\pounds W$. Assuming that the non-syndicate punters choose their numbers independently and at random, find the most probable number of winning tickets and show that the expected net loss of the syndicate is approximately

$$N - \frac{5\left(1 - e^{-2.8}\right)}{14}\, W\,.$$

13 **(i)** The life times of a large batch of electric light bulbs are independently and identically distributed. The probability that the life time, T hours, of a given light bulb is greater than t hours is given by

$$\mathrm{P}(T > t) = \frac{1}{(1 + kt)^{\alpha}}\,,$$

where α and k are constants, and $\alpha > 1$. Find the median M and the mean m of T in terms of α and k.

(ii) Nine randomly selected bulbs are switched on simultaneously and are left until all have failed. The fifth failure occurs at 1000 hours and the mean life time of all the bulbs is found to be 2400 hours. Show that $\alpha \approx 2$ and find the approximate value of k. Hence estimate the probability that, if a randomly selected bulb is found to last M hours, it will last a further $m - M$ hours.

14 Two coins A and B are tossed together. A has probability p of showing a head, and B has probability $2p$, independent of A, of showing a head, where $0 < p < \frac{1}{2}$. The random variable X takes the value 1 if A shows a head and it takes the value 0 if A shows a tail. The random variable Y takes the value 1 if B shows a head and it takes the value 0 if B shows a tail. The random variable T is defined by

$$T = \lambda X + \tfrac{1}{2}(1-\lambda)Y.$$

(i) Show that $\mathrm{E}(T) = p$ and find an expression for $\mathrm{Var}\,(T)$ in terms of p and λ.

(ii) Show that as λ varies, the minimum of $\mathrm{Var}\,(T)$ occurs when

$$\lambda = \frac{1-2p}{3-4p}\,.$$

(iii) The two coins are tossed n times, where $n > 30$, and $\overline{T}$ is the mean value of T. Let b be a fixed positive number. Show that the maximum value of $\mathrm{P}\big(|\overline{T} - p| < b\big)$ as λ varies is approximately $2\Phi(b/s) - 1$, where Φ is the cumulative distribution function of a standard normal variate and

$$s^2 = \frac{p(1-p)(1-2p)}{(3-4p)n}\,.$$

Section A: Pure Mathematics

1 **(i)** A number of the form $1/N$, where N is an integer greater than 1, is called a *unit fraction*. Noting that

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \quad \text{and} \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12},$$

guess a general result of the form

$$\frac{1}{N} = \frac{1}{a} + \frac{1}{b} \qquad (*)$$

and hence prove that any unit fraction can be expressed as the sum of two distinct unit fractions.

(ii) By writing $(*)$ in the form

$$(a-N)(b-N) = N^2$$

and by considering the factors of N^2, show that if N is prime, then there is only one way of expressing $1/N$ as the sum of two distinct unit fractions.

(iii) Prove similarly that any fraction of the form $2/N$, where N is prime number greater than 2, can be expressed uniquely as the sum of two distinct unit fractions.

2 **(i)** Prove that if $(x-a)^2$ is a factor of the polynomial $\mathrm{p}(x)$, then $\mathrm{p}'(a) = 0$. Prove a corresponding result if $(x-a)^4$ is a factor of $\mathrm{p}(x)$.

(ii) Given that the polynomial

$$x^6 + 4x^5 - 5x^4 - 40x^3 - 40x^2 + 32x + k$$

has a factor of the form $(x-a)^4$, find k.

3 The lengths of the sides BC, CA, AB of the triangle ABC are denoted by a, b, c, respectively.

(i) Given that

$$b = 8 + \epsilon_1,\ c = 3 + \epsilon_2,\ A = \tfrac{1}{3}\pi + \epsilon_3,$$

where ϵ_1, ϵ_2, and ϵ_3 are small, show that $a \approx 7 + \eta$, where $\eta = \big(13\,\epsilon_1 - 2\,\epsilon_2 + 24\sqrt{3}\,\epsilon_3\big)/14$.

(ii) Given now that

$$|\epsilon_1| \leqslant 2 \times 10^{-3}, \quad |\epsilon_2| \leqslant 4 \cdot 9 \times 10^{-2}, \quad |\epsilon_3| \leqslant \sqrt{3} \times 10^{-3},$$

find the range of possible values of η.

4 **(i)** Prove that

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$$

and that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

(ii) By considering $(5-\mathrm{i})^2(1+\mathrm{i})$, or otherwise, prove that

$$\arctan\left(\frac{7}{17}\right) + 2\arctan\left(\frac{1}{5}\right) = \frac{\pi}{4}\,.$$

(iii) Prove also that

$$3\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) = \frac{\pi}{4}\,.$$

[Note that $\arctan\theta$ is another notation for $\tan^{-1}\theta$.]

5 It is required to approximate a given function $\mathrm{f}(x)$, over the interval $0 \leqslant x \leqslant 1$, by the linear function λx, where λ is chosen to minimise

$$\int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \, \mathrm{d}x.$$

(i) Show that

$$\lambda = 3\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x.$$

(ii) The residual error, R, of this approximation process is such that

$$R^2 = \int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \, \mathrm{d}x.$$

Show that

$$R^2 = \int_0^1 \big(\mathrm{f}(x)\big)^2 \, \mathrm{d}x - \tfrac{1}{3}\lambda^2.$$

(iii) Given now that $\mathrm{f}(x) = \sin(\pi x/n)$, show that **(i)** for large n, $\lambda \approx \pi/n$ and **(ii)** $\lim_{n\to\infty} R = 0$. Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.]

6 **(i)** Show that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \frac{1+\cos\theta}{\sin\theta} = \tan(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta),$$

where $t = \tan\frac{1}{2}\theta$.

(ii) Use the substitution $t = \tan\frac{1}{2}\theta$ to show that, for $0 < \alpha < \frac{1}{2}\pi$,

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+\cos\alpha\sin\theta}\,\mathrm{d}\theta = \frac{\alpha}{\sin\alpha},$$

and deduce a similar result for

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+\sin\alpha\cos\theta}\,\mathrm{d}\theta.$$

7 The line l has vector equation $\mathbf{r} = \lambda \mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3}\,)\,\mathbf{i} + (\sqrt{2}\ \sin\theta)\,\mathbf{j} + (\cos\theta - \sqrt{3}\,)\,\mathbf{k}$$

and λ is a scalar parameter.

(i) Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$.

(ii) Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\pi/6$.

(iii) A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

8 **(i)** Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\mathrm{e}^{-x^2}(y+3)^{\frac{1}{2}} = 0 (x \geqslant 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \to 1$ as $x \to \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x\mathrm{e}^{6x^2}(y+3)^{1-k} = 0 (x \geqslant 0).$$

Find a value of k such that, as $x \to \infty$, $\mathrm{e}^{-3x^2}y$ tends to a finite non-zero limit, which you should determine.

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ may be assumed.]

Section B: Mechanics

9 In an aerobatics display, Jane and Karen jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v, Jane experiences air resistance kv per unit mass but Karen, who spread-eagles, experiences air resistance $kv + (2k^2/g)v^2$ per unit mass.

(i) Show that Jane's speed can never reach g/k. Obtain the corresponding result for Karen.

(ii) Jane opens her parachute when her speed is $g/(3k)$. Show that she has then been in free fall for time $k^{-1}\ln(3/2)$.

(iii) Karen also opens her parachute when her speed is $g/(3k)$. Find the time she has then been in free fall.

10 A long light inextensible string passes over a fixed smooth light pulley. A particle of mass 4 kg is attached to one end A of this string and the other end is attached to a second smooth light pulley. A long light inextensible string BC passes over the second pulley and has a particle of mass 2 kg attached at B and a particle of mass of 1 kg attached at C.

(i) The system is held in equilibrium in a vertical plane. The string BC is then released from rest. Find the accelerations of the two moving particles.

(ii) After T seconds, the end A is released so that all three particles are now moving in a vertical plane. Find the accelerations of A, B and C in this second phase of the motion. Find also, in terms of g and T, the speed of A when B has moved through a total distance of $0.6gT^2$ metres.

11 The string AP has a natural length of $1 \cdot 5$ metres and modulus of elasticity equal to $5g$ newtons. The end A is attached to the ceiling of a room of height $2{\cdot}5$ metres and a particle of mass $0{\cdot}5$ kg is attached to the end P. The end P is released from rest at a point $0{\cdot}5$ metres above the floor and vertically below A.

(i) Show that the string becomes slack, but that P does not reach the ceiling.

(ii) Show also that while the string is in tension, P executes simple harmonic motion, and that the time in seconds that elapses from the instant when P is released to the instant when P first returns to its original position is

$$\left(\frac{8}{3g}\right)^{\frac{1}{2}} + \left(\frac{3}{5g}\right)^{\frac{1}{2}} \Big(\pi - \arccos(3/7)\Big).$$

[Note that $\arccos x$ is another notation for $\cos^{-1} x$.]

Section C: Probability and Statistics

12 *Tabulated values of* $\Phi(\cdot)$*, the cumulative distribution function of a standard normal variable, should not be used in this question.* Henry the commuter lives in Cambridge and his working day starts at his office in London at 0900. He catches the 0715 train to King's Cross with probability p, or the 0720 to Liverpool Street with probability $1-p$. Measured in minutes, journey times for the first train are $N(55,25)$ and for the second are $N(65,16)$. Journey times from King's Cross and Liverpool Street to his office are $N(30,144)$ and $N(25,9)$, respectively.

(i) Show that Henry is more likely to be late for work if he catches the first train.

(ii) Henry makes M journeys, where M is large. Writing A for $1-\Phi(20/13)$ and B for $1-\Phi(2)$, find, in terms of A, B, M and p, the expected number, L, of times that Henry will be late and show that for all possible values of p,

$$BM \leqslant L \leqslant AM.$$

(iii) Henry noted that in 3/5 of the occasions when he was late, he had caught the King's Cross train. Obtain an estimate of p in terms of A and B.

[A random variable is said to be $N\left(\mu,\sigma^2\right)$ if it has a normal distribution with mean μ and variance σ^2.]

13 A group of biologists attempts to estimate the magnitude, N, of an island population of voles (*Microtus agrestis*). Accordingly, the biologists capture a random sample of 200 voles, mark them and release them. A second random sample of 200 voles is then taken of which 11 are found to be marked. Show that the probability, p_N, of this occurrence is given by

$$p_N = k\frac{\big((N-200)!\big)^2}{N!(N-389)!},$$

where k is independent of N.

(i) The biologists then estimate N by calculating the value of N for which p_N is a maximum. Find this estimate.

(ii) All unmarked voles in the second sample are marked and then the entire sample is released. Subsequently a third random sample of 200 voles is taken. Write down the probability that this sample contains exactly j marked voles, leaving your answer in terms of binomial coefficients.

(iii) Deduce that

$$\sum_{j=0}^{200}\binom{389}{j}\binom{3247}{200-j} = \binom{3636}{200}.$$

14 The random variables X_1, X_2, … , X_{2n+1} are independently and uniformly distributed on the interval $0 \leqslant x \leqslant 1$. The random variable Y is defined to be the median of X_1, X_2, … , X_{2n+1}.

(i) Given that the probability density function of Y is $\mathrm{g}(y)$, where

$$\mathrm{g}(y) = \begin{cases} ky^n(1-y)^n & \text{if } 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

use the result

$$\int_0^1 y^r(1-y)^s\,\mathrm{d}y = \frac{r!s!}{(r+s+1)!}$$

to show that $k = (2n+1)!/(n!)^2$, and evaluate $\mathrm{E}(Y)$ and $\mathrm{Var}\,(Y)$.

(ii) Hence show that, for any given positive number d, the inequality

$$\mathrm{P}\left(|Y - 1/2| < d/\sqrt{n}\right) < \mathrm{P}\left(|\bar{X} - 1/2| < d/\sqrt{n}\right)$$

holds provided n is large enough, where $\bar{X}$ is the mean of X_1, X_2, … , X_{2n+1}.

[You may assume that Y and $\bar{X}$ are normally distributed for large n.]

Section A: Pure Mathematics

1 Let $x = 10^{100}$, $y = 10^x$, $z = 10^y$, and let

$$a_1 = x!, \quad a_2 = x^y, \quad a_3 = y^x, \quad a_4 = z^x, \quad a_5 = \mathrm{e}^{xyz}, \quad a_6 = z^{1/y}, \quad a_7 = y^{z/x}.$$

(i) Use Stirling's approximation $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} \mathrm{e}^{-n}$, which is valid for large n, to show that $\log_{10}(\log_{10} a_1) \approx 102$.

(ii) Arrange the seven numbers $a_1, \ldots, a_7$ in ascending order of magnitude, justifying your result.

2 Consider the quadratic equation

$$nx^2 + 2x\sqrt{pn^2 + q} + rn + s = 0, \tag{$*$}$$

where $p > 0$, $p \neq r$ and $n = 1, 2, 3, \ldots$.

(i) For the case where $p = 3$, $q = 50$, $r = 2$, $s = 15$, find the set of values of n for which equation $(*)$ has no real roots.

(ii) Prove that if $p < r$ and $4q(p - r) > s^2$, then $(*)$ has no real roots for any value of n.

(iii) If $n = 1$, $p - r = 1$ and $q = s^2/8$, show that $(*)$ has real roots if, and only if, $s \leqslant 4 - 2\sqrt{2}$ or $s \geqslant 4 + 2\sqrt{2}$.

3 Let

$$\mathrm{S}_n(x) = \mathrm{e}^{x^3} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^3}).$$

(i) Show that $\mathrm{S}_2(x) = 9x^4 - 6x$ and find $\mathrm{S}_3(x)$.

(ii) Prove by induction on n that $\mathrm{S}_n(x)$ is a polynomial. By means of your induction argument, determine the order of this polynomial and the coefficient of the highest power of x.

(iii) Show also that if $\dfrac{\mathrm{d}S_n}{\mathrm{d}x} = 0$ for some value a of x, then $S_n(a)S_{n+1}(a) \leqslant 0$.

4 By considering the expansions in powers of x of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^{n}\binom{n}{s}^2 = \binom{2n}{n},$$

where $\displaystyle\binom{n}{s} = \frac{n!}{s!\,(n-s)!}$.

By considering similar identities, or otherwise, show also that:

(i) if n is an even integer, then

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}^2 = (-1)^{n/2}\binom{n}{n/2};$$

(ii) $\displaystyle\sum_{t=1}^{n}2t\binom{n}{t}^2 = n\binom{2n}{n}.$

5 **(i)** Show that if α is a solution of the equation

$$5\cos x + 12\sin x = 7,$$

then either

$$\cos\alpha = \frac{35-12\sqrt{120}}{169}$$

or $\cos\alpha$ has one other value which you should find.

(ii) Prove carefully that if $\frac{1}{2}\pi < \alpha < \pi$, then $\alpha < \frac{3}{4}\pi$.

6 **(i)** Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if

$$y = \frac{ax+b}{cx+d}. \qquad (*)$$

(ii) By using changes of variable of the form $(*)$, or otherwise, show that

$$\int_0^1 \frac{1}{(x+3)^2}\ \ln\left(\frac{x+1}{x+3}\right)\mathrm{d}x = \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12},$$

and evaluate the integrals

$$\int_0^1 \frac{1}{(x+3)^2}\ \ln\left(\frac{x^2+3x+2}{(x+3)^2}\right)\mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{1}{(x+3)^2}\ \ln\left(\frac{x+1}{x+2}\right)\mathrm{d}x.$$

7 The curve C has equation

$$y = \frac{x}{\sqrt{x^2-2x+a}}\ ,$$

where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point.

Sketch C when **(i)** $a = 2$ and **(ii)** $a = 1$.

8 Prove that

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}\ . \qquad (*)$$

(i) Deduce that, when n is large,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) \approx \frac{2n}{\pi}\ .$$

(ii) By differentiating $(*)$ with respect to θ, or otherwise, show that, when n is large,

$$\sum_{k=0}^{n} k\sin^2\left(\frac{k\pi}{2n}\right) \approx \left(\frac{1}{4}+\frac{1}{\pi^2}\right)n^2\ .$$

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ may be assumed.]

Section B: Mechanics

9 In the Z–universe, a star of mass M suddenly blows up, and the fragments, with various initial speeds, start to move away from the centre of mass G which may be regarded as a fixed point. In the subsequent motion the acceleration of each fragment is directed towards G. Moreover, in accordance with the laws of physics of the Z–universe, there are positive constants k_1, k_2 and R such that when a fragment is at a distance x from G, the magnitude of its acceleration is k_1x^3 if $x < R$ and is k_2x^{-4} if $x \geqslant R$. The initial speed of a fragment is denoted by u.

(i) For $x < R$, write down a differential equation for the speed v, and hence determine v in terms of u, k_1 and x for $x < R$.

(ii) Show that if $u < a$, where $2a^2 = k_1R^4$, then the fragment does not reach a distance R from G.

(iii) Show that if $u \geqslant b$, where $6b^2 = 3k_1R^4 + 4k_2/R^3$, then from the moment of the explosion the fragment is always moving away from G.

(iv) If $a < u < b$, determine in terms of k_2, b and u the maximum distance from G attained by the fragment.

10 N particles P_1, P_2, P_3, ..., P_N with masses m, qm, q^2m, ... , $q^{N-1}m$, respectively, are at rest at distinct points along a straight line in gravity-free space. The particle P_1 is set in motion towards P_2 with velocity V and in every subsequent impact the coefficient of restitution is e, where $0 < e < 1$.

(i) Show that after the first impact the velocities of P_1 and P_2 are

$$\left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad \left(\frac{1+e}{1+q}\right)V,$$

respectively.

(ii) Show that if $q \leqslant e$, then there are exactly $N-1$ impacts and that if $q = e$, then the total loss of kinetic energy after all impacts have occurred is equal to

$$\frac{1}{2}me\big(1-e^{N-1}\big)V^2.$$

11 An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω. A shell is fired from O, the centre of this circle, with initial speed V and angle of elevation α. Show that if $V^2 < gR$, then no matter what the value of α, or what vertical plane the shell is fired in, the shell cannot hit the target.

(i) Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

(ii) Deduce that there are exactly two possible values of β.

(iii) Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R\sin\left(\frac{\omega}{g}\sqrt{V^2 - Rg}\right).$$

Section C: Probability and Statistics

12 **(i)** It is known that there are three manufacturers A, B, C, who can produce micro chip MB666. The probability that a randomly selected MB666 is produced by A is $2p$, and the corresponding probabilities for B and C are p and $1-3p$, respectively, where $0 \leqslant p \leqslant \frac{1}{3}$. It is also known that 70% of MB666 micro chips from A are sound and that the corresponding percentages for B and C are 80% and 90%, respectively.

Find in terms of p, the conditional probability, $\mathrm{P}(A|S)$, that if a randomly selected MB666 chip is found to be sound then it came from A, and also the conditional probability, $\mathrm{P}(C|S)$, that if it is sound then it came from C.

(ii) A quality inspector took a random sample of one MB666 micro chip and found it to be sound. She then traced its place of manufacture to be A, and so estimated p by calculating the value of p that corresponds to the greatest value of $\mathrm{P}(A|S)$. A second quality inspector also a took random sample of one MB666 chip and found it to be sound. Later he traced its place of manufacture to be C and so estimated p by applying the procedure of his colleague to $\mathrm{P}(C|S)$.

Determine the values of the two estimates and comment briefly on the results obtained.

13 A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio, R, of the length of the shorter piece to the length of the longer piece is less than r.

Find the probability density function for R, and calculate the mean and variance of R.

14 You play the following game. You throw a six-sided fair die repeatedly. You may choose to stop after any throw, except that you must stop if you throw a 1. Your score is the number obtained on your last throw. Determine the strategy that you should adopt in order to maximize your expected score, explaining your reasoning carefully.

Section A: Pure Mathematics

1 **(i)** Show that, if n is an integer such that

$$(n-3)^3 + n^3 = (n+3)^3, \qquad (*)$$

then n is even and n^2 is a factor of 54. Deduce that there is no integer n which satisfies the equation $(*)$.

(ii) Show that, if n is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \qquad (**)$$

then n is even. Deduce that there is no integer n which satisfies the equation $(**)$.

2 **(i)** Use the first four terms of the binomial expansion of $(1-1/50)^{1/2}$, writing $1/50 = 2/100$ to simplify the calculation, to derive the approximation $\sqrt{2} \approx 1.414214$.

(ii) Calculate similarly an approximation to the cube root of 2 to six decimal places by considering $(1+N/125)^a$, where a and N are suitable numbers.

[You need not justify the accuracy of your approximations.]

3 **(i)** Show that the sum S_N of the first N terms of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots+\frac{2n-1}{n(n+1)(n+2)}+\cdots$$

is

$$\frac{1}{2}\left(\frac{3}{2}+\frac{1}{N+1}-\frac{5}{N+2}\right).$$

What is the limit of S_N as $N\to\infty$?

(ii) The numbers a_n are such that

$$\frac{a_n}{a_{n-1}}=\frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for a_n/a_1 and hence, or otherwise, evaluate $\sum\limits_{n=1}^{\infty}a_n$ when $a_1=\dfrac{2}{9}$.

4 The integral I_n is defined by

$$I_n=\int_0^{\pi}(\pi/2-x)\sin(nx+x/2)\,\mathrm{cosec}\,(x/2)\,\mathrm{d}x,$$

where n is a positive integer. Evaluate I_n-I_{n-1}, and hence evaluate I_n leaving your answer in the form of a sum.

5 **(i)** Define the modulus of a complex number z and give the geometric interpretation of $|\,z_1-z_2\,|$ for two complex numbers z_1 and z_2. On the basis of this interpretation establish the inequality

$$|\,z_1+z_2\,|\leqslant|\,z_1\,|+|\,z_2\,|.$$

(ii) Use this result to prove, by induction, the corresponding inequality for $|\,z_1+\cdots+z_n\,|$.

(iii) The complex numbers a_1, a_2, $\dots$, a_n satisfy $|a_i|\leqslant 3$ $(i=1,2,\dots,n)$. Prove that the equation

$$a_1z+a_2z^2\cdots+a_nz^n=1$$

has no solution z with $|\,z\,|\leqslant 1/4$.

6 Two curves are given parametrically by

$$x_1 = (\theta + \sin\theta), \qquad y_1 = (1 + \cos\theta), \tag{1}$$

and

$$x_2 = (\theta - \sin\theta), \qquad y_1 = -(1 + \cos\theta), \tag{2}$$

(i) Find the gradients of the tangents to the curves at the points where $\theta = \pi/2$ and $\theta = 3\pi/2$.

(ii) Sketch, using the same axes, the curves for $0 \leqslant \theta \leqslant 2\pi$.

(iii) Find the equation of the normal to the curve (1) at the point with parameter θ. Show that this normal is a tangent to the curve (2).

7

$$\begin{aligned} \mathrm{f}(x) &= \tan x - x, \\ \mathrm{g}(x) &= 2 - 2\cos x - x\sin x, \\ \mathrm{h}(x) &= 2x + x\cos 2x - \tfrac{3}{2}\sin 2x, \\ \mathrm{F}(x) &= \frac{x(\cos x)^{1/3}}{\sin x}. \end{aligned}$$

(i) By considering $\mathrm{f}(0)$ and $\mathrm{f}'(x)$, show that $\mathrm{f}(x) > 0$ for $0 < x < \frac{1}{2}\pi$.

(ii) Show similarly that $\mathrm{g}(x) > 0$ for $0 < x < \frac{1}{2}\pi$.

(iii) Show that $\mathrm{h}(x) > 0$ for $0 < x < \frac{1}{4}\pi$, and hence that

$$x(\sin^2 x + 3\cos^2 x) - 3\sin x\cos x > 0$$

for $0 < x < \frac{1}{4}\pi$.

(iv) By considering $\dfrac{\mathrm{F}'(x)}{\mathrm{F}(x)}$, show that $\mathrm{F}'(x) < 0$ for $0 < x < \frac{1}{4}\pi$.

8 Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in three dimensions have coordinate vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively.

(i) Show that the lines joining the vertices of the triangle ABC to the mid-points of the opposite sides meet at a point R.

(ii) P is a point which is **not** in the plane ABC. Lines are drawn through the mid-points of BC, CA and AB parallel to PA, PB and PC respectively. Write down the vector equations of the lines and show by inspection that these lines meet at a common point Q.

(iii) Prove further that the line PQ meets the plane ABC at R.

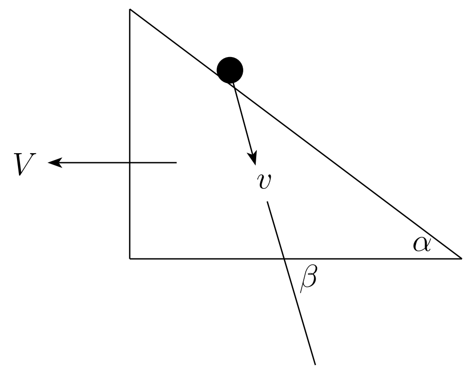
Section B: Mechanics

9 A light smoothly jointed planar framework in the form of a regular hexagon $ABCDEF$ is suspended smoothly from A and a weight 1kg is suspended from C. The framework is kept rigid by three light rods BD, BE and BF. What is the direction and magnitude of the supporting force which must be exerted on the framework at A?

Indicate on a labelled diagram which rods are in thrust (compression) and which are in tension.

Find the magnitude of the force in BE.

10 A wedge of mass M rests on a smooth horizontal surface. The face of the wedge is a smooth plane inclined at an angle α to the horizontal. A particle of mass m slides down the face of the wedge, starting from rest. At a later time t, the speed V of the wedge, the speed v of the particle and the angle β of the velocity of the particle below the horizontal are as shown in the diagram.



Let y be the vertical distance descended by the particle. Derive the following results, stating in **(ii)** and **(iii)** the mechanical principles you use:

(i) $V \sin\alpha = v\sin(\beta - \alpha)$;

(ii) $\tan\beta = (1 + m/M)\tan\alpha$;

(iii) $2gy = v^2(M + m\cos^2\beta)/M$.

Write down a differential equation for y and hence show that

$$y = \frac{gMt^2\sin^2\beta}{2\,(M + m\cos^2\beta)}.$$

11 **(i)** A fielder, who is perfectly placed to catch a ball struck by the batsman in a game of cricket, watches the ball in flight. Assuming that the ball is struck at the fielder's eye level and is caught just in front of her eye, show that $\frac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant, where θ is the angle between the horizontal and the fielder's line of sight.

(ii) In order to catch the next ball, which is also struck towards her but at a different velocity, the fielder runs at constant speed v towards the batsman. Assuming that the ground is horizontal, show that the fielder should choose v so that $\frac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ remains constant.

Section C: Probability and Statistics

12 **(i)** The diagnostic test AL has a probability 0.9 of giving a positive result when applied to a person suffering from the rare disease mathematitis. It also has a probability $1/11$ of giving a false positive result when applied to a non-sufferer. It is known that only 1% of the population suffer from the disease. Given that the test AL is positive when applied to Frankie, who is chosen at random from the population, what is the probability that Frankie is a sufferer?

(ii) In an attempt to identify sufferers more accurately, a second diagnostic test STEP is given to those for whom the test AL gave a positive result. The probablility of STEP giving a positive result on a sufferer is 0.9, and the probability that it gives a false positive result on a non-sufferer is p. Half of those for whom AL was positive and on whom STEP then also gives a positive result are sufferers. Find p.

13 A random variable X has the probability density function

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & x \geqslant 0, \\ 0 & x < 0. \end{cases}$$

(i) Show that

$$\mathrm{P}(X > s + t \,|\, X > t) = \mathrm{P}(X > s).$$

(ii) The time it takes an assistant to serve a customer in a certain shop is a random variable with the above distribution and the times for different customers are independent. If, when I enter the shop, the only two assistants are serving one customer each, what is the probability that these customers are both still being served at time t after I arrive?

(iii) One of the assistants finishes serving his customer and immediately starts serving me. What is the probability that I am still being served when the other customer has finished being served?

14 The staff of Catastrophe College are paid a salary of A pounds per year. With a Teaching Assessment Exercise impending it is decided to try to lower the student failure rate by offering each lecturer an alternative salary of $B/(1+X)$ pounds, where X is the number of his or her students who fail the end of year examination. Dr Doom has N students, each with independent probability p of failure.

(i) Show that she should accept the new salary scheme if

$$A(N+1)p < B(1-(1-p)^{N+1}).$$

(ii) Under what circumstances could X, for Dr Doom, be modelled by a Poisson random variable?

(iii) What would Dr Doom's expected salary be under this model?

Section A: Pure Mathematics

1 Find the sum of those numbers between 1000 and 6000 every one of whose digits is one of the numbers 0, 2, 5 or 7, giving your answer as a product of primes.

2 Suppose that

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \cdots .$$

Guess an expression, in terms of n, for x_n. Then, by induction or otherwise, prove the correctness of your guess.

3 **(i)** Find constants a, b, c and d such that

$$\frac{ax+b}{x^2+2x+2} + \frac{cx+d}{x^2-2x+2} = \frac{1}{x^4+4}.$$

(ii) Show that

$$\int_0^1 \frac{\mathrm{d}x}{x^4+4} = \frac{1}{16}\ln 5 + \frac{1}{8}\tan^{-1} 2.$$

4 Show that, when the polynomial $\mathrm{p}(x)$ is divided by $(x-a)$, where a is a real number, the remainder is $\mathrm{p}(a)$.

(i) When the polynomial $\mathrm{p}(x)$ is divided by $x-1$, $x-2$, $x-3$ the remainders are 3,1,5 respectively. Given that

$$\mathrm{p}(x) = (x-1)(x-2)(x-3)\mathrm{q}(x) + \mathrm{r}(x),$$

where $\mathrm{q}(x)$ and $\mathrm{r}(x)$ are polynomials with $\mathrm{r}(x)$ having degree less than three, find $\mathrm{r}(x)$.

(ii) Find a polynomial $\mathrm{P}(x)$ of degree $n+1$, where n is a given positive integer, such that for each integer a satisfying $0 \leqslant a \leqslant n$, the remainder when $\mathrm{P}_n(x)$ is divided by $x-a$ is a.

5 The complex numbers $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$ are related by the equation

$$z = (\cos v + \mathrm{i}\sin v)\mathrm{e}^u.$$

(i) Find all w which correspond to $z = \mathrm{i}\,\mathrm{e}$.

(ii) Find the loci in the x–y plane corresponding to the lines $u =$ constant in the u–v plane. Find also the loci corresponding to the lines $v =$ constant. Illustrate your answers with clearly labelled sketches.

(iii) Identify two subsets W_1 and W_2 of the u–v plane each of which is in one-to-one correspondence with the first quadrant $\{(x, y) : x > 0, y > 0\}$ of the x–y plane. Identify also two subsets W_3 and W_4 each of which is in one-to-one correspondence with the set $\{z : 0 < |z| < 1\}$.

[**NB** 'one-to-one' means here that to each value of w there is only one corresponding value of z, and vice-versa.]

6 **(i)** Show that, if $\tan^2\phi = 2\tan\phi + 1$, then $\tan 2\phi = -1$.

(ii) Find all solutions of the equation

$$\tan\theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$, expressing your answers as rational multiples of π.

(iii) Find all solutions of the equation the equation

$$\cot\theta = 2 + \cot 3\theta$$

which satisfy

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}.$$

7 Let

$$y^2 = x^2(a^2 - x^2),$$

where a is a real constant. Find, in terms of a, the maximum and minimum values of y.

Sketch carefully on the same axes the graphs of y in the cases $a = 1$ and $a = 2$.

8 **(i)** If $\mathrm{f}(t) \geqslant \mathrm{g}(t)$ for $a \leqslant t \leqslant b$, explain very briefly why $\int_a^b \mathrm{f}(t)\mathrm{d}t \geqslant \int_a^b \mathrm{g}(t)\mathrm{d}t$.

(ii) Prove that if $p > q > 0$ and $x \geqslant 1$ then

$$\frac{x^p - 1}{p} \geqslant \frac{x^q - 1}{q}.$$

(iii) Show that this inequality also holds when $p > q > 0$ and $0 \leqslant x \leqslant 1$.

(iv) Prove that, if $p > q > 0$ and $x \geqslant 0$, then

$$\frac{1}{p}\left(\frac{x^p}{p+1} - 1\right) \geqslant \frac{1}{q}\left(\frac{x^q}{q+1} - 1\right).$$

Section B: Mechanics

9 A uniform solid sphere of diameter d and mass m is drawn slowly and without slipping from horizontal ground onto a step of height $d/4$ by a horizontal force which is always applied to the highest point of the sphere and is always perpendicular to the vertical plane which forms the face of the step.

(i) Find the maximum horizontal force throughout the movement,

(ii) and prove that the coefficient of friction between the sphere and the edge of the step must exceed $1/\sqrt{3}$.

10 *In this question the effect of gravity is to be neglected.*

A small body of mass M is moving with velocity v along the axis of a long, smooth, fixed, circular cylinder of radius L. An internal explosion splits the body into two spherical fragments, with masses qM and $(1-q)M$, where $q \leqslant \frac{1}{2}$.

(i) After bouncing perfectly elastically off the cylinder (one bounce each) the fragments collide and coalesce at a point $\frac{1}{2}L$ from the axis. Show that $q = \frac{3}{8}$.

(ii) The collision occurs at a time $5L/v$ after the explosion. Find the energy imparted to the fragments by the explosion, and find the velocity after coalescence.

11 A tennis player serves from height H above horizontal ground, hitting the ball downwards with speed v at an angle α below the horizontal. The ball just clears the net of height h at horizontal distance a from the server and hits the ground a further horizontal distance b beyond the net.

(i) Show that

$$v^2 = \frac{g(a+b)^2(1+\tan^2\alpha)}{2[H-(a+b)\tan\alpha]}$$

and

$$\tan\alpha = \frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h\,.$$

(ii) By considering the signs of v^2 and $\tan\alpha$, find upper and lower bounds on H for such a serve to be possible.

Section C: Probability and Statistics

12 The game of Cambridge Whispers starts with the first participant Albert flipping an un-biased coin and whispering to his neighbour Bertha whether it fell 'heads' or 'tails'. Bertha then whispers this information to her neighbour, and so on. The game ends when the final player Zebedee whispers to Albert and the game is won, by all players, if what Albert hears is correct. The acoustics are such that the listeners have, independently at each stage, only a probability of 2/3 of hearing correctly what is said.

(i) Find the probability that the game is won when there are just three players.

(ii) By considering the binomial expansion of $(a+b)^n + (a-b)^n$, or otherwise, find a concise expression for the probability P that the game is won when is it played by n players each having a probability p of hearing correctly.

(iii) To avoid the trauma of a lost game, the rules are now modified to require Albert to whisper to Bertha what he hears from Zebedee, and so keep the game going, if what he hears from Zebedee is not correct. Find the expected total number of times that Albert whispers to Bertha before the modified game ends.

[You may use without proof the fact that $\sum_1^\infty kx^{k-1} = (1-x)^{-2}$ for $|x| < 1$.]

13 A needle of length two cm is dropped at random onto a large piece of paper ruled with parallel lines two cm apart.

(i) By considering the angle which the needle makes with the lines, find the probability that the needle crosses the nearest line given that its centre is x cm from it, where $0 < x < 1$.

(ii) Given that the centre of the needle is x cm from the nearest line and that the needle crosses that line, find the cumulative distribution function for the length of the shorter segment of the needle cut off by the line.

(iii) Find the probability that the needle misses all the lines.

14 Traffic enters a tunnel which is 9600 metres long, and in which overtaking is impossible. The number of vehicles which enter in any given time is governed by the Poisson distribution with mean 6 cars per minute. All vehicles travel at a constant speed until forced to slow down on catching up with a slower vehicle ahead. I enter the tunnel travelling at 30 $\mathrm{m\,s^{-1}}$ and all the other traffic is travelling at 32 $\mathrm{m\,s^{-1}}$.

(i) What is the expected number of vehicles in the queue behind me when I leave the tunnel?

(ii) Assuming again that I travel at 30 $\mathrm{m\,s^{-1}}$, but that all the other vehicles are independently equally likely to be travelling at 30 $\mathrm{m\,s^{-1}}$ or 32 $\mathrm{m\,s^{-1}}$, find the probability that exactly two vehicles enter the tunnel within 20 seconds of my doing so and catch me up before I leave it.

(iii) Find also the probability that there are exactly two vehicles queuing behind me when I leave the tunnel.

[Ignore the lengths of the vehicles.]

Section A: Pure Mathematics

1 **(i)** Find the coefficient of x^6 in

$$(1-2x+3x^2-4x^3+5x^4)^3.$$

You should set out your working clearly.

(ii) By considering the binomial expansions of $(1+x)^{-2}$ and $(1+x)^{-6}$, or otherwise, find the coefficient of x^6 in

$$(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6)^3.$$

2 Consider the system of equations

$$\begin{aligned} 2yz+zx-5xy&=2\\ yz-zx+2xy&=1\\ yz-2zx+6xy&=3 \end{aligned}$$

Show that

$$xyz=\pm 6$$

and find the possible values of x, y and z.

3 The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geqslant 1$.

(i) Show that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and compute F_5, F_6 and F_7.

(ii) Compute $F_{n+1}F_{n-1} - F_n^2$ for a few values of n; guess a general formula and prove it by induction, or otherwise.

(iii) By induction on k, or otherwise, show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

for all positive integers n and k.

4 **(i)** Show that $\cos 4u = 8\cos^4 u - 8\cos^2 u + 1$.

(ii) If

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\,\mathrm{d}x,$$

show, by using the change of variable $x = \cos t$, that

$$I = \int_0^{\pi} \frac{\sin t}{4\cos^2\left(\frac{t}{4} - \frac{\pi}{8}\right)}\,\mathrm{d}t.$$

(iii) By using the further change of variable $u = \frac{t}{4} - \frac{\pi}{8}$, or otherwise, show that

$$I = 4\sqrt{2} - \pi - 2.$$

[You may assume that $\tan \frac{\pi}{8} = \sqrt{2} - 1$.]

5 **(i)** If

$$z^4 + z^3 + z^2 + z + 1 = 0 \qquad (*)$$

and $u = z + z^{-1}$, find the possible values of u. Hence find the possible values of z. [Do not try to simplify your answers.]

(ii) Show that, if z satisfies $(*)$, then

$$z^5 - 1 = 0.$$

(iii) Hence write the solutions of $(*)$ in the form $z = r(\cos\theta + i\sin\theta)$ for suitable real r and θ.

(iv) Deduce that

$$\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{10+2\sqrt{5}}}{4} \quad \text{and} \quad \cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}.$$

6 A *proper factor* of a positive integer N is an integer M, with $M \neq 1$ and $M \neq N$, which divides N without remainder. Show that 12 has 4 proper factors and 16 has 3.

Suppose that N has the prime factorisation

$$N = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where p_1, p_2, $\dots$, p_r are distinct primes and m_1, m_2, $\dots$, m_r are positive integers. How many proper factors does N have and why?

Find:

(i) the smallest positive integer which has precisely 12 proper factors;

(ii) the smallest positive integer which has at least 12 proper factors.

7 Consider a fixed square $ABCD$ and a variable point P in the plane of the square. We write the perpendicular distance from P to AB as p, from P to BC as q, from P to CD as r and from P to DA as s. (Remember that distance is never negative, so $p, q, r, s \geqslant 0$.) If $pr = qs$, show that the only possible positions of P lie on two straight lines and a circle and that every point on these two lines and a circle is indeed a possible position of P.

8 **(i)** Suppose that

$$f''(x) + f(-x) = x + 3\cos(2x)$$

and $f(0) = 1$, $f'(0) = -1$. If $g(x) = f(x) + f(-x)$, find $g(0)$ and show that $g'(0) = 0$.

(ii) Show that

$$g''(x) + g(x) = 6\cos(2x),$$

and hence find $g(x)$.

(iii) Similarly, if $h(x) = f(x) - f(-x)$, find $h(x)$ and show that

$$f(x) = 2\cos(x) - \cos(2x) - x.$$

Section B: Mechanics

9 A child's toy consists of a solid cone of height λa and a solid hemisphere of radius a, made out of the same uniform material and fastened together so that their plane faces coincide. (Thus the diameter of the hemisphere is equal to that of the base of the cone.)

(i) Show that if $\lambda < \sqrt{3}$ the toy will always move to an upright position if placed with the surface of the hemisphere on a horizontal table, but that if $\lambda > \sqrt{3}$ the toy may overbalance.

(ii) Show, however, that if the toy is placed with the surface of the cone touching the table it will remain there whatever the value of λ.

[The centre of gravity of a uniform solid cone of height h is a height $h/4$ above its base. The centre of gravity of a uniform solid hemisphere of radius a is at distance $3a/8$ from the centre of its base.]

10 The plot of 'Rhode Island Red and the Henhouse of Doom' calls for the heroine to cling on to the circumference of a fairground wheel of radius a rotating with constant angular velocity ω about its horizontal axis and then let go. Let ω_0 be the largest value of ω for which it is not possible for her subsequent path to carry her higher than the top of the wheel.

(i) Find ω_0 in terms of a and g.

(ii) If $\omega > \omega_0$ show that the greatest height above the top of the wheel to which she can rise is

$$\frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

11 A particle hangs in equilibrium from the ceiling of a stationary lift, to which it is attached by an elastic string of natural length l extended to a length $l+a$. The lift now descends with constant acceleration f such that $0<f<g/2$.

(i) Show that the extension y of the string from its equilibrium length satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\frac{g}{a}y=g-f.$$

(ii) Hence show that the string never becomes slack and the amplitude of the oscillation of the particle is af/g.

(iii) After a time T the lift stops accelerating and moves with constant velocity. Show that the string never becomes slack and the amplitude of the oscillation is now

$$\frac{2af}{g}|\sin\tfrac{1}{2}\omega T|,$$

where $\omega^2=g/a$.

Section C: Probability and Statistics

12 **(i)** Let X_1, X_2, ..., X_n be independent random variables each of which is uniformly distributed on $[0,1]$. Let Y be the largest of X_1, X_2, ..., X_n. By using the fact that $Y < \lambda$ if and only if $X_j < \lambda$ for $1 \leqslant j \leqslant n$, find the probability density function of Y. Show that the variance of Y is

$$\frac{n}{(n+2)(n+1)^2}.$$

(ii) The probability that a neon light switched on at time 0 will have failed by a time $t > 0$ is $1 - \mathrm{e}^{-t/\lambda}$ where $\lambda > 0$. I switch on n independent neon lights at time zero. Show that the expected time until the first failure is λ/n.

13 **(i)** By considering the coefficients of t^n in the equation

$$(1+t)^n(1+t)^n = (1+t)^{2n},$$

or otherwise, show that

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{r}\binom{n}{n-r} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

(ii) The large American city of Triposville is laid out in a square grid with equally spaced streets running east-west and avenues running north-south. My friend is staying at a hotel n avenues west and n streets north of my hotel. Both hotels are at intersections. We set out from our own hotels at the same time. We walk at the same speed, taking 1 minute to go from one intersection to the next. Every time I reach an intersection I go north with probability $1/2$ or west with probability $1/2$. Every time my friend reaches an intersection she goes south with probability $1/2$ or east with probability $1/2$. Our choices are independent of each other and of our previous decisions. Indicate by a sketch or by a brief description the set of points where we could meet. Find the probability that we meet.

(iii) Suppose that I oversleep and leave my hotel $2k$ minutes later than my friend leaves hers, where k is an integer and $0 \leqslant 2k \leqslant n$. Find the probability that we meet. Have you any comment? If $n = 1$ and I leave my hotel 1 minute later than my friend leaves hers, what is the probability that we meet and why?

14 The random variable X is uniformly distributed on $[0, 1]$. A new random variable Y is defined by the rule

$$Y = \begin{cases} 1/4 & \text{if } X \leqslant 1/4, \\ X & \text{if } 1/4 \leqslant X \leqslant 3/4 \\ 3/4 & \text{if } X \geqslant 3/4. \end{cases}$$

(i) Find $\mathrm{E}(Y^n)$ for all integers $n \geqslant 1$.

(ii) Show that $\mathrm{E}(Y) = \mathrm{E}(X)$ and that

$$\mathrm{E}(X^2) - \mathrm{E}(Y^2) = \frac{1}{24}.$$

(iii) By using the fact that $4^n = (3+1)^n$, or otherwise, show that $\mathrm{E}(X^n) > \mathrm{E}(Y^n)$ for $n \geqslant 2$.

(iv) Suppose that Y_1, Y_2, ... are independent random variables each having the same distribution as Y. Find, to a good approximation, K such that

$$\mathrm{P}(Y_1 + Y_2 + \cdots + Y_{240000} < K) = 3/4.$$

Section A: Pure Mathematics

1 **(i)** By considering $(1 + x + x^2 + \cdots + x^n)(1 - x)$ show that, if $x \neq 1$,

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(ii) By differentiating both sides and setting $x = -1$ show that

$$1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n$$

takes the value $-n/2$ is n is even and the value $(n+1)/2$ if n is odd.

(iii) Show that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}(An^2 + Bn)$$

where the constants A and B are to be determined.

2 I have n fence posts placed in a line and, as part of my spouse's birthday celebrations, I wish to paint them using three different colours red, white and blue in such a way that no adjacent fence posts have the same colours. (This allows the possibility of using fewer than three colours as well as exactly three.) Let r_n be the number of ways (possibly zero) that I can paint them if I paint the first and the last post red and let s_n be the number of ways that I can paint them if I paint the first post red but the last post either of the other two colours. Explain why $r_{n+1} = s_n$ and find $r_n + s_n$. Hence find the value of $r_{n+1} + r_n$ for all $n \geqslant 1$.

(i) Prove, by induction, that

$$r_n = \frac{2^{n-1} + 2(-1)^{n-1}}{3}.$$

(ii) Find the number of ways of painting n fence posts (where $n \geqslant 3$) placed in a circle using three different colours in such a way that no adjacent fence posts have the same colours.

3 The Tour de Clochemerle is not yet as big as the rival Tour de France. This year there were five riders, Arouet, Barthes, Camus, Diderot and Eluard, who took part in five stages. The winner of each stage got 5 points, the runner up 4 points and so on down to the last rider who got 1 point. The total number of points acquired over the five states was the rider's score. Each rider obtained a different score overall and the riders finished the whole tour in alphabetical order with Arouet gaining a magnificent 24 points. Camus showed consistency by gaining the same position in four of the five stages and Eluard's rather dismal performance was relieved by a third place in the fourth stage and first place in the final stage. Explain why Eluard must have received 11 points in all and find the scores obtained by Barthes, Camus and Diderot.

Where did Barthes come in the final stage?

4 **(i)** Let

$$u_n = \int_0^{\frac{1}{2}\pi} \sin^n t \, \mathrm{d}t$$

for each integer $n \geqslant 0$. By integrating

$$\int_0^{\frac{1}{2}\pi} \sin t \sin^{n-1} t \, \mathrm{d}t$$

by parts, or otherwise, obtain a formula connecting u_n and u_{n-2} when $n \geqslant 2$ and deduce that

$$n u_n u_{n-1} = (n-1)\, u_{n-1} u_{n-2}$$

for all $n \geqslant 2$. Deduce that

$$n u_n u_{n-1} = \tfrac{1}{2}\pi.$$

(ii) Sketch graphs of $\sin^n t$ and $\sin^{n-1} t$, for $0 \leqslant t \leqslant \frac{1}{2}\pi$, on the same diagram and explain why $0 < u_n < u_{n-1}$. By using the result of the previous paragraph show that

$$n u_n^2 < \tfrac{1}{2}\pi < n u_{n-1}^2$$

for all $n \geqslant 1$. Hence show that

$$\left(\frac{n}{n+1}\right) \tfrac{1}{2}\pi < n u_n^2 < \tfrac{1}{2}\pi$$

and deduce that $n u_n^2 \to \frac{1}{2}\pi$ as $n \to \infty$.

5 The famous film star Birkhoff Maclane is sunning herself by the side of her enormous circular swimming pool (with centre O) at a point A on its circumference. She wants a drink from a small jug of iced tea placed at the diametrically opposite point B. She has three choices:

(i) to swim directly to B.

(ii) to choose θ with $0 < \theta < \pi$, to run round the pool to a point X with $\angle AOX = \theta$ and then to swim directly from X to B.

(iii) to run round the pool from A to B.

She can run k times as fast as she can swim and she wishes to reach her tea as fast as possible. Explain, with reasons, which of **(i)**, **(ii)** and **(iii)** she should choose for each value of k. Is there one choice from **(i)**, **(ii)** and **(iii)** she will never take whatever the value of k?

6 **(i)** If u and v are the two roots of $z^2 + az + b = 0$, show that $a = -u - v$ and $b = uv$.

(ii) Let $\alpha = \cos(2\pi/7) + \mathrm{i}\sin(2\pi/7)$. Show that α is a root of $z^6 - 1 = 0$ and express the roots in terms of α. The number $\alpha + \alpha^2 + \alpha^4$ is a root of a quadratic equation

$$z^2 + Az + B = 0$$

where A and B are real. By guessing the other root, or otherwise, find the numerical values of A and B.

(iii) Show that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2},$$

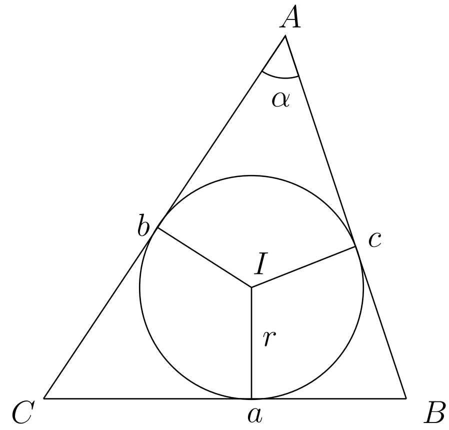
and evaluate

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7},$$

making it clear how you determine the sign of your answer.

7 The diagram shows a circle, of radius r and centre I, touching the three sides of a triangle ABC. We write a for the length of BC and α for the angle $\angle BAC$ and so on.

Let $s = \frac{1}{2}(a + b + c)$ and let $\triangle$ be the area of the triangle.



(i) By considering the area of the triangles AIB, BIC and CIA, or otherwise, show that $\Delta = rs$.

(ii) By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that

$$\Delta^2 = \tfrac{1}{16}[4b^2c^2 - (2bc\cos\alpha)^2].$$

Now use the formula $a^2 = b^2 + c^2 - 2bc\cos\alpha$ to show that

$$\Delta^2 = \tfrac{1}{16}[(a^2 - (b-c)^2)((b+c)^2 - a^2)]$$

and deduce that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(iii) A hole in the shape of the triangle ABC is cut in the top of a level table. A sphere of radius R rests in the hole. Find the height of the centre of the sphere above the level of the table top, expressing your answer in terms of a, b, c, s and R.

8 **(i)** If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

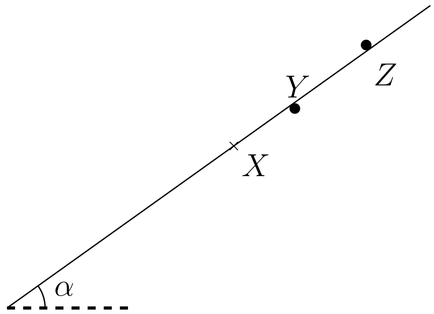
$$x(t) = K + K\frac{1 - \mathrm{e}^{-2Kt}}{1 + \mathrm{e}^{-2Kt}}.$$

(ii) Sketch, for $t \geqslant 0$, the graph of x against t. What happens to $x(t)$ as $t \to \infty$?

(iii) Now suppose that the situation is as described in the first paragraph, except that we remove the bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x. By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \to K + \alpha$ as $t \to \infty$.

Section B: Mechanics

9



Two thin horizontal bars are parallel and fixed at a distance d apart, and the plane containing them is at an angle α to the horizontal. A thin uniform rod rests in equilibrium in contact with the bars under one and above the other and perpendicular to both. The diagram shows the bards (in cross section and exaggerated in size) with the rod over one bar at Y and under the other at Z. (Thus YZ has length d.) The centre of the rod is at X and XZ has length l. The coefficient of friction between the rod and each bar is μ.

(i) Explain why we must have $l \leqslant d$.

(ii) Find, in terms of d, l and α, the least possible value of μ.

(iii) Verify that, when $l = 2d$, your result shows that

$$\mu \geqslant \frac{1}{3} \tan \alpha.$$

10 Three small spheres of masses m_1, m_2 and m_3, move in a straight line on a smooth horizontal table. (Their order on the straight line is the order given.) The coefficient of restitution between any two spheres is e. The first moves with velocity u towards the second whilst the second and third are at rest. After the first collision the second sphere hits the third after which the velocity of the second sphere is u. Find m_1 in terms of m_2, m_3 and e. deduce that

$$m_2 e > m_3(1 + e + e^2).$$

Suppose that the relation between m_1, m_2 and m_3 is that in the formula you found above, but that now the first sphere initially moves with velocity u and the other two spheres with velocity v, all in the same direction along the line. If $u > v > 0$ use the first part to find the velocity of the second sphere after two collisions have taken place. (You should not need to make any substantial computations but you should state your argument clearly.)

11 Two identical particles of unit mass move under gravity in a medium for which the magnitude of the retarding force on a particle is k times its speed. The first particle is allowed to fall from rest at a point A whilst, at the same time, the second is projected upwards with speed u from a point B a positive distance d vertically above A. Find their distance apart after a time t and show that this distance tends to the value

$$d + \frac{u}{k}$$

as $t \to \infty$.

Section C: Probability and Statistics

12 Bread roll throwing duels at the Drones' Club are governed by a strict etiquette. The two duellists throw alternatively until one is hit, when the other is declared the winner. If Percy has probability $p > 0$ of hitting his target and Rodney has probability $r > 0$ of hitting his, show that, if Percy throws first, the probability that he beats Rodney is

$$\frac{p}{p + r - pr}.$$

Algernon, Bertie and Cuthbert decide to have a three sided duel in which they throw in order $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$ except that anyone who is hit must leave the game. Cuthbert always his target, Bertie hits his target with probability $3/5$ and Algernon hits his target with probability $2/5$. Bertie and Cuthbert will always aim at each other if they are both still in the duel. Otherwise they aim at Algernon. With his first shot Algernon may aim at either Bertie or Cuthbert or deliberately miss both. Faced with only one opponent Algernon will aim at him. What are Algernon's changes of winning if he:

(i) hits Cuthbert with his first shot?

(ii) hits Bertie with his first shot?

(iii) misses with his first shot?

Advise Algernon as to his best plan and show that, if he uses this plan, his probability of winning is $226/475$.

13 Fly By Night Airlines run jumbo jets which seat N passengers. From long experience they know that a very small proportion ϵ of their passengers fail to turn up. They decide to sell $N+k$ tickets for each flight. If k is very small compared with N explain why they might expect

$$\mathrm{P}(r \text{ passengers fail to turn up}) = \frac{\lambda^r}{r!}\mathrm{e}^{-\lambda}$$

approximately, with $\lambda = N\epsilon$. For the rest of the question you may assume that the formula holds exactly.

Each ticket sold represents £A profit, but the airline must pay each passenger that it cannot fly £B where $B > A > 0$. Explain why, if r passengers fail to turn up, its profit, in pounds, is

$$A(N+k) - B\max(0, k-r),$$

where $\max(0, k-r)$ is the larger of 0 and $k-r$.

(i) Write down the expected profit u_k when $k = 0, 1, 2$ and 3.

(ii) Find $v_k = u_{k+1} - u_k$ for general k and show that $v_k > v_{k+1}$.

(iii) Show also that

$$v_k \to A - B$$

as $k \to \infty$.

(iv) Advise Fly By Night on how to choose k to maximise its expected profit u_k.

14 Suppose X is a random variable with probability density

$$\mathrm{f}(x) = Ax^2\exp(-x^2/2)$$

for $-\infty < x < \infty$. Find A.

You belong to a group of scientists who believe that the outcome of a certain experiment is a random variable with the probability density just given, while other scientists believe that the probability density is the same except with different mean (i.e. the probability density is $\mathrm{f}(x-\mu)$ with $\mu \neq 0$). In each of the following two cases decide whether the result given would shake your faith in your hypothesis, and justify your answer.

(i) A single trial produces the result 87.3.

(ii) 1000 independent trials produce results having a mean value 0.23.

[Great weight will be placed on clear statements of your reasons and none on the mere repetition of standard tests, however sophisticated, if unsupported by argument. There are several possible approaches to this question. For some of them it is useful to know that if Z is normal with mean 0 and variance 1 then $\mathrm{E}(Z^4) = 3$.]

Section A: Pure Mathematics

1 In this question we consider only positive, non-zero integers written out in the usual (decimal) way. We say, for example, that 207 ends in 7 and that 5310 ends in 1 followed by 0. Show that, if n does not end in 5 or an even number, then there exists m such that $n \times m$ ends in 1.

(i) Show that, given any n, we can find m such that $n \times m$ ends either in 1 or in 1 followed by one or more zeros.

(ii) Show that, given any n which ends in 1 or in 1 followed by one or more zeros, we can find m such that $n \times m$ contains all the digits $0, 1, 2, \ldots, 9$.

2 **(i)** If Q is a polynomial, m is an integer, $m \geqslant 1$ and $\mathrm{P}(x) = (x-a)^m \mathrm{Q}(x)$, show that

$$\mathrm{P}'(x) = (x-a)^{m-1}\mathrm{R}(x)$$

where R is a polynomial. Explain why $\mathrm{P}^{(r)}(a) = 0$ whenever $1 \leqslant r \leqslant m-1$. ($\mathrm{P}^{(r)}$ is the rth derivative of P.)

(ii) If

$$\mathrm{P}_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

for $n \geqslant 1$ show that P_n is a polynomial of degree n. By repeated integration by parts, or otherwise, show that, if $n-1 \geqslant m \geqslant 0$,

$$\int_{-1}^{1} x^m \mathrm{P}_n(x)\,\mathrm{d}x = 0$$

and find the value of

$$\int_{-1}^{1} x^n \mathrm{P}_n(x)\,\mathrm{d}x.$$

[**Hint.** *You may use the formula*

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,\mathrm{d}t = \frac{(2^{2n})(n!)^2}{(2n+1)!}$$

without proof if you need it. However some ways of doing this question do not use this formula.]

3 The function f satisfies $\mathrm{f}(0) = 1$ and

$$\mathrm{f}(x-y) = \mathrm{f}(x)\mathrm{f}(y) - \mathrm{f}(a-x)\mathrm{f}(a+y)$$

for some fixed number a and all x and y. Without making any further assumptions about the nature of the function show that $\mathrm{f}(a) = 0$.

Show that, for all t,

(i) $\mathrm{f}(t) = \mathrm{f}(-t)$,

(ii) $\mathrm{f}(2a) = -1$,

(iii) $\mathrm{f}(2a - t) = -\mathrm{f}(t)$,

(iv) $\mathrm{f}(4a + t) = \mathrm{f}(t)$.

Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = \pi/2$. Give an example of an non-constant function satisfying the conditions of the first paragraph with $a = -2$.

4 **(i)** By considering the area of the region defined in terms of Cartesian coordinates (x, y) by

$$\{(x,y):\ x^2+y^2=1,\ 0\leqslant y,\ 0\leqslant x\leqslant c\},$$

show that

$$\int_0^c (1-x^2)^{\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{2}[c(1-c^2)^{\frac{1}{2}}+\sin^{-1}c],$$

if $0 < c \leqslant 1$.

(ii) Show that the area of the region defined by

$$\left\{(x,y):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0\leqslant y,\ 0\leqslant x\leqslant c\right\},$$

is

$$\frac{ab}{2}\left[\frac{c}{a}\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}+\sin^{-1}\left(\frac{c}{a}\right)\right],$$

if $0 < c \leqslant a$ and $0 < b$.

(iii) Suppose that $0 < b \leqslant a$. Show that the area of intersection $E \cap F$ of the two regions defined by

$$E=\left\{(x,y):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\} \qquad \text{and} \qquad F=\left\{(x,y):\ \frac{x^2}{b^2}+\frac{y^2}{a^2}\leqslant 1\right\}$$

is

$$4ab\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right).$$

5 **(i)** Show that the equation

$$(x-1)^4+(x+1)^4=c$$

has exactly two real roots if $c > 2$, one root if $c = 2$ and no roots if $c < 2$.

(ii) How many real roots does the equation $(x-3)^4+(x-1)^4=c$ have?

(iii) How many real roots does the equation $|x-3|+|x-1|=c$ have?

(iv) How many real roots does the equation $(x-3)^3+(x-1)^3=c$ have?

[The answers to parts **(ii)**, **(iii)** and **(iv)** may depend on the value of c. You should give reasons for your answers.]

6 **(i)** Prove by induction, or otherwise, that, if $0 < \theta < \pi$,

$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r} = \frac{1}{\theta} - \cot\theta.$$

7 **(i)** Show that the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

where $a > 0$ and $f^2 + g^2 > ac$ represents a circle in Cartesian coordinates and find its centre.

(ii) The smooth and level parade ground of the First Ruritanian Infantry Division is ornamented by two tall vertical flagpoles of heights h_1 and h_2 a distance d apart. As part of an initiative test a soldier has to march in such a way that he keeps the angles of elevation of the tops of the two flagpoles equal to one another. Show that if the two flagpoles are of different heights he will march in a circle. What happens if the two flagpoles have the same height?

(iii) To celebrate the King's birthday a third flagpole is added. Soldiers are then assigned to each of the three different pairs of flagpoles and are told to march in such a way that they always keep the tops of their two assigned flagpoles at equal angles of elevation to one another. Show that, if the three flagpoles have different heights h_1, h_2 and h_3 and the circles in which the soldiers march have centres of (x_{ij}, y_{ij}) (for the flagpoles of height h_i and h_j) relative to Cartesian coordinates fixed in the parade ground, then the x_{ij} satisfy

$$h_3^2\left(h_1^2 - h_2^2\right)x_{12} + h_1^2\left(h_2^2 - h_3^2\right)x_{23} + h_2^2\left(h_3^2 - h_1^2\right)x_{31} = 0,$$

and the same equation connects the y_{ij}. Deduce that the three centres lie in a straight line.

8 **(i)** '24 Hour Spares' stocks a small, widely used and cheap component. Every T hours X units arrive by lorry from the wholesaler, for which the owner pays a total $\pounds(a+qX)$. It costs the owner $\pounds b$ per hour to store one unit. If she has the units in stock she expects to sell r units per hour at $\pounds(p+q)$ per unit. The other running costs of her business remain at $\pounds c$ pounds an hour irrespective of whether she has stock or not. (All of the quantities T, X, a, b, r, q, p and c are greater than 0.) Explain why she should take $X \leqslant rT$.

(ii) Given that the process may be assumed continuous (the items are very small and she sells many each hour), sketch $S(t)$ the amount of stock remaining as a function of t the time from the last delivery. Compute the total profit over each period of T hours. Show that, if T is fixed with $T \geqslant p/b$, the business can be made profitable if

$$p^2 > 2\frac{(a+cT)b}{r}.$$

Section B: Mechanics

9 A light rod of length $2a$ is hung from a point O by two light inextensible strings OA and OB each of length b and each fixed at O. A particle of mass m is attached to the end A and a particle of mass $2m$ is attached to the end B.

(i) Show that, in equilibrium, the angle θ that the rod makes the horizontal satisfies the equation

$$\tan\theta = \frac{a}{3\sqrt{b^2 - a^2}}.$$

(ii) Express the tension in the string AO in terms of m, g, a and b.

10 A truck is towing a trailer of mass m across level ground by means of an elastic rope of natural length l whose modulus of elasticity is λ. At first the rope is slack and the trailer stationary. The truck then accelerates until the rope becomes taut and thereafter the truck travels in a straight line at a constant speed u. Assuming that the effect of friction on the trailer is negligible, show that the trailer will collide with the truck at a time

$$\pi\left(\frac{lm}{\lambda}\right)^{\frac{1}{2}} + \frac{l}{u}$$

after the rope first becomes taut.

11 As part of a firework display a shell is fired vertically upwards with velocity v from a point on a level stretch of ground. When it reaches the top of its trajectory an explosion it splits into two equal fragments each travelling at speed u but (since momentum is conserved) in exactly opposite (not necessarily horizontal) directions. Show, neglecting air resistance, that the greatest possible distance between the points where the two fragments hit the ground is $2uv/g$ if $u \leqslant v$ and $(u^2 + v^2)/g$ if $v \leqslant u$.

Section C: Probability and Statistics

12 **(i)** Calamity Jane sits down to play the game of craps with Buffalo Bill. In this game she rolls two fair dice. If, on the first throw, the sum of the dice is $2, 3$ or 12 she loses, while if it is 7 or 11 she wins. Otherwise Calamity continues to roll the dice until either the first sum is repeated, in which case she wins, or the sum is 7, in which case she loses. Find the probability that she wins on the first throw.

(ii) Given that she throws more than once, show that the probability that she wins on the nth throw is

$$\frac{1}{48}\left(\frac{3}{4}\right)^{n-2} + \frac{1}{27}\left(\frac{13}{18}\right)^{n-2} + \frac{25}{432}\left(\frac{25}{36}\right)^{n-2}.$$

(iii) Given that she throws more than m times, where $m > 1$, what is the probability that she wins on the nth throw?

13 The makers of Cruncho ('The Cereal Which Cares') are giving away a series of cards depicting n great mathematicians. Each packet of Cruncho contains one picture chosen at random. Show that when I have collected r different cards the expected number of packets I must open to find a new card is $n/(n-r)$ where $0 \leqslant r \leqslant n-1$.

(i) Show by means of a diagram, or otherwise, that

$$\frac{1}{r+1} \leqslant \int_r^{r+1} \frac{1}{x}\,\mathrm{d}x \leqslant \frac{1}{r}$$

and deduce that

$$\sum_{r=2}^{n} \frac{1}{r} \leqslant \ln n \leqslant \sum_{r=1}^{n-1} \frac{1}{r}$$

for all $n \geqslant 2$.

(ii) My children will give me no peace until we have the complete set of cards, but I am the only person in our household prepared to eat Cruncho and my spouse will only buy the stuff if I eat it. If n is large, roughly how many packets must I expect to consume before we have the set?

14 When Septimus Moneybags throws darts at a dart board they are certain to end on the board (a disc of radius a) but, it must be admitted, otherwise are uniformly randomly distributed over the board.

(i) Show that the distance R that his shot lands from the centre of the board is a random variable with variance $a^2/18$.

(ii) At a charity fête he can buy m throws for $£(12+m)$, but he must choose m before he starts to throw. If at least one of his throws lands with $a/\sqrt{10}$ of the centre he wins back $£12$. In order to show that a good sport he is, he is determined to play but, being a careful man, he wishes to choose m so as to minimise his expected loss. What values of m should he choose?

Section A: Pure Mathematics

1 In the game of "Colonel Blotto" there are two players, Adam and Betty. First Adam chooses three non-negative integers a_1, a_2 and a_3, such that $a_1 + a_2 + a_3 = 9$, and then Betty chooses non-negative integers b_1, b_2 and b_3, such that $b_1 + b_2 + b_3 = 9$. If $a_1 > b_1$ then Adam scores one point; if $a_1 < b_1$ then Betty scores one point; and if $a_1 = b_1$ no points are scored. Similarly for a_2, b_2 and a_3, b_3. The winner is the player who scores the greater number of points: if the socres are equal then the game is drawn.

(i) Show that, if Betty knows the numbers a_1, a_2 and a_3, she can always choose her numbers so that she wins.

(ii) Show that Adam can choose a_1, a_2 and a_3 in such a way that he will never win no matter what Betty does.

(iii) Now suppose that Adam is allowed to write down two triples of numbers and that Adam wins unless Betty can find one triple that beats both of Adam's choices (knowing what they are). Confirm that Adam wins by writing down $(5, 3, 1)$ and $(3, 1, 5)$.

2 **(i)** Evaluate

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,\mathrm{d}x,$$

where m, n are integers, taking into account any special cases that arise.

(ii) Find $\int \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x$.

3 **(i)** Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y - 3y^2 = -2$$

by making the substitution $y = -\frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}$.

(ii) Solve the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^2y^2 = 1$$

by making the substitution

$$y = \frac{1}{x} + \frac{1}{v},$$

where v is a function of x.

4 Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

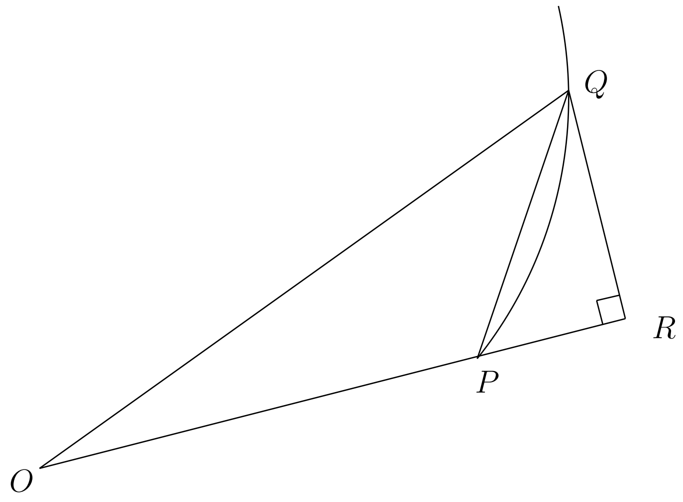
$$\mathbf{p}_1 = (2,1,-1) \qquad \mathbf{p}_2 = (1,0,-2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4,3,0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0,-3,1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2)\cdot(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

5



In the diagram, O is the origin, P is a point of a curve $r = r(\theta)$ with coordinates (r, θ) and Q is another point of the curve, close to P, with coordinates $(r + \delta r, \theta + \delta\theta)$. The angle $\angle PRQ$ is a right angle.

(i) By calculating $\tan \angle QPR$, show that the angle at which the curve cuts OP is

$$\tan^{-1}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right).$$

(ii) Let α be a constant angle, $0 < \alpha < \frac{1}{2}\pi$. The curve with the equation

$$r = \mathrm{e}^{\theta \cot \alpha}$$

in polar coordinates is called an *equiangular spiral*. Show that it cuts every radius line at an angle α. Sketch the spiral.

(iii) Find the length of the complete turn of the spiral beginning at $r = 1$ and going outwards. What is the total length of the part of the spiral for which $r \leqslant 1$?

[You may assume that the arc length s of the curve satisfies

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2.]$$

6 *In this question,* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{X}$ *are non-zero* 2×2 *real matrices.*

Are the following assertions true or false? You must provide a proof or a counterexample in each case.

(i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.

(ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

(iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

7 The integers a, b and c satisfy

$$2a^2 + b^2 = 5c^2.$$

(i) By considering the possible values of $a \pmod 5$ and $b \pmod 5$, show that a and b must both be divisible by 5.

(ii) By considering how many times a, b and c can be divided by 5, show that the only solution is $a = b = c = 0$.

8 **(i)** Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leqslant \left(\frac{a_1 + a_2}{2}\right)^2 .$$

(ii) Prove by induction that for all positive integers m

$$a_1 \cdots a_{2^m} \leqslant \left(\frac{a_1 + \cdots + a_{2^m}}{2^m}\right)^{2^m} . \qquad (*)$$

(iii) If $n < 2^m$, put $b_1 = a_2$, $b_2 = a_2, \cdots, b_n = a_n$ and $b_{n+1} = \cdots = b_{2^m} = A$, where

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

By applying $(*)$ to the b_i, show that

$$a_1 \cdots a_n A^{(2^m - n)} \leqslant A^{2^m}$$

(notice that $b_1 + \cdots + b_n = nA$). Deduce the (arithmetic mean)/(geometric mean) inequality

$$(a_1 \cdots a_n)^{1/n} \leqslant \frac{a_1 + \cdots + a_n}{n}.$$

9 *In this question, the argument of a complex number is chosen to satisfy* $0 \leqslant \arg z < 2\pi$.

(i) Let z be a complex number whose imaginary part is positive. What can you say about $\arg z$?

(ii) The complex numbers z_1, z_2 and z_3 all have positive imaginary part and $\arg z_1 < \arg z_2 < \arg z_3$.

Draw a diagram that shows why

$$\arg z_1 < \arg(z_1 + z_2 + z_3) < \arg z_3.$$

(iii) Prove that $\arg(z_1 z_2 z_3)$ is never equal to $\arg(z_1 + z_2 + z_3)$.

10 **(i)** Verify that if

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}$$

then $\mathbf{PAP}$ is a diagonal matrix.

(ii) Put $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. By writing

$$\mathbf{x} = \mathbf{P}\mathbf{x}_1 + \mathbf{a}$$

for a suitable vector $\mathbf{a}$, show that the equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0,$$

where $\mathbf{b} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, becomes

$$3x_1^2 - y_1^2 = c$$

for some constant c (which you should find).

Section B: Mechanics

11 *In this question, take the value of g to be* $10\ \mathrm{ms}^{-2}$.

A body of mass m kg is dropped vertically into a deep pool of liquid. Once in the liquid, it is subject to gravity, an upward buoyancy force of $\frac{6}{5}$ times its weight, and a resistive force of $2mv^2\mathrm{N}$ opposite to its direction of travel when it is travelling at speed $v\ \mathrm{ms}^{-1}$.

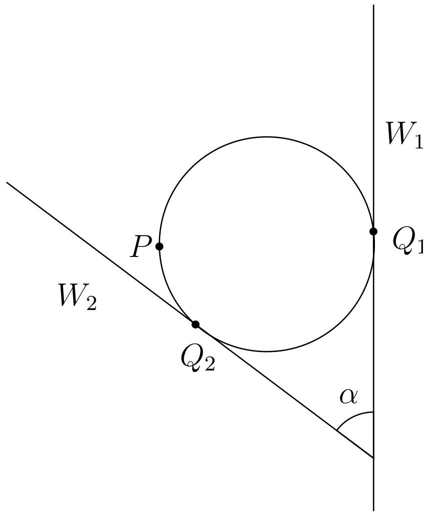
(i) Show that the body stops sinking less than $\frac{1}{4}\pi$ seconds after it enters the pool.

(ii) Suppose now that the body enters the liquid with speed $1\ \mathrm{ms}^{-1}$. Show that the body descends to a depth of $\frac{1}{4}\ln 2$ metres and that it returns to the surface with speed $\frac{1}{\sqrt{2}}\ \mathrm{ms}^{-1}$, at a time

$$\frac{\pi}{8} + \frac{1}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

seconds after entering the pool.

12



A uniform sphere of mass M and radius r rests between a vertical wall W_1 and an inclined plane W_2 that meets W_1 at an angle α. Q_1 and Q_2 are the points of contact of the sphere with W_1 and W_2 resectively, as shown in the diagram. A particle of mass m is attached to the sphere at P, where PQ_1 is a diameter, and the system is released. The sphere is on the point of slipping at Q_1 and at Q_2.

(i) Show that if the coefficients of friction between the sphere and W_1 and W_2 are μ_1 and μ_2 respectively, then

$$m = \frac{\mu_2 + \mu_1 \cos\alpha - \mu_1\mu_2 \sin\alpha}{(2\mu_1\mu_2 + 1)\sin\alpha + (\mu_2 - 2\mu_1)\cos\alpha - \mu_2} M.$$

(ii) If the sphere is on the point of rolling about Q_2 instead of slipping, show that

$$m = \frac{M}{\sec\alpha - 1}.$$

13 The force F of repulsion between two particles with positive charges Q and Q' is given by $F = kQQ'/r^2$, where k is a positive constant and r is the distance between the particles. Two small beads P_1 and P_2 are fixed to a straight horizontal smooth wire, a distance d apart. A third bead P_3 of mass m is free to move along the wire between P_1 and P_3. The beads carry positive electrical charges Q_1, Q_2 and Q_3.

(i) If P_3 is in equilibrium at a distance a from P_1, show that

$$a = \frac{d\sqrt{Q_1}}{\sqrt{Q_1} + \sqrt{Q_2}}.$$

(ii) Suppose that P_3 is displaced slightly from its equilibrium position and released from rest. Show that it performs approximate simple harmonic motion with period

$$\frac{\pi d}{(\sqrt{Q_1} + \sqrt{Q_2})^2}\sqrt{\frac{2md\sqrt{Q_1 Q_2}}{kQ_3}}.$$

[You may use the fact that $\dfrac{1}{(a+y)^2} \approx \dfrac{1}{a^2} - \dfrac{2y}{a^3}$ for small y.]

14 A ball of mass m is thrown vertically upwards from the floor of a room of height h with speed $\sqrt{2kgh}$, where $k > 1$. The coefficient of restitution between the ball and the ceiling or floor is a. Both the ceiling and floor are level.

(i) Show that the kinetic energy of the ball immediately before hitting the ceiling for the nth time is

$$mgh\left(a^{4n-4}(k-1) + \frac{a^{4n-4} - 1}{a^2 + 1}\right).$$

(ii) Hence show that the number of times the ball hits the ceiling is at most

$$1 - \frac{\ln[a^2(k-1) + k]}{4\ln a}.$$

Section C: Probability and Statistics

15 Two computers, LEP and VOZ are programmed to add numbers after first approximating each number by an integer. LEP approximates the numbers by rounding: that is, it replaces each number by the nearest integer. VOZ approximates by truncation: that is, it replaces each number by the largest integer less than or equal to the number. The fractional parts of the numbers to be added are uniformly and independently distributed. (The fractional part of a number a is $a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ is the largest integer less than or equal to a.) Both computers approximate and add 1500 numbers.

(i) For each computer, find the probability that the magnitude of error in the answer will exceed 15.

(ii) How many additions can LEP perform before the probability that the magnitude of error is less than 10 drops below 0.9?

16 At the terminus of a bus route, passengers arrive at an average rate of 4 per minute according to a Poisson process. Each minute, on the minute, one bus arrives with probability $\frac{1}{4}$, independently of the arrival of passengers or previous buses. Just after eight o'clock there is no-one at the bus stop.

(i) What is the probability that the first bus arrives at n minutes past 8?

(ii) If the first bus arrives at 8:05, what is the probability that there are m people waiting for it?

(iii) Each bus can take 25 people and, since it is the terminus, the bus arrive empty. Explain carefully how you would calculate, to two significant figures, the probability that when the first bus arrives it is unable to pick up all the passengers. Your method should need the use of a calculator and standard tables only. There is no need to carry out the calculation.

Section A: Pure Mathematics

1 Find the limit, as $n \to \infty$, of each of the following. You should explain your reasoning briefly.

$$\textbf{(i)}\ \frac{n}{n+1}, \qquad \textbf{(ii)}\ \frac{5n+1}{n^2-3n+4}, \qquad \textbf{(iii)}\ \frac{\sin n}{n},$$

$$\textbf{(iv)}\ \frac{\sin(1/n)}{(1/n)}, \qquad \textbf{(v)}\ (\arctan n)^{-1}, \qquad \textbf{(vi)}\ \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}.$$

2 Suppose that y satisfies the differential equation

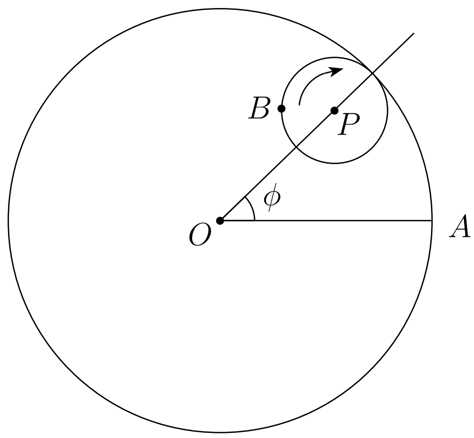
$$y = x\frac{\mathrm{d}y}{\mathrm{d}x} - \cosh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right). \qquad (*)$$

(i) By differentiating both sides of $(*)$ with respect to x, show that either

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad \text{or} \qquad x - \sinh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

(ii) Find the general solutions of each of these two equations. Determine the solutions of $(*)$.

3 In the figure, the large circle with centre O has radius 4 and the small circle with centre P has radius 1. The small circle rolls around the inside of the larger one. When P was on the line OA (before the small circle began to roll), the point B was in contact with the point A on the large circle.



(i) Sketch the curve C traced by B as the circle rolls. Show that if we take O to be the origin of cartesian coordinates and the line OA to be the x-axis (so that A is the point $(4, 0)$) then B is the point

$$(3\cos\phi + \cos 3\phi, 3\sin\phi - \sin 3\phi).$$

(ii) It is given that the area of the region enclosed by the curve C is

$$\int_0^{2\pi} x\frac{\mathrm{d}y}{\mathrm{d}\phi}\,\mathrm{d}\phi,$$

where B is the point (x, y). Calculate this area.

4 $\Diamond$ is an operation which take polynomials in x to polynomials in x; that is, given a polynomial $\mathrm{h}(x)$ there is another polynomial called $\Diamond\mathrm{h}(x)$. It is given that, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are any two polynomials in x, the following are always true:

(i) $\Diamond(\mathrm{f}(x)\mathrm{g}(x)) = \mathrm{g}(x)\Diamond\mathrm{f}(x) + \mathrm{f}(x)\Diamond\mathrm{g}(x)$,

(ii) $\Diamond(\mathrm{f}(x) + \mathrm{g}(x)) = \Diamond\mathrm{f}(x) + \Diamond\mathrm{g}(x)$,

(iii) $\Diamond x = 1$

(iv) if λ is a constant then $\Diamond(\lambda\mathrm{f}(x)) = \lambda\Diamond\mathrm{f}(x)$.

(i) Show that, if $\mathrm{f}(x)$ is a constant (i.e., a polynomial of degree zero), then $\Diamond\mathrm{f}(x) = 0$.

(ii) Calculate $\Diamond x^2$ and $\Diamond x^3$.

(iii) Prove that $\Diamond\mathrm{h}(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{h}(x))$ for any polynomial $\mathrm{h}(x)$.

5 **(i)** Explain what is meant by the order of an element g of a group G.

(ii) The set S consists of all 2×2 matrices whose determinant is 1.

(iii) Find the inverse of the element $\mathbf{A}$ of S, where

$$\mathbf{A} = \begin{pmatrix} w & x \\ y & x \end{pmatrix}.$$

(iv) Show that S is a group under matrix multiplication (you may assume that matrix multiplication is associative).

(v) For which elements $\mathbf{A}$ is $\mathbf{A}^{-1} = \mathbf{A}$?

(vi) Which element or elements have order 2?

(vii) Show that the element $\mathbf{A}$ of S has order 3 if, and only if, $w + z + 1 = 0$.

(viii) Write down one such element.

6 **(i)** Sketch the graphs of $y = \sec x$ and $y = \ln(2\sec x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2\sec x)$$

has no solution with $0 \leqslant x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large.

(ii) Explain why there is a number k_0 such that

$$k_0 x = \ln(2\sec x)$$

has exactly one solution with $0 \leqslant x < \frac{1}{2}\pi$.

(iii) Let x_0 be this solution, so that $0 \leqslant x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2\sec x)$. Show that

$$x_0 = \cot x_0 \ln(2\sec x_0).$$

(iv) Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0.

7 The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c. Express p, q and r in terms of a, b and c.

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

8 Calculate the following integrals

(i) $\displaystyle\int \frac{x}{(x-1)(x^2-1)}\,\mathrm{d}x;$

(ii) $\displaystyle\int \frac{1}{3\cos x + 4\sin x}\,\mathrm{d}x;$

(iii) $\displaystyle\int \frac{1}{\sinh x}\,\mathrm{d}x.$

9 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane.

(i) Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters.

(ii) By writing this equation in the form $\mathbf{r}\cdot\mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda+\mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) + \mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

for all scalars λ, μ.

(iii) Deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

(iv) Say briefly what happens if A, B, C and O are all in the same plane.

10 Let α be a fixed angle, $0 < x \leqslant \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $\arg\left(\dfrac{z-1}{z}\right) = \alpha,$

(ii) $\arg\left(\dfrac{z-1}{z}\right) = \alpha - \pi,$

(iii) $\left|\dfrac{z-1}{z}\right| = 1.$

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

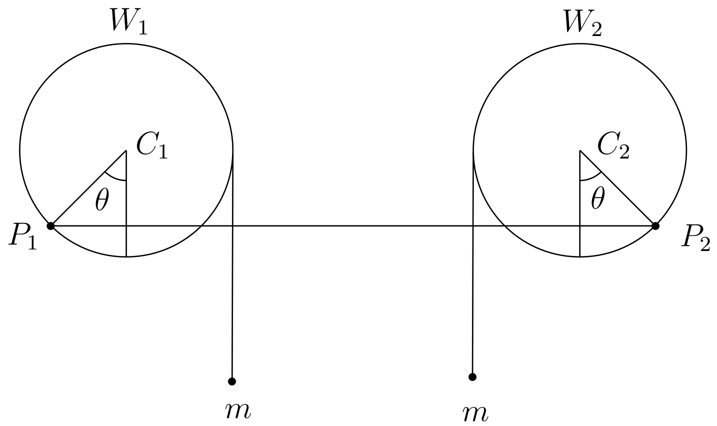
show, by considering $\arg w$, that w is real.

Section B: Mechanics

11 I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h. It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V, at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geqslant g(2h + d).$$

12 In the figure, W_1 and W_2 are wheels, both of radius r. Their centres C_1 and C_2 are fixed at the same height, a distance d apart, and each wheel is free to rotate, without friction, about its centre. Both wheels are in the same vertical plane. Particles of mass m are suspended from W_1 and W_2 as shown, by light inextensible strings would round the wheels. A light elastic string of natural length d and modulus elasticity λ is fixed to the rims of the wheels at the points P_1 and P_2. The lines joining C_1 to P_1 and C_2 to P_2 both make an angle θ with the vertical. The system is in equilibrium.



Show that

$$\sin 2\theta = \frac{mgd}{\lambda r}.$$

For what value or values of λ (in terms of m, d, r and g) are there

(i) no equilibrium positions,

(ii) just one equilibrium position,

(iii) exactly two equilibrium positions,

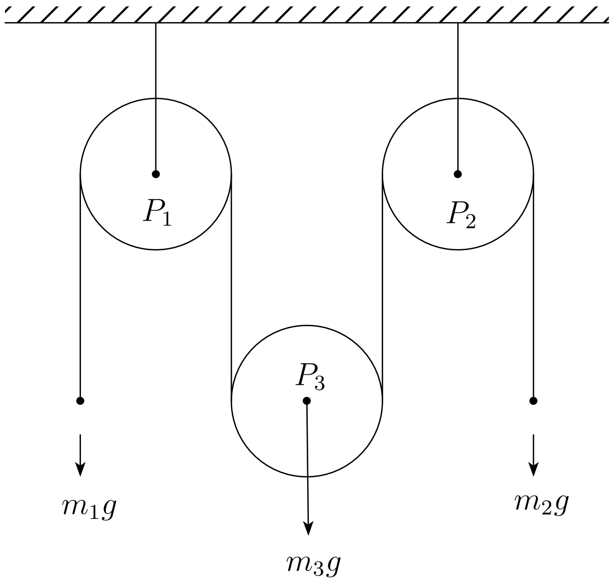
(iv) more than two equilibrium positions?

13 Two particles P_1 and P_2, each of mass m, are joined by a light smooth inextensible string of length ℓ. P_1 lies on a table top a distance d from the edge, and P_2 hangs over the edge of the table and is suspended a distance b above the ground. The coefficient of friction between P_1 and the table top is μ, and $\mu < 1$. The system is released from rest. Show that P_1 will fall off the edge of the table if and only if

$$\mu < \frac{b}{2d - b}.$$

Suppose that $\mu > b/(2d - b)$, so that P_1 comes to rest on the table, and that the coefficient of restitution between P_2 and the floor is e. Show that, if $e > 1/(2\mu)$, then P_1 comes to rest before P_2 bounces a second time.

14



In the diagram P_1 and P_2 are smooth light pulleys fixed at the same height, and P_3 is a third smooth light pulley, freely suspended. A smooth light inextensible string runs over P_1, under P_3 and over P_2, as shown: the parts of the string not in contact with any pulley are vertical. A particle of mass m_3 is attached to P_3. There is a particle of mass m_1 attached to the end of the string below P_1 and a particle of mass m_2 attached to the other end, below P_2. The system is released from rest. Find the tension in the string, and show that the pulley P_3 will remain at rest if

$$4m_1m_2 = m_3(m_1 + m_2).$$

Section C: Probability and Statistics

15 A point moves in unit steps on the x-axis starting from the origin. At each step the point is equally likely to move in the positive or negative direction. The probability that after s steps it is at one of the points $x = 2, x = 3, x = 4$ or $x = 5$ is $\mathrm{P}(s)$.

(i) Show that $\mathrm{P}(5) = \frac{3}{16}$, $\mathrm{P}(6) = \frac{21}{64}$ and

$$\mathrm{P}(2k) = \binom{2k+1}{k-1}\left(\frac{1}{2}\right)^{2k}$$

where k is a positive integer. Find a similar expression for $\mathrm{P}(2k+1)$.

(ii) Determine the values of s for which $\mathrm{P}(s)$ has its greatest value.

16 A taxi driver keeps a packet of toffees and a packet of mints in her taxi. From time to time she takes either a toffee (with probability p) or mint (with probability $q = 1 - p$). At the beginning of the week she has n toffees and m mints in the packets. On the Nth occasion that she reaches for a sweet, she discovers (for the first time) that she has run out of that kind of sweet. What is the probability that she was reaching for a toffee?

Section A: Pure Mathematics

1 **(i)** Let $\mathrm{h}(x) = ax^2 + bx + c$, where a, b and c are constants, and $a \neq 0$. Give a condition which a, b and c must satisfy in order that $\mathrm{h}(x)$ can be written in the form

$$a(x+k)^2, \qquad (*)$$

where k is a constant.

(ii) If $\mathrm{f}(x) = 3x^2+4x$ and $\mathrm{g}(x) = x^2-2$, find the two constant values of λ such that $\mathrm{f}(x)+\lambda\mathrm{g}(x)$ can be written in the form $(*)$. Hence, or otherwise, find constants A, B, C, D, m and n such that

$$\mathrm{f}(x) = A(x+m)^2 + B(x+n)^2$$
$$\mathrm{g}(x) = C(x+m)^2 + D(x+n)^2.$$

(iii) If $\mathrm{f}(x) = 3x^2 + 4x$ and $\mathrm{g}(x) = x^2 + \alpha$ and it is given by that there is only one value of λ for which $\mathrm{f}(x) + \lambda\mathrm{g}(x)$ can be written in the form $(*)$, find α.

2 The equation of a hyperbola (with respect to axes which are displaced and rotated with respect to the standard axes) is

$$3y^2 - 10xy + 3x^2 + 16y - 16x + 15 = 0. \qquad (\dagger)$$

(i) By differentiating $(\dagger)$, or otherwise, show that the equation of the tangent through the point (s, t) on the curve is

$$y = \left(\frac{5t-3s+8}{3t-5s+8}\right)x - \left(\frac{8t-8s+15}{3t-5s+8}\right).$$

Show that the equations of the asymptote (the limiting tangents as $s \to \infty$) are

$$y = 3x - 4 \qquad \text{and} \qquad 3y = x - 4.$$

[**Hint**: You will need to find a relationship between s and t which is valid in the limit as $s \to \infty$.]

(ii) Show that the angle between one asymptote and the x-axis is the same as the angle between the other asymptote and the y-axis. Deduce the slopes of the lines that bisect the angles between the asymptotes and find the equations of the axes of the hyperbola.

3 It is given that x, y and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \dfrac{1}{y}$ is either $+1$ or -1.

4 **(i)** Let $y = \cos\phi + \cos 2\phi$, where $\phi = \dfrac{2\pi}{5}$. Verify by direct substitution that y satisfies the quadratic equation $2y^2 = 3y + 2$ and deduce that the value of y is $-\frac{1}{2}$.

(ii) Let $\theta = \dfrac{2\pi}{17}$. Show that

$$\sum_{k=0}^{16} \cos k\theta = 0.$$

(iii) If $z = \cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, show that the value of z is $-(1 - \sqrt{17})/4$.

5 **(i)** Give a rough sketch of the function $\tan^k \theta$ for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ in the two cases $k = 1$ and $k \gg 1$ (i.e. k is much greater than 1).

(ii) Show that for any positive integer n

$$\int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \, \mathrm{d}\theta = (-1)^n \left(\tfrac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m} \right),$$

and deduce that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m} = \tfrac{1}{2}\ln 2.$$

(iii) Show similarly that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} = \frac{\pi}{4}.$$

6 **(i)** Show by means of a sketch, or otherwise, that if $0 \leqslant \mathrm{f}(y) \leqslant \mathrm{g}(y)$ for $0 \leqslant y \leqslant x$ then

$$0 \leqslant \int_0^x \mathrm{f}(y)\,\mathrm{d}y \leqslant \int_0^x \mathrm{g}(y)\,\mathrm{d}y.$$

(ii) Starting from the inequality $0 \leqslant \cos y \leqslant 1$, or otherwise, prove that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $0 \leqslant \sin x \leqslant x$ and $\cos x \geqslant 1 - \frac{1}{2}x^2$. Deduce that

$$\frac{1}{1800} \leqslant \int_0^{\frac{1}{10}} \frac{x}{(2+\cos x)^2}\,\mathrm{d}x \leqslant \frac{1}{1797}.$$

(iii) Show further that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $\sin x \geqslant x - \frac{1}{6}x^3$. Hence prove that

$$\frac{1}{3000} \leqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1-x+\sin x)^2}\,\mathrm{d}x \leqslant \frac{2}{5999}.$$

7 The function g satisfies, for all positive x and y,

$$\mathrm{g}(x) + \mathrm{g}(y) = \mathrm{g}(z), \qquad (*)$$

where $z = xy/(x+y+1)$.

(i) By treating y as a constant, show that

$$\mathrm{g}'(x) = \frac{y^2+y}{(x+y+1)^2}\mathrm{g}'(z) = \frac{z(z+1)}{x(x+1)}\mathrm{g}'(z),$$

and deduce that $2\mathrm{g}'(1) = (u^2+u)\mathrm{g}'(u)$ for all u satisfying $0 < u < 1$.

(ii) Now by treating u as a variable, show that

$$\mathrm{g}(u) = A\ln\left(\frac{u}{u+1}\right) + B,$$

where A and B are constants. Verify that g satisfies $(*)$ for a suitable value of B. Can A be determined from $(*)$?

(iii) The function f satisfies, for all positive x and y,

$$\mathrm{f}(x) + \mathrm{f}(y) = \mathrm{f}(z)$$

where $z = xy$. Show that $\mathrm{f}(x) = C\ln x$ where C is a constant.

8 **(i)** Solve the quadratic equation $u^2 + 2u\sinh x - 1 = 0$, giving u in terms of x.

(ii) Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$.

(iii) Find the solution of the differential equation

$$\sinh x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

9 **(i)** Let G be the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b and c are integers modulo 5, and $a \neq 0 \neq c$.

Show that G forms a group under matrix multiplication (which may be assumed to be associative).

(ii) What is the order of G? Determine whether or not G is commutative.

(iii) Determine whether or not the set consisting of all elements in G of order 1 or 2 is a subgroup of G.

10 A straight stick of length h stands vertically. On a sunny day, the stick casts a shadow on flat horizontal ground. In cartesian axes based on the centre of the Earth, the position of the Sun may be taken to be $R(\cos\theta, \sin\theta, 0)$ where θ varies but R is constant. The positions of the base and tip of the stick are $a(0, \cos\phi, \sin\phi)$ and $b(0, \cos\phi, \sin\phi)$, respectively, where $b - a = h$. Show that the displacement vector from the base of the stick to the tip of the shadow is

$$Rh(R\cos\phi\sin\theta - b)^{-1}\begin{pmatrix} -\cos\theta \\ -\sin^2\phi\sin\theta \\ \cos\phi\sin\phi\sin\theta \end{pmatrix}.$$

['Stands vertically' means that the centre of the Earth, the base of the stick and the tip of the stick are collinear, 'horizontal' means perpendicular to the stick.

Section B: Mechanics

11 The Ruritanian army is supplied with shells which may explode at any time in flight but not before the shell reaches its maximum height. The effect of the explosion on any observer depends only on the distance between the exploding shell and the observer (and decreases with distance). Ruritanian guns fire the shells with fixed muzzle speed, and it is the policy of the gunners to fire the shell at an angle of elevation which minimises the possible damages to themselves (assuming the ground is level) - i.e. they aim so that the point on the descending trajectory that is nearest to them is as far away as possible. With that intention, they choose the angle of elevation that minimises the damage to themselves if the shell explodes at its maximum height. What angle do they choose?

Does the shell then get any nearer to the gunners during its descent?

12 A particle is attached to one end B of a light elastic string of unstretched length a. Initially the other end A is at rest and the particle hangs at rest at a distance $a + c$ vertically below A. At time $t = 0$, the end A is forced to oscillate vertically, its downwards displacement at time t being $b \sin pt$.

(i) Let $x(t)$ be the downwards displacement of the particle at time t from its initial equilibrium position. Show that, while the string remains taut, $x(t)$ satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -n^2(x - b\sin pt),$$

where $n^2 = g/c$, and that if $0 < p < n$, $x(t)$ is given by

$$x(t) = \frac{bn}{n^2 - p^2}(n \sin pt - p \sin nt).$$

(ii) Write down a necessary and sufficient condition that the string remains taut throughout the subsequent motion, and show that it is satisfied if $pb < (n - p)c$.

13 A non-uniform rod AB of mass m is pivoted at one end A so that it can swing freely in a vertical plane. Its centre of mass is a distance d from A and its moment of inertia about any axis perpendicular to the rod through A is mk^2. A small ring of mass αm is free to slide along the rod and the coefficient of friction between the ring and rod is μ. The rod is initially held in a horizontal position with the ring a distance x from A.

(i) If $k^2 > xd$, show that when the rod is released, the ring will start to slide when the rod makes an angle θ with the downward vertical, where

$$\mu \tan\theta = \frac{3\alpha x^2 + k^2 + 2xd}{k^2 - xd}.$$

(ii) Explain what will happen if (i) $k^2 = xd$ and (ii) $k^2 < xd$.

14 The current in a straight river of constant width h flows at uniform speed αv parallel to the river banks, where $0 < \alpha < 1$. A boat has to cross from a point A on one bank to a point B on the other bank directly opposite to A. The boat moves at constant speed v relative to the water. When the position of the boat is (x, y), where x is the perpendicular distance from the opposite bank and y is the distance downstream from AB, the boat is pointing in a direction which makes an angle θ with AB. Determine the velocity vector of the boat in terms of v, θ and α.

(i) The pilot of the boat steers in such a way that the boat always points exactly towards B. Show that the velocity vector of the boat is

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \tan\theta \dfrac{\mathrm{d}x}{\mathrm{d}t} + x\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{pmatrix}.$$

(ii) By comparing this with your previous expression deduce that

$$\alpha \frac{\mathrm{d}x}{\mathrm{d}\theta} = -x\sec\theta$$

and hence show that

$$(x/h)^\alpha = (\sec\theta + \tan\theta)^{-1}.$$

(iii) Let $s(t)$ be a new variable defined by $\tan\theta = \sinh(\alpha s)$. Show that $x = he^{-s}$, and that

$$he^{-s}\cosh(\alpha s)\frac{\mathrm{d}s}{\mathrm{d}t} = v.$$

Hence show that the time of crossing is $hv^{-1}(1-\alpha^2)^{-1}$.

Section C: Probability and Statistics

15 Integers $n_1, n_2, \ldots, n_r$ (possibly the same) are chosen independently at random from the integers $1, 2, 3, \ldots, m$. Show that the probability that $|n_1 - n_2| = k$, where $1 \leqslant k \leqslant m-1$, is $2(m-k)/m^2$ and show that the expectation of $|n_1 - n_2|$ is $(m^2-1)/(3m)$. Verify, for the case $m = 2$, the result that the expection of $|n_1 - n_2| + |n_2 - n_3|$ is $2(m^2-1)/(3m)$. Write down the expectation, for general m, of

$$|n_1 - n_2| + |n_2 - n_3| + \cdots + |n_{r-1} - n_r|\,.$$

Desks in an examination hall are placed a distance d apart in straight lines. Each invigilator looks after one line of m desks. When called by a candidate, the invigilator walks to that candidate's desk, and stays there until called again. He or she is equally likely to be called by any of the m candidates in the line but candidates never call simultaneously or while the invigilator is attending to another call. At the beginning of the examination the invigilator stands by the first desk. Show that the expected distance walked by the invigilator in dealing with $N+1$ calls is

$$\frac{d(m-1)}{6m}[2N(m+1)+3m].$$

16 Each time it rains over the Cabbibo dam, a volume V of water is deposited, almost instantaneously, in the reservoir. Each day (midnight to midnight) water flows from the reservoir at a constant rate u units of volume per day. An engineer, if present, may choose to alter the value of u at any midnight.

(i) Suppose that it rains at most once in any day, that there is a probability p that it will rain on any given day and that, if it does, the rain is equally likely to fall at any time in the 24 hours (i.e. the time at which the rain falls is a random variable uniform on the interval $[0, 24]$). The engineers decides to take two days' holiday starting at midnight. If at this time the volume of water in the reservoir is V below the top of the dam, find an expression for u such that the probability of overflow in the two days is Q, where $Q < p^2$.

For the engineer's summer holidays, which last 18 days, the reservoir is drained to a volume kV below the top of the dam and the rate of outflow u is set to zero. The engineer wants to drain off as little as possible, consistent with the requirement that the probability that the dam will overflow is less than $\frac{1}{10}$. In the case $p = \frac{1}{3}$, find by means of a suitable approximation the required value of k.

(ii) Suppose instead that it may rain at most once before noon and at most once after noon each day, that the probability of rain in any given half-day is $\frac{1}{6}$ and that it is equally likely to rain at any time in each half-day. Is the required value of k lower or higher?

Section A: Pure Mathematics

1 **(i)** Prove that both $x^4-2x^3+x^2$ and $x^2-8x+17$ are non-negative for all real x. By considering the intervals $x \leqslant 0$, $0 < x \leqslant 2$ and $x > 2$ separately, or otherwise, prove that the equation

$$x^4 - 2x^3 + x^2 - 8x + 17 = 0$$

has no real roots.

(ii) Prove that the equation $x^4 - x^3 + x^2 - 4x + 4 = 0$ has no real roots.

2 **(i)** Prove that if $A + B + C + D = \pi$, then

$$\sin(A+B)\sin(A+D) - \sin B \sin D = \sin A \sin C.$$

The points P, Q, R and S lie, in that order, on a circle of centre O.

(ii) Prove that

$$PQ \times RS + QR \times PS = PR \times QS.$$

3 **(i)** Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

$$\textbf{(i)}\ \ 2b < c\sqrt{3}, \quad \textbf{(ii)}\ \ 2b = c\sqrt{3} \neq 0, \quad \textbf{(iii)}\ \ c\sqrt{3} < 2b < 2c, \quad \textbf{(iv)}\ \ b = c \neq 0,$$
$$\textbf{(v)}\ \ b > c > 0, \quad \textbf{(vi)}\ \ c = 0, b \neq 0, \quad \textbf{(vii)}\ \ c = b = 0.$$

(ii) Sketch also the curves given by $y^2 = x^3 - 2bx^2 + c^2x$ in the cases **(i)**, **(v)** and **(vii)**.

4 A plane contains n distinct given lines, no two of which are parallel, and no three of which intersect at a point.

(i) By first considering the cases $n = 1, 2, 3$ and 4, provide and justify, by induction or otherwise, a formula for the number of line segments (including the infinite segments).

(ii) Prove also that the plane is divided into $\frac{1}{2}(n^2 + n + 2)$ regions (including those extending to infinity).

5 The distinct points L, M, P and Q of the Argand diagram lie on a circle S centred on the origin and the corresponding complex numbers are l, m, p and q. By considering the perpendicular bisectors of the chords, or otherwise, prove that the chord LM is perpendicular to the chord PQ if and only if $lm + pq = 0$.

Let A_1, A_2 and A_3 be three distinct points on S. For any given point A_1' on S, the points A_2', A_3' and A_1'' are chosen on S such that $A_1'A_2', A_2'A_3'$ and $A_3'A_1''$ are perpendicular to A_1A_2, A_2A_3 and A_3A_1, respectively. Show that for exactly two positions of A_1', the points A_1' and A_1'' coincide.

If, instead, A_1, A_2, A_3 and A_4 are four given distinct points on S and, for any given point A_1', the points A_2', A_3', A_4' and A_1'' are chosen on S such that $A_1'A_2', A_2'A_3', A_3'A_4'$ and $A_4'A_1''$ are respectively perpendicular to A_1A_2, A_2A_3, A_3A_4 and A_4A_1, show that A_1' coincides with A_1''.

Give the corresponding result for n distinct points on S.

6 Let a, b, c, d, p and q be positive integers. Prove that:

(i) if $b > a$ and $c > 1$, then $bc \geqslant 2c \geqslant 2 + c$;

(ii) if $a < b$ and $d < c$, then $bc - ad \geqslant a + c$;

(iii) if $\dfrac{a}{b} < p < \dfrac{c}{d}$, then $(bc - ad)\,p \geqslant a + c$;

(iv) if $\dfrac{a}{b} < \dfrac{p}{q} < \dfrac{c}{d}$, then $p \geqslant \dfrac{a+c}{bc-ad}$ and $q \geqslant \dfrac{b+d}{bc-ad}$.

Hence find all fractions with denominators less than 20 which lie between $8/9$ and $9/10$.

7 A damped system with feedback is modelled by the equation

$$\mathrm{f}'(t) + \mathrm{f}(t) - k\mathrm{f}(t-1) = 0, \qquad (\dagger)$$

where k is a given non-zero constant.

(i) Show that (non-zero) solutions for f of the form $\mathrm{f}(t) = A\mathrm{e}^{pt}$, where A and p are constants, are possible provided p satisfies

$$p + 1 = k\mathrm{e}^{-p}. \qquad (*)$$

(ii) Show also, by means of a sketch, or otherwise, that equation $(*)$ can have $0, 1$ or 2 real roots, depending on the value of k, and find the set of values of k for which such solutions of $(\dagger)$ exist.

(iii) For what set of values of k do such solutions tend to zero as $t \to +\infty$?

8 The functions x and y are related by

$$\mathrm{x}(t) = \int_0^t \mathrm{y}(u)\,\mathrm{d}u,$$

so that $\mathrm{x}'(t) = \mathrm{y}(t)$.

(i) Show that

$$\int_0^1 \mathrm{x}(t)\mathrm{y}(t)\,\mathrm{d}t = \tfrac{1}{2}\left[\mathrm{x}(1)\right]^2.$$

(ii) In addition, it is given that y(t) satisfies

$$\mathrm{y}'' + (\mathrm{y}^2 - 1)\mathrm{y}' + \mathrm{y} = 0, \ (*)$$

with $\mathrm{y}(0) = \mathrm{y}(1)$ and $\mathrm{y}'(0) = \mathrm{y}'(1)$. By integrating $(*)$, prove that $\mathrm{x}(1) = 0$.

(iii) By multiplying $(*)$ by $\mathrm{x}(t)$ and integrating by parts, prove the relation

$$\int_0^1 \left[\mathrm{y}(t)\right]^2\,\mathrm{d}t = \tfrac{1}{3}\int_0^1 \left[\mathrm{y}(t)\right]^4\,\mathrm{d}t.$$

(iv) Prove also the relation

$$\int_0^1 \left[\mathrm{y}'(t)\right]^2\,\mathrm{d}t = \int_0^1 \left[\mathrm{y}(t)\right]^2\,\mathrm{d}t.$$

9 **(i)** Show by means of a sketch that the parabola $r(1+\cos\theta) = 1$ cuts the interior of the cardioid $r = 4(1+\cos\theta)$ into two parts.

(ii) Show that the total length of the boundary of the part that includes the point $r = 1, \theta = 0$ is $18\sqrt{3} + \ln(2+\sqrt{3})$.

10 **(i)** Two square matrices $\mathbf{A}$ and $\mathbf{B}$ satisfies $\mathbf{AB} = \mathbf{0}$. Show that either $\det\mathbf{A} = 0$ or $\det\mathbf{B} = 0$ or $\det\mathbf{A} = \det\mathbf{B} = 0$. If $\det\mathbf{B} \neq 0$, what must $\mathbf{A}$ be? Give an example to show that the condition $\det\mathbf{A} = \det\mathbf{B} = 0$ is not sufficient for the equation $\mathbf{AB} = \mathbf{0}$ to hold.

(ii) Find real numbers p, q and r such that

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} + p\mathbf{I})(\mathbf{M} + q\mathbf{I})(\mathbf{M} + r\mathbf{I}),$$

where $\mathbf{M}$ is any square matrix and $\mathbf{I}$ is the appropriate identity matrix.

(iii) Hence, or otherwise, find all matrices $\mathbf{M}$ of the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ which satisfy the equation

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = \mathbf{0}.$$

Section B: Mechanics

11 A disc is free to rotate in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2. Along one diameter is a narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is at a distance a from the axis and is moving towards the axis with speed V, where $k^2V^2 = \Omega^2 a^2(k^2 + a^2)$.

(i) Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\Omega r(k^2 + a^2)}{k(k^2 + r^2)^{\frac{1}{2}}}.$$

(ii) Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where θ is the angle through which the disc has turned at time t.

(iii) By making the substitution $u = 1/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Hence, or otherwise, show that the particle never reaches the axis.

12 A straight staircase consists of N smooth horizontal stairs each of height h. A particle slides over the top stair at speed U, with velocity perpendicular to the edge of the stair, and then falls down the staircase, bouncing once on every stair. The coefficient of restitution between the particle and each stair is e, where $e < 1$.

(i) Show that the horizontal distance d_n travelled between the nth and $(n + 1)$th bounces is given by

$$d_n = U\left(\frac{2h}{g}\right)^{\frac{1}{2}}(e\alpha_n + \alpha_{n+1}),$$

where $\alpha_n = \left(\frac{1 - e^{2n}}{1 - e^2}\right)^{\frac{1}{2}}$.

(ii) If N is very large, show that U must satisfy

$$U = \left(\frac{L^2 g}{2h}\right)^{\frac{1}{2}}\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}},$$

where L is the horizontal distance between the edges of successive stairs.

13 A thin non-uniform rod PQ of length $2a$ has its centre of gravity a distance $a + d$ from P. It hangs (not vertically) in equilibrium suspended from a small smooth peg O by means of a light inextensible string of length $2b$ which passes over the peg and is attached at its ends to P and Q.

(i) Express OP and OQ in terms of a, b and d.

(ii) By considering the angle POQ, or otherwise, show that $d < a^2/b$.

14 The identical uniform smooth spherical marbles $A_1, A_2, \ldots, A_n$, where $n \geqslant 3$, each of mass m, lie in that order in a smooth straight trough, with each marble touching the next. The marble A_{n+1}, which is similar to A_n but has mass λm, is placed in the trough so that it touches A_n. Another marble A_0, identical to A_n, slides along the trough with speed u and hits A_1. It is given that kinetic energy is conserved throughout.

(i) Show that if $\lambda < 1$, there is a possible subsequent motion in which only A_n and A_{n+1} move (and A_0 is reduced to rest), but that if $\lambda > 1$, such a motion is not possible.

(ii) If $\lambda > 1$, show that a subsequent motion in which only A_{n-1}, A_n and A_{n+1} move is not possible.

(iii) If $\lambda > 1$, find a possible subsequent motion in which only two marbles move.

Section C: Probability and Statistics

15 A target consists of a disc of unit radius and centre O. A certain marksman never misses the target, and the probability of any given shot hitting the target within a distance t from O it t^2, where $0 \leqslant t \leqslant 1$. The marksman fires n shots independently. The random variable Y is the radius of the smallest circle, with centre O, which encloses all the shots.

(i) Show that the probability density function of Y is $2ny^{2n-1}$ and find the expected area of the circle.

(ii) The shot which is furthest from O is rejected. Show that the expected area of the smallest circle, with centre O, which encloses the remaining $(n-1)$ shots is

$$\left(\frac{n-1}{n+1}\right)\pi.$$

16 Each day, I choose at random between my brown trousers, my grey trousers and my expensive but fashionable designer jeans. Also in my wardrobe, I have a black silk tie, a rather smart brown and fawn polka-dot tie, my regimental tie, and an elegant powder-blue cravat which I was given for Christmas. With my brown or grey trousers, I choose ties (including the cravat) at random, except of course that I don't wear the cravat with the brown trousers or the polka-dot tie with the grey trousers. With the jeans, the choice depends on whether it is Sunday or one of the six weekdays: on weekdays, half the time I wear a cream-coloured sweat-shirt with $E = mc^2$ on the front and no tie; otherwise, and on Sundays (when naturally I always wear a tie), I just pick at random from my four ties.

This morning, I received through the post a compromising photograph of myself. I often receive such photographs and they are equally likely to have been taken on any day of the week. However, in this particular photograph, I am wearing my black silk tie.

(i) Show that, on the basis of this information, the probability that the photograph was taken on Sunday is $11/68$.

(ii) I should have mentioned that on Mondays I lecture on calculus and I therefore always wear my jeans (to make the lectures seem easier to understand). Find, on the basis of the complete information, the probability that the photograph was taken on Sunday.

[The phrase 'at random' means 'with equal probability'.]

Section A: Pure Mathematics

1 Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Show how the cubic equation

$$24x^3 - 72x^2 + 66x - 19 = 0 \qquad (*)$$

can be reduced to the form

$$4z^3 - 3z = k$$

by means of the substitution $y = x + a$ and $z = by$, for suitable values of the constants a and b. Hence find the three roots of the equation $(*)$, to three significant figures.

Show, by means of a counterexample, or otherwise, that not all cubic equations of the form

$$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

can be solved by this method.

2 Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{for small } x,$$

$$x\cot x = 1 + \sum_{n=1}^{\infty} b_n x^n \quad \text{for small } x \text{ and not zero.}$$

Using the relation

$$\cot x - \tan x = 2\cot 2x, \qquad (*)$$

or otherwise, prove that $a_{n-1} = (1 - 2^n)b_n$, for $n \geqslant 1$.

Let

$$x\mathrm{cosec}x = 1 + \sum_{n=1}^{\infty} c_n x^n \quad \text{for small } x \neq 0.$$

Using a relation similar to $(*)$ involving $2\mathrm{cosec}2x$, or otherwise, prove that

$$c_n = \frac{2^{n-1} - 1}{2^n - 1}\frac{1}{2^{n-1}}a_{n-1} \qquad (n \geqslant 1).$$

3 The real numbers x and y are related to the real numbers u and v by

$$2(u+\mathrm{i}v) = \mathrm{e}^{x+\mathrm{i}y} - \mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x = a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2 = 1$$

in the u-v plane. Show also that the line given by $y = b$, where b is a constant and $0 < \sin b < 1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point.

Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x = 0$;

(ii) $y = \frac{1}{2}\pi, \quad 0 \leqslant x \leqslant a.$

4 The function f is defined by

$$\mathrm{f}(x) = \frac{(x-a)\,(x-b)}{(x-c)\,(x-d)} \qquad (x \neq c,\ x \neq d)\,,$$

where a, b, c and d are real and distinct, and $a + d \neq c + d$. Show that

$$\frac{x\mathrm{f}'(x)}{\mathrm{f}(x)} = \left(1-\frac{a}{x}\right)^{-1} + \left(1-\frac{b}{x}\right)^{-1} - \left(1-\frac{c}{x}\right)^{-1} - \left(1-\frac{d}{x}\right)^{-1},$$

$(x \neq 0, x \neq a, x \neq b)$ and deduce that when $|x|$ is much larger than each of $|a|\,, |b|\,, |c|$ and $|d|\,$, the gradient of $\mathrm{f}(x)$ has the same sign as $(a + b - c - d)$.

It is given that there is a real value of real value of x for which $\mathrm{f}(x)$ takes the real value z if and only if

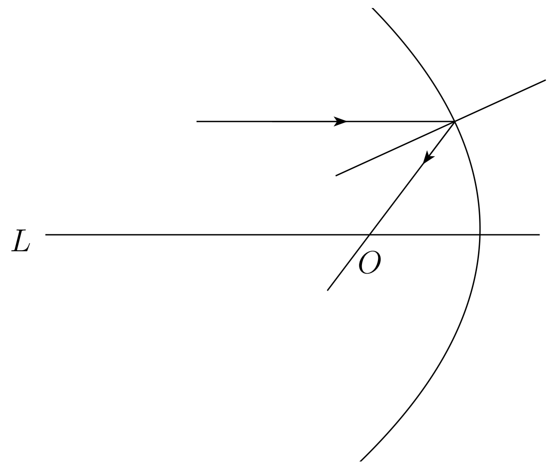
$$\left[(c-d)^2 z + (a-c)\,(b-d) + (a-d)\,(b-c)\right]^2 \geqslant 4\,(a-c)\,(b-d)\,(a-d)\,(b-c)\,.$$

Describe briefly a method by which this result could be proved, but do not attempt to prove it.

Given that $a < b$ and $a < c < d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$.

5 **(i)** Show that in polar coordinates, the gradient of any curve at the point (r, θ) is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r\right)\Big/\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta\right).$$



(ii) A mirror is designed so that any ray of light which hits one side of the mirror and which is parallel to a certain fixed line L is reflected through a fixed point O on L. For any ray hitting the mirror, the normal to the mirror at the point of reflection bisects the angle between the incident ray and the reflected ray, as shown in the figure. Prove that the mirror intersects any plane containing L in a parabola.

6 The function f satisfies the condition $\mathrm{f}'(x) > 0$ for $a \leqslant x \leqslant b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y,$$

where $\alpha = \mathrm{f}(a)$ and $\beta = \mathrm{f}(b)$. Interpret this formula geometrically, in the case where α and a are both positive.

Prove similarly and interpret (for $\alpha > 0$ and $a > 0$) the formula

$$2\pi\int_a^b x\mathrm{f}(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi\int_\alpha^\beta [\mathrm{g}(y)]^2\,\mathrm{d}y.$$

7 By means of the substitution x^α, where α is a suitably chosen constant, find the general solution for $x > 0$ of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y = 0,$$

where b is a constant and $b > -1$.

Show that, if $b > 0$, there exist solutions which satisfy $y \to 1$ and $\mathrm{d}y/\mathrm{d}x \to 0$ as $x \to 0$, but that these conditions do not determine a unique solution. For what values of b do these conditions determine a unique solution?

8 Let $\Omega = \exp(\mathrm{i}\pi/3)$. Prove that $\Omega^2 - \Omega + 1 = 0$.

Two transformations, R and T, of the complex plane are defined by

$$R : z \longmapsto \Omega^2 z \qquad \text{and} \qquad T : z \longmapsto \frac{\Omega z + \Omega^2}{2\Omega^2 z + 1}.$$

Verify that each of R and T permute the four point $z_0 = 0$, $z_1 = 1$, $z_2 = \Omega^2$ and $z_3 = -\Omega$. Explain, without explicitly producing a group multiplication table, why the smallest group of transformations which contains elements R and T has order at least 12.

Are there any permutations of these points which cannot be produced by repeated combinations of R and T?

9 The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n,$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$, and t is a variable scalar. Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I} \cosh t + \mathbf{A} \sinh t.$$

Show also that $\mathbf{F}^{-1} = \mathbf{I} \cosh t - \mathbf{A} \sinh t$, and that $\dfrac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{FA}$.

The vector $\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{Ar} = \mathbf{0},$$

with $x = \alpha$ and $y = \beta$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$x(t) = \alpha \cosh t + (3\alpha + \beta) \sinh t$$
$$y(t) = \beta \cosh t - (8\alpha + 3\beta) \sinh t.$$

10 State carefully the conditions which the fixed vectors $\mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ must satisfy in order to ensure that the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$ intersects the line $\mathbf{r} = \mathbf{b} + \mu \mathbf{v}$ in exactly one point.

Find the two values of the fixed scalar b for which the planes with equations

$$\left.\begin{aligned} x + y + bz &= b + 2 \\ bx + by + z &= 2b + 1 \end{aligned}\right\} \qquad (*)$$

do not intersect in a line. For other values of b, express the line of intersection of the two planes in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$, where $\mathbf{a} \cdot \mathbf{u} = 0$.

Find the conditions which b and the fixed scalars c and d must satisfy to ensure that there is exactly one point on the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \mu \begin{pmatrix} 1 \\ d \\ 0 \end{pmatrix}$$

whose coordinates satisfy both equations $(*)$.

Section B: Mechanics

11 A lift of mass M and its counterweight of mass M are connected by a light inextensible cable which passes over a light frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is h. Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than h. A small tile of mass m becomes detached from the ceiling of the lift. Show that the time taken for it to fall to the floor is

$$t = \sqrt{\frac{2\,(M-m)\,h}{Mg}}.$$

The collision between the tile and the lift floor is perfectly inelastic. Show that the lift is reduced to rest by the collision, and that the loss of energy of the system is mgh.

12 A uniform rectangular lamina of sides $2a$ and $2b$ rests in a vertical plane. It is supported in equilibrium by two smooth pegs fixed in the same horizontal plane, a distance d apart, so that one corner of the lamina is below the level of the pegs. Show that if the distance between this (lowest) corner and the peg upon which the side of length $2a$ rests is less than a, then the distance between this corner and the other peg is less than b.

Show also that

$$b\cos\theta - a\sin\theta = d\cos 2\theta,$$

where θ is the acute angle which the sides of length $2b$ make with the horizontal.

13 A body of mass m and centre of mass O is said to be *dynamically equivalent* to a system of particles of total mass m and centre of mass O if the moment of inertia of the system of particles is the same as the moment of inertia of the body, about any axis through O. Show that this implies that the moment of inertia of the system of particles is the same as that of the body about *any* axis.

Show that a uniform rod of length $2a$ and mass m is dynamically equivalent to a suitable system of three particles, one at each end of the rod, and one at the midpoint.

Use this result to deduce that a uniform rectangular lamina of mass M is dynamically equivalent to a system consisting of particles each of mass $\frac{1}{36}M$ at the corners, particles each of mass $\frac{1}{9}M$ at the midpoint of each side, and a particle of mass $\frac{4}{9}M$ at the centre. Hence find the moment of inertia of a square lamina, of side $2a$ and mass M, about one of its diagonals.

The mass per unit length of a thin rod of mass m is proportional to the distance from one end of the rod, and a dynamically equivalent system consists of one particle at each end of the rod and one at the midpoint. Write down a set of equations which determines these masses, and show that, in fact, only two particles are required.

14 One end of a light inextrnsible string of length l is fixed to a point on the upper surface of a thin, smooth, horizontal table-top, at a distance $(l-a)$ from one edge of the table-top. A particle of mass m is fixed to the other end of the string, and held a distance a away from this edge of the table-top, so that the string is horizontal and taut. The particle is then released. Find the tension in the string after the string has rotated through an angle θ, and show that the largest magnitude of the force on the edge of the table top is $8mg/\sqrt{3}$.

Section C: Probability and Statistics

15 Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. What is the probability that exactly one of them lies on the diameter?

Let the area of the triangle formed by the two points and the midpoint of the diameter be denoted by the random variable A.

(i) Given that exactly one point lies on the diameter, show that the expected value of A is $(2\pi)^{-1}$.

(ii) Given that neither point lies on the diameter, show that the expected value of A is π^{-1}. [You may assume that if two points are chosen at random on a line of length π units, the probability density function for the distance X between the two points is $2\left(\pi - x\right)/\pi^2$ for $0 \leqslant x \leqslant \pi$.]

Using these results, or otherwise, show that the expected value of A is $(2+\pi)^{-1}$.

16 Widgets are manufactured in batches of size $(n+N)$. Any widget has a probability p of being faulty, independent of faults in other widgets. The batches go through a quality control procedure in which a sample of size n, where $n \geqslant 2$, is taken from each batch and tested. If two or more widgets in the sample are found to be faulty, all widgets in the batch are tested and all faults corrected. If fewer than two widgets in the sample are found to be faulty, the sample is replaced in the batch and no faults are corrected. Show that the probability that the batch contains exactly k, where $k \leqslant N$, faulty widgets after quality control is

$$\frac{\left[N+1+k\left(n-1\right)\right]N!}{(N-k+1)!k!}p^k\left(1-p\right)^{N+n-k},$$

and verify that this formula also gives the correct answer for $k = N+1$.

Show that the expected number of faulty widgets in a batch after quality control is

$$\left[N+n+pN(n-1)\right]p(1-p)^{n-1}.$$

Section A: Pure Mathematics

1 The function f is defined, for $x \neq 1$ and $x \neq 2$ by

$$\mathrm{f}(x) = \frac{1}{(x-1)(x-2)}$$

(i) Show that for $|x| < 1$

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

(ii) and that for $1 < |x| < 2$

$$\mathrm{f}(x) = -\sum_{n=1}^{\infty} x^{-n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

(iii) Find an expression for f(x) which is valid for $|x| > 2$.

2 The numbers x, y and z are non-zero, and satisfy

$$2a - 3y = \frac{(z-x)^2}{y} \quad \text{and} \quad 2a - 3z = \frac{(x-y)^2}{z},$$

for some number a.

(i) If $y \neq z$, prove that

$$x + y + z = a,$$

and that

$$2a - 3x = \frac{(y-z)^2}{x}.$$

(ii) Determine whether this last equation holds *only* if $y \neq z$.

3 The quadratic equation $x^2 + bx + c = 0$, where b and c are real, has the properly that if k is a (possibly complex) root, then k^{-1} is a root.

(i) Determine carefully the restriction that this property places on b and c.

(ii) If, in addition to this property, the equation has the further property that if k is a root, then $1 - k$ is a root, find b and c.

(iii) Show that

$$x^3 - \tfrac{3}{2}x^2 - \tfrac{3}{2}x + 1 = 0$$

is the only cubic equation of the form $x^3 + px^2 + qx + r = 0$, where p, q and r are real, which has both the above properties.

4 The complex number w is such that $w^2 - 2x$ is real.

(i) Sketch the locus of w in the Argand diagram.

(ii) If $w^2 = x + \mathrm{i}y$, describe fully and sketch the locus of points (x, y) in the x-y plane.

The complex number t is such that $t^2 - 2t$ is imaginary. If $t^2 = p + \mathrm{i}q$, sketch the locus of points (p, q) in the p-q plane.

5 **(i)** By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

(ii) Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

(iii) Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

6 **(i)** Show that the following functions are positive when x is positive:

(a) $x - \tanh x$

(b) $x \sinh x - 2\cosh x + 2$

(c) $2x\cosh 2x - 3\sinh 2x + 4x$.

(ii) The function f is defined for $x > 0$ by

$$\mathrm{f}(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}.$$

Show that $\mathrm{f}(x)$ has no turning points when $x > 0$, and sketch $\mathrm{f}(x)$ for $x > 0$.

7 The integral I is defined by

$$I = \int_1^2 \frac{(2 - 2x + x^2)^k}{x^{k+1}}\,\mathrm{d}x$$

where k is a constant.

(i) Show that

$$I = \int_0^1 \frac{(1+x^2)^k}{(1+x)^{k+1}}\,\mathrm{d}x = \int_0^{\frac{1}{4}\pi} \frac{\mathrm{d}\theta}{\left[\sqrt{2}\cos\theta\cos\left(\frac{1}{4}\pi - \theta\right)\right]^{k+1}} = 2\int_0^{\frac{1}{8}\pi} \frac{\mathrm{d}\theta}{\left[\sqrt{2}\cos\theta\cos\left(\frac{1}{4}\pi - \theta\right)\right]^{k+1}}.$$

(ii) Hence show that

$$I = 2\int_0^{\sqrt{2}-1} \frac{(1+x^2)^k}{(1+x)^{k+1}}\,\mathrm{d}x$$

(iii) Deduce that

$$\int_1^{\sqrt{2}} \left(\frac{2 - 2x^2 + x^4}{x^2}\right)^k \frac{1}{x}\,\mathrm{d}x = \int_1^{\sqrt{2}} \left(\frac{2 - 2x + x^2}{x}\right)^k \frac{1}{x}\,\mathrm{d}x$$

8 In a crude model of population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year at $\frac{1}{4}A + \frac{3}{4}B$ and $\frac{3}{2}B - \frac{1}{2}A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values.

Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a-b plane, mark the regions which correspond to the following situations

(i) an equilibrium population is reached as time $t \to \infty$;

(ii) buffaloes become extinct after a finite time;

(iii) buffaloes approach extinction as $t \to \infty$.

9 **(i)** Give a careful argument to show that, if G_1 and G_2 are subgroups of a finite group G such that every element of G is either in G_1 or in G_2, then either $G_1 = G$ or $G_2 = G$.

(ii) Give an example of a group H which has three subgroups H_1, H_2 and H_3 such that every element of H is either in H_1, H_2 or H_3 and $H_1 \neq H, H_2 \neq H, H_3 \neq H$.

10 The surface S in 3-dimensional space is described by the equation

$$\mathbf{a} \cdot \mathbf{r} + ar = a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq \mathbf{0})$ is the position vector of a fixed point, $r = |\mathbf{r}|$ and $a = |\mathbf{a}|$.

(i) Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r} \cdot \mathbf{a} = a^2$.

(ii) The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q.

(iii) The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm \frac{1}{\sqrt{2ap - a^2}} \mathbf{a} \times \mathbf{p},$$

where $p = |\mathbf{p}|$.

(iv) Show that the area of the triangle PQT is

$$\frac{ap^2}{2p - a}.$$

Section B: Mechanics

11 A heavy particle lies on a smooth horizontal table, and is attached to one end of a light inextensible string of length L. The other end of the string is attached to a point P on the circumference of the base of a vertical post which is fixed into the table. The base of the post is a circle of radius a with its centre at a point O on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the line OP. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time t, a length $a\theta(t)$ $(< L)$ of the string is in contact with the post.

(i) Using cartesian axes with origin O, find the position and velocity vectors of the particle at time t in terms of a, L, θ and $\dot{\theta}$, and

(ii) hence show that the speed of the particle is $(L - a\theta)\dot{\theta}$.

(iii) If the initial speed of the particle is v, show that the particle hits the post at a time $L^2/(2av)$.

12 One end of a thin uniform inextensible, but perfectly flexible, string of length l and uniform mass per unit length is held at a point on a smooth table a distance $d(< l)$ away from a small vertical hole in the surface of the table. The string passes through the hole so that a length $l - d$ of the string hangs vertically. The string is released from rest. Assuming that the height of the table is greater than l, find the time taken for the end of the string to reach the top of the hole.

13 A librarian wishes to pick up a row of identical books from a shelf, by pressing her hands on the outer covers of the two outermost books and lifting the whole row together. The covers of the books are all in parallel vertical planes, and the weight of each book is W. With each arm, the librarian can exert a maximum force of P in the vertical direction, and, independently, a maximum force of Q in the horizontal direction. The coefficient of friction between each pair of books and also between each hand and a book is μ. Derive an expression for the maximum number of books that can be picked up without slipping, using this method.

[You may assume that the books are thin enough for the rotational effect of the couple on each book to be ignored.]

14 Two particles of mass M and m $(M > m)$ are attached to the ends of a light rod of length $2l$. The rod is fixed at its midpoint to a point O on a horizontal axle so that the rod can swing freely about O in a vertical plane normal to the axle. The axle rotates about a *vertical* axis through O at a constant angular speed ω such that the rod makes a constant angle α $(0 < \alpha < \frac{1}{2}\pi)$ with the vertical.

(i) Show that

$$\omega^2 = \left(\frac{M-m}{M+m}\right)\frac{g}{l\cos\alpha}.$$

(ii) Show also that the force of reaction of the rod on the axle is inclined at an angle

$$\tan^{-1}\left[\left(\frac{M-m}{M+m}\right)^2\tan\alpha\right]$$

with the downward vertical.

Section C: Probability and Statistics

15 An examination consists of several papers, which are marked independently. The mark given for each paper can be an integer from 0 to m inclusive, and the total mark for the examination is the sum of the marks on the individual papers. In order to make the examination completely fair, the examiners decide to allocate the mark for each paper at random, so that the probability that any given candidate will be allocated k marks $(0 \leqslant k \leqslant m)$ for a given paper is $(m+1)^{-1}$.

(i) If there are just two papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{2m-n+1}{(m+1)^2}$$

for $m < n \leqslant 2m$, and find the corresponding result for $0 \leqslant n \leqslant m$.

(ii) If the examination consists of three papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{6mn-4m^2-2n^2+3m+2}{2\,(m+1)^2}$$

in the case $m < n \leqslant 2m$.

(iii) Find the corresponding result for $0 \leqslant n \leqslant m$, and deduce the result for $2m < n \leqslant 3m$.

16 Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots when B is normally distributed with zero mean and unit variance.

Given that the two roots X_1 and X_2 are real, find:

(i) the probability that both X_1 and X_2 are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$;

giving your answers to three significant figures.

Section A: Pure Mathematics

1 Prove that:

(i) if $a + 2b + 3c = 7x$, then

$$a^2 + b^2 + c^2 = (x - a)^2 + (2x - b)^2 + (3x - c)^2\,;$$

(ii) if $2a + 3b + 3c = 11x$, then

$$a^2 + b^2 + c^2 = (2x - a)^2 + (3x - b)^2 + (3x - c)^2\,.$$

Give a general result of which **(i)** and **(ii)** are special cases.

2 Show that if at least one of the four angles $A \pm B \pm C$ is a multiple of π, then

$$\sin^4 A + \sin^4 B + \sin^4 C - 2\sin^2 B \sin^2 C - 2\sin^2 C \sin^2 A \\ - 2\sin^2 A \sin^2 B + 4\sin^2 A \sin^2 B \sin^2 C = 0.$$

3 Let a and b be positive integers such that $b < 2a - 1$. For any given positive integer n, the integers N and M are defined by

$$[a + \sqrt{a^2 - b}]^n = N - r,$$

$$[a - \sqrt{a^2 - b}]^n = M + s,$$

where $0 \leqslant r < 1$ and $0 \leqslant s < 1$. Prove that

(i) $M = 0$,

(ii) $r = s$,

(iii) $r^2 - Nr + b^n = 0$.

(iv) Show that for large n, $\left(8 + 3\sqrt{7}\right)^n$ differs from an integer by about 2^{-4n}.

4 **(i)** Explain the geometrical relationship between the points in the Argand diagram represented by the complex numbers z and $z\mathrm{e}^{\mathrm{i}\theta}$.

(ii) Write down necessary and sufficient conditions that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle taken in anticlockwise order.

(iii) Show that α, β and γ represent the vertices of an equilateral triangle (taken in any order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

(iv) Find necessary and sufficient conditions on the complex coefficients a, b and c for the roots of the equation

$$z^3 + az^2 + bz + c = 0$$

to lie at the vertices of an equilateral triangle in the Argand digram.

5 If $y = \mathrm{f}(x)$, then the inverse of f (when it exists) can be obtained from *Lagrange's identity*. This identity, which you may use without proof, is

$$\mathrm{f}^{-1}(y) = y + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} \left[y - \mathrm{f}\left(y\right)\right]^n,$$

provided the series converges.

(i) Verify Lagrange's identity when $\mathrm{f}(x) = \alpha x$, $(0 < \alpha < 2)$.

(ii) Show that one root of the equation

$$\tfrac{1}{2} = x - \tfrac{1}{4}x^3$$

is

$$x = \sum_{n=0}^{\infty} \frac{(3n)!}{n!\,(2n+1)!2^{4n+1}}$$

(iii) Find a solution for x, as a series in λ, of the equation

$$x = \mathrm{e}^{\lambda x}.$$

[You may assume that the series in part **(ii)** converges, and that the series in part **(iii)** converges for suitable λ.]

6 Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1 - \sin\theta \sin 2\alpha}\, \mathrm{d}\theta\,,$$

where $0 < \alpha < \frac{1}{4}\pi$.

(i) Show that

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1 + \sin\theta \sin 2\alpha}\, \mathrm{d}\theta\,,$$

and hence that

$$I = \frac{\pi}{\sin^2 2\alpha} - \cot^2 2\alpha \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1 + \cos^2 2\alpha \tan^2\theta}\, \mathrm{d}\theta.$$

(ii) Show that $I = \frac{1}{2}\pi \sec^2\alpha$, and state the value of I if $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

7 A definite integral can be evaluated approximately by means of the Trapezium rule:

$$\int_{x_0}^{x_N} \mathrm{f}(x)\,\mathrm{d}x \approx \tfrac{1}{2}h\left\{\mathrm{f}\left(x_0\right) + 2\mathrm{f}\left(x_1\right) + \ldots + 2\mathrm{f}\left(x_{N-1}\right) + \mathrm{f}\left(x_N\right)\right\},$$

where the interval length h is given by $Nh = x_N - x_0$, and $x_r = x_0 + rh$.

(i) Justify briefly this approximation.

(ii) Use the Trapezium rule with intervals of unit length to evaluate approximately the integral

$$\int_1^n \ln x\,\mathrm{d}x,$$

where $n(> 2)$ is an integer.

(iii) Deduce that $n! \approx \mathrm{g}(n)$, where

$$\mathrm{g}(n) = n^{n+\frac{1}{2}}\mathrm{e}^{1-n},$$

and show by means of a sketch, or otherwise, that

$$n! < \mathrm{g}(n).$$

(iv) By using the Trapezium rule on the above integral with intervals of width k^{-1}, where k is a positive integer, show that

$$(kn)! \approx k!n^{kn+\frac{1}{2}}\left(\frac{\mathrm{e}}{k}\right)^{k(1-n)}.$$

Determine whether this approximation or $\mathrm{g}(kn)$ is closer to $(kn)!$.

8 Let $\mathbf{r}$ be the position vector of a point in three-dimensional space. Describe fully the locus of the point whose position vector is $\mathbf{r}$ in each of the following four cases:

(i) $(\mathbf{a}-\mathbf{b})\cdot\mathbf{r} = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2)$;

(ii) $(\mathbf{a}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{r}) = 0$;

(iii) $|\mathbf{r}-\mathbf{a}|^2 = \frac{1}{2}|\mathbf{a}-\mathbf{b}|^2$;

(iv) $|\mathbf{r}-\mathbf{b}|^2 = \frac{1}{2}|\mathbf{a}-\mathbf{b}|^2$.

Prove algebraically that the equations **(i)** and **(ii)** together are equivalent to **(iii)** and **(iv)** together. Explain carefully the geometrical meaning of this equivalence.

9 For any square matrix $\mathbf{A}$ such that $\mathbf{I}-\mathbf{A}$ is non-singular (where $\mathbf{I}$ is the unit matrix), the matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}.$$

Prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ if and only if $\mathbf{A}+\mathbf{A}^{\mathrm{T}} = \mathbf{O}$ (where $\mathbf{O}$ is the zero matrix), explaining clearly each step of your proof.

[*You may quote standard results about matrices without proof.*]

10 The set S consists of $N(>2)$ elements $a_1, a_2, \ldots, a_N$. S is acted upon by a binary operation $\circ$, defined by

$$a_j \circ a_k = a_m,$$

where m is equal to the greater of j and k.

(i) Determine, giving reasons, which of the four group axioms hold for S under $\circ$, and which do not.

(ii) Determine also, giving reasons, which of the group axioms hold for S under $*$, where $*$ is defined by

$$a_j * a_k = a_n,$$

where $n = |j-k|+1$.

Section B: Mechanics

11 A rough ring of radius a is fixed so that it lies in a plane inclined at an angle α to the horizontal. A uniform heavy rod of length $b(> a)$ has one end smoothly pivoted at the centre of the ring, so that the rod is free to move in any direction. It rests on the circumference of the ring, making an angle θ with the radius to the highest point on the circumference. Find the relation between α, θ and the coefficient of friction, μ, which must hold when the rod is in limiting equilibrium.

12 A long, inextensible string passes through a small fixed ring. One end of the string is attached to a particle of mass m, which hangs freely. The other end is attached to a bead also of mass m which is threaded on a smooth rigid wire fixed in the same vertical plane as the ring. The curve of the wire is such that the system can be in static equilibrium for all positions of the bead. The shortest distance between the wire and the ring is $d(> 0)$.

(i) Using plane polar coordinates centred on the ring, find the equation of the curve.

(ii) The bead is set in motion. Assuming that the string remains taut, show that the speed of the bead when it is a distance r from the ring is

$$\left(\frac{r}{2r-d}\right)^{\frac{1}{2}} v,$$

where v is the speed of the bead when $r = d$.

13 Ice snooker is played on a rectangular horizontal table, of length L and width B, on which a small disc (the *puck*) slides without friction. The table is bounded by smooth vertical walls (the *cushions*) and the coefficient of restitution between the puck and any cushion is e.

(i) If the puck is hit so that it bounces off two adjacent cushions, show that its final path (after two bounces) is parallel to its original path.

(ii) The puck rests against the cushion at a point which divides the side of length L in the ratio $z : 1$. Show that it is possible, whatever z, to hit the puck so that it bounces off the three other cushions in succession clockwise and returns to the spot at which it started.

(iii) By considering these paths as z varies, explain briefly why there are two different ways in which, starting at any point away from the cushions, it is possible to perform a shot in which the puck bounces off all four cushions in succession clockwise and returns to its starting point.

14 A thin uniform elastic band of mass m, length l and modulus of elasticity λ is pushed on to a smooth circular cone of vertex angle 2α, in such a way that all elements of the band are the same distance from the vertex. It is then released from rest. Let $x(t)$ be the length of the band at time t after release, and let t_0 be the time at which the band becomes slack.

(i) Assuming that a small element of the band which subtends an angle $\delta\theta$ at the axis of the cone experiences a force, due to the tension T in the band, of magnitude $T\delta\theta$ directed towards the axis, and ignoring the effects of gravity, show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{4\pi^2\lambda}{ml}(x-l)\sin^2\alpha = 0, \qquad (0 < t < t_0).$$

(ii) Find the value of t_0.

Section C: Probability and Statistics

15 A train of length l_1 and a lorry of length l_2 are heading for a level crossing at speeds u_1 and u_2 respectively. Initially the front of the train and the front of the lorry are at distances d_1 and d_2 from the crossing.

(i) Find conditions on u_1 and u_2 under which a collision will occur. On a diagram with u_1 and u_2 measured along the x and y axes respectively, shade in the region which represents collision.

(ii) Hence show that if u_1 and u_2 are two independent random variables, both uniformly distributed on $(0, V)$, then the probability of a collision in the case when initially the back of the train is nearer to the crossing than the front of the lorry is

$$\frac{l_1 l_2 + l_2 d_1 + l_1 d_2}{2d_2 (l_2 + d_2)}.$$

(iii) Find the probability of a collision in each of the other two possible cases.

16 My two friends, who shall remain nameless, but whom I shall refer to as P and Q, both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because P tells the truth with a probability of only p, while Q (independently) tells the truth with probability q. I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in it as not. Clearly, in view of what P and Q told me, I must revise this estimate.

(i) Explain carefully why my new estimate of the probability of there being a body in the fridge should be

$$\frac{pq}{1 - p - q + 2pq}.$$

(ii) I have now been to look in the fridge, and there is indeed a body in it; perhaps more than one. It seems to me that only my enemy A, or my enemy B, or (with a bit of luck) both A and B could be in my fridge, and this morning I would have judged these three possibilities to be equally likely. But tonight I asked P and Q separately whether or not A was in the fridge, and they each said that he was. What should be my new estimate of the probability that both A and B are in my fridge?

Of course, I tell the truth always.

Section A: Pure Mathematics

1 Given that

$$f(x) = \frac{3x^2 + 2(a+b)x + ab}{x^3 + (a+b)x^2 + abx}, \qquad \text{where } a \text{ and } b \text{ are non-zero}$$

express $f(x)$ in partial fractions, considering any special case which may arise.

If x, a and b are positive integers, show that $f(x)$ takes the value 1 for only a finite number of values of x, a and b.

2 Given that x, y and z satisfy the equations

$$x^2 - yz = a,$$
$$y^2 - zx = b,$$
$$z^2 - xy = c,$$

where a, b and c are positive distinct real numbers, show that

$$\frac{y-z}{b-c} = \frac{z-x}{c-a} = \frac{x-y}{a-b} = \frac{1}{x+y+z}.$$

By considering

$$(y-z)^2 + (z-x)^2 + (x-y)^2,$$

or otherwise, show that

$$x + y + z = \triangle,$$

where

$$\triangle^2 = \frac{a^2 + b^2 + c^2 - bc - ca - ab}{a+b+c}.$$

Hence solve the given equations for x, y and z.

3 Prove de Moivre's theorem, that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta,$$

where n is a positive integer.

Find all real numbers x and y which satisfy

$$x^3\cos 3y + 2x^2\cos 2y + 2x\cos y = -1,$$
$$x^3\sin 3y + 2x^2\sin 2y + 2x\sin y = 0.$$

4 **(i)** Show that

$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x),$$

when principal values only are considered.

(ii) Show that

$$\sinh^{-1}(\tan y) = \tanh^{-1}(\sin y),$$

when $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

Sketch the graphs of $\sinh^{-1}(\tan y)$ and $\tanh^{-1}(\sin y)$ in the interval $-\pi < y < \pi$ and find the relationship between the two expressions when $\frac{1}{2}\pi < y < \pi$.

5 Explain, by means of a sketch, or otherwise, why

$$\sum_{r=n}^{\infty}\frac{1}{r^2} > \int_n^{\infty}\frac{\mathrm{d}x}{x^2} > \sum_{r=n+1}^{\infty}\frac{1}{r^2}.$$

Deduce that

$$\frac{1}{n} > A - \sum_{r=1}^{n}\frac{1}{r^2} > \frac{1}{n+1}, \qquad \text{where } A = \sum_{r=1}^{\infty}\frac{1}{r^2}.$$

Find the smallest value of n for which $\sum_{r=1}^{n}\frac{1}{r^2}$ approximates A with an error of less than 10^{-4}. Show that, for this n,

$$\frac{1}{n+1} + \sum_{r=1}^{n}\frac{1}{r^2}$$

approximates A with an error of less than 10^{-8}.

6 *In this question, standard properties of exponential, logarithmic and trigonometric functions should not be used.*

A function f satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)] = \mathrm{f}(x)$$

with $\mathrm{f}(0) = 1$. For any fixed number a, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(a-x)\mathrm{f}(x)] = 0,$$

and deduce that $\mathrm{f}(x)\mathrm{f}(y) = \mathrm{f}(x+y)$ for all x and y.

Functions c and s satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{c}(x)] = -\mathrm{s}(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{s}(x)] = \mathrm{c}(x),$$

with $\mathrm{s}(0) = 0$ and $\mathrm{c}(0) = 1$. Show that

$$\mathrm{c}(x+y) = \mathrm{c}(x)\mathrm{c}(y) - \mathrm{s}(x)\mathrm{s}(y).$$

7 Let

$$I = \int_0^{\ln K} [\mathrm{e}^x]\,\mathrm{d}x,$$

where the notation $[y]$ means the largest integer less than or equal to y. Show that

$$I = N\ln K - \ln(N!),$$

where $N = [K]$.

8 Let S be the set of consecutive integers $1, 2, \ldots, (N-1)$, where $N \geqslant 3$, and let G be a subset of S which forms a group under multiplication modulo N. Show that if $(N-1) \in G$, then the order of $(N-1)$ is 2.

Let m and n be elements of G, with orders p and q respectively, such that $m + n = N$. Explain your reasoning carefully, show that

(i) if p and q are both even, then $p = q$,

(ii) if p is even and q is off, then $p = 2q$,

(iii) it is impossible for both p and q to be odd.

Now suppose that

$$G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\},$$

which may be assumed to form a group under multiplication modulo 21. Calculate the order of the elements 2 and 5 of this group. By making deductions about the orders of all other elements of G, or otherwise, prove that G is not isomorphic to the cyclic group of order 12.

9 **(i)** Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(ii) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

10 The distinct island of Amphibia is populated by speaking frogs and toads. They spend much of their time in small groups, making statements about themselves. Toads always tell the truth and frogs always lie. In each of the following four scenes from Amphibian life, decide which characters mentioned are frogs and which are toads, explaining your reasoning carefully:

(i) A: "Both B and myself are frogs"

(ii) C: "At least one of D and myself is a frog."

(iii) E: "Both G and H are toads."

G: "This is true."

H: "No, that is not true."

(iv) I and J talking about I, J and K:

I: "All of us are frogs."

J: "Exactly one of us is a toad."

Section B: Mechanics

11 Two points A and B are at a distance a apart on a horizontal plane. A particle of mass m is projected from A with speed V, at an angle of elevation of $45°$ to the line AB. Another particle, also of mass m, is projected from B with speed U at an angle of elevation of $30°$ to the line BA so that the two particles collide at the instant when each particle is at the highest point of its trajectory.

Show that $U^2 = 2V^2$ and that

$$a = \frac{V^2}{2g}(1 + \sqrt{3}).$$

At impact the two particles coalesce. When the combined particle strikes the horizontal plane the velocity of the particle is inclined at an angle ϕ to the horizontal. Show that $\tan\phi = 1 + \sqrt{3}$.

12 A thin smooth wire in the form of a circle, of radius a and centre O, is fixed in a horizontal plane. Two small beads A and B, each of mass m, are threaded on the wire and are connected by a light straight spring of natural length $2a\sin\alpha$ and modulus λ, where $0 < \alpha < \frac{1}{4}\pi$. The spring is compressed so that the angle AOB is 2β and the beads are then released from rest. Show that in the ensuing motion

$$ma\dot{\theta}^2\sin\alpha + \lambda(\sin\theta - \sin\alpha)^2 = \lambda(\sin\beta - \sin\alpha)^2$$

where 2θ denotes the angle AOB at time t after release.

(i) If $\beta - \alpha$ is small, show that T, the preiod of oscillations, is given approximately by

$$T = 2\pi\sqrt{\frac{ma\sin\alpha}{\lambda\cos^2\alpha}}.$$

(ii) If $\beta - \alpha$ is not small, write down an expression, in the form of a definite integral, for the exact period of oscillations, in each of the two cases (a) $\sin\beta > 2\sin\alpha - 1$ and (b) $\sin\beta < 2\sin\alpha - 1$.

13 A chocolate orange consists of a solid sphere of uniform chocolate of mass M and radius a, sliced into segments by planes through its axis. It stands on a horizontal table with its axis vertical, and it is held together only by a narrow ribbon round its equator.

Show that the tension in the ribbon is at least $\frac{3}{32}Mg$.

[*You may assume that the centre of mass of a segment of angle* 2θ *is at a distance* $\dfrac{3\pi a \sin\theta}{16\theta}$ *from the axis.*]

14 A uniform disc of mass M and radius a is free to rotate in a horizontal plane about a fixed vertical axis through the centre, O, of the disc. A particle of mass $\frac{1}{2}M$ is attached by a light straight wire of length $a/2$ to the vertical axis at O, so that the particle can rotate freely about the vertical axis. The particle, initially at rest, is placed gently on the disc at time $t = 0$, when the disc is spinning with angular speed Ω. Relative motion between the particle and disc is opposed by a frictional force of magnitude $Mak(\omega_1 - \omega_2)$, where, at time t, ω_1 is the angular speed of the disc, ω_2 is the angular speed of the wire, and k is a constant. Derive equations for the rate of change of ω_1 and ω_2, and show that

$$4\omega_1 + \omega_2 = 4\Omega.$$

Show further that

$$\omega_1 = \frac{\Omega}{5}(4 + \mathrm{e}^{-5kt}).$$

Section C: Probability and Statistics

15 The King of Smorgasbrod wishes to raise as much money as possible by fining people who sell underweight cartons of kippers. The weight of a kipper is normally distributed with mean 200 grams and standard deviation 10 grams. Kippers are packed in cartons of 625, and vast quantities of them are sold.

Every day a carton is to be selected at random from each vendor of Kippers. Three schemes for determining the fines are proposed:

(i) Weight the entire carton, and find the vendor 1500 crowns if the average weight of a kipper is less than 199 grams.

(ii) Weigh 25 kippers selected at random from the carton and fine the vendor 100 crowns if the average weight of a kipper is less than 198 grams.

(iii) Remove kippers one at a time and at random from the carton until a kipper has been found which weighs *more* than 200 grams and fine the vendor $3n(n-1)$ crowns, where n is the number of kippers removed.

Determine which scheme the king should select, justifying your answer.

16 A tennis tournament is arranged for 2^n players. It is organised as a knockout tournament so that only the winners in any given round proceed to the next round. Opponents in each round except the final are drawn at random, and in any match either player has a probability of $\frac{1}{2}$ of winning. Two players are chosen at random before the draw for the first round. Find the probabilities that they play each other:

(i) in the first round,

(ii) in the final,

(iii) in the tournament.

Section A: Pure Mathematics

1 **(i)** The function f is given by

$$\mathrm{f}(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

Find the stationary point of the curve $y = \mathrm{f}(\beta)$ and sketch the curve.

Sketch also the curve $y = \mathrm{g}(\beta)$, where

$$\mathrm{g}(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

(ii) Let u and v be the roots of the equation

$$x^2 + \alpha x + \beta = 0\,,$$

where $\beta \neq 0$. Obtain expressions in terms of α and β for $u + v + \dfrac{1}{uv}$ and $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

(iii) Given that $u + v + \dfrac{1}{uv} = -1$, and that u and v are real, show that $\dfrac{1}{u} + \dfrac{1}{v} + uv \leqslant -1$.

(iv) Given instead that $u + v + \dfrac{1}{uv} = 3$, and that u and v are real, find the greatest value of $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

2 The sequence of functions y_0, y_1, y_2, ... is defined by $y_0 = 1$ and, for $n \geqslant 1$,

$$y_n = (-1)^n \frac{1}{z}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}\,,$$

where $z = \mathrm{e}^{-x^2}$.

(i) Show that $\dfrac{\mathrm{d}y_n}{\mathrm{d}x} = 2xy_n - y_{n+1}$ for $n \geqslant 1$.

(ii) Prove by induction that, for $n \geqslant 1$,

$$y_{n+1} = 2xy_n - 2ny_{n-1}\,.$$

Deduce that, for $n \geqslant 1$,

$$y_{n+1}^2 - y_n y_{n+2} = 2n\big(y_n^2 - y_{n-1}y_{n+1}\big) + 2y_n^2\,.$$

(iii) Hence show that $y_n^2 - y_{n-1}y_{n+1} > 0$ for $n \geqslant 1$.

3 Show that the second-order differential equation

$$x^2y'' + (1-2p)x\,y' + (p^2-q^2)\,y = \mathrm{f}(x)\,,$$

where p and q are constants, can be written in the form

$$x^a\big(x^b(x^cy)'\big)' = \mathrm{f}(x)\,, \qquad (*)$$

where a, b and c are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = 0$$

in the different cases that arise according to the values of p and q.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of p and n.

4 The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at P can be written as

$$bx - ay\sin\theta = ab\cos\theta\,.$$

(i) This tangent meets the lines $\dfrac{x}{a} = \dfrac{y}{b}$ and $\dfrac{x}{a} = -\dfrac{y}{b}$ at S and T, respectively.

How is the mid-point of ST related to P?

(ii) The point $Q(a\sec\phi, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at P and Q are perpendicular. These two tangents intersect at (x, y).

Obtain expressions for x^2 and y^2 in terms of a, θ and ϕ.

Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.

5 The real numbers a_1, a_2, a_3, ... are all positive. For each positive integer n, A_n and G_n are defined by

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad G_n = \big(a_1 a_2 \cdots a_n\big)^{1/n}.$$

(i) Show that, for any given positive integer k,

$$(k+1)(A_{k+1} - G_{k+1}) \geqslant k(A_k - G_k)$$

if and only if

$$\lambda_k^{k+1} - (k+1)\lambda_k + k \geqslant 0\,,$$

where $\lambda_k = \left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$.

(ii) Let

$$\mathrm{f}(x) = x^{k+1} - (k+1)x + k\,,$$

where $x > 0$ and k is a positive integer. Show that $\mathrm{f}(x) \geqslant 0$ and that $\mathrm{f}(x) = 0$ if and only if $x = 1$.

(iii) Deduce that:

(a) $A_n \geqslant G_n$ for all n;

(b) if $A_n = G_n$ for some n, then $a_1 = a_2 = \cdots = a_n$.

6 **(i)** The distinct points A, Q and C lie on a straight line in the Argand diagram, and represent the distinct complex numbers a, q and c, respectively. Show that $\dfrac{q-a}{c-a}$ is real and hence that $(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$.

Given that $aa^* = cc^* = 1$, show further that

$$q + acq^* = a + c\,.$$

(ii) The distinct points A, B, C and D lie, in anticlockwise order, on the circle of unit radius with centre at the origin (so that, for example, $aa^* = 1$). The lines AC and BD meet at Q. Show that

$$(ac - bd)q^* = (a + c) - (b + d)\,,$$

where b and d are complex numbers represented by the points B and D respectively, and show further that

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab)\,.$$

(iii) The lines AB and CD meet at P, which represents the complex number p. Given that p is real, show that $p(1 + ab) = a + b$. Given further that $ac - bd \neq 0$, show that

$$p(q + q^*) = 2\,.$$

7 **(i)** Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then

$$\frac{(\cot\theta + \mathrm{i})^{2n+1} - (\cot\theta - \mathrm{i})^{2n+1}}{2\mathrm{i}} = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta},$$

for any positive integer n.

Deduce that the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right)$$

where $m = 1,\ 2,\ \ldots,\ n$.

(ii) Hence show that

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}.$$

(iii) Given that $0 < \sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta.$$

Hence show that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

8 In this question, you should ignore issues of convergence.

(i) Let

$$I = \int_0^1 \frac{\mathrm{f}(x^{-1})}{1+x}\,\mathrm{d}x,$$

where $\mathrm{f}(x)$ is a function for which the integral exists.

Show that

$$I = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\mathrm{f}(y)}{y(1+y)}\,\mathrm{d}y$$

and deduce that, if $\mathrm{f}(x) = \mathrm{f}(x+1)$ for all x, then

$$I = \int_0^1 \frac{\mathrm{f}(x)}{1+x}\,\mathrm{d}x.$$

(ii) The *fractional part*, $\{x\}$, of a real number x is defined to be $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example $\{3.2\} = 0.2$ and $\{3\} = 0$.

Use the result of part (i) to evaluate

$$\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x.$$

Section B: Mechanics

9 A particle P of mass m is projected with speed u_0 along a smooth horizontal floor directly towards a wall. It collides with a particle Q of mass km which is moving directly away from the wall with speed v_0. In the subsequent motion, Q collides alternately with the wall and with P. The coefficient of restitution between Q and P is e, and the coefficient of restitution between Q and the wall is 1.

Let u_n and v_n be the velocities of P and Q, respectively, towards the wall after the nth collision between P and Q.

(i) Show that, for $n \geqslant 2$,

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0\,. \qquad (*)$$

(ii) You are now given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, and that the solution of $(*)$ is of the form

$$u_n = A\left(\tfrac{7}{10}\right)^n + B\left(\tfrac{5}{7}\right)^n \qquad (n \geqslant 0)\,,$$

where A and B are independent of n. Find expressions for A and B in terms of u_0 and v_0.

Show that, if $0 < 6u_0 < v_0$, then u_n will be negative for large n.

10 A uniform disc with centre O and radius a is suspended from a point A on its circumference, so that it can swing freely about a horizontal axis L through A. The plane of the disc is perpendicular to L. A particle P is attached to a point on the circumference of the disc. The mass of the disc is M and the mass of the particle is m.

(i) In equilibrium, the disc hangs with OP horizontal, and the angle between AO and the downward vertical through A is β. Find $\sin\beta$ in terms of M and m and show that

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}\,.$$

(ii) The disc is rotated about L and then released. At later time t, the angle between OP and the horizontal is θ; when P is higher than O, θ is positive and when P is lower than O, θ is negative. Show that

$$\tfrac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta\,(1-\cos\theta)$$

is constant during the motion, where I is the moment of inertia of the disc about L.

(iii) Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$, show that the period of small oscillations is

$$3\pi\sqrt{\frac{3a}{5g}}\,.$$

11 A particle is attached to one end of a light inextensible string of length b. The other end of the string is attached to a fixed point O. Initially the particle hangs vertically below O. The particle then receives a horizontal impulse.

(i) The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is α, at which time it becomes slack. Express V, the speed of the particle when the string becomes slack, in terms of b, g and α.

(ii) Show that the string becomes taut again a time T later, where

$$gT = 4V\sin\alpha\,,$$

and that just before this time the trajectory of the particle makes an angle β with the horizontal where $\tan\beta = 3\tan\alpha$.

(iii) When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}\,.$$

Section C: Probability and Statistics

12 A random process generates, independently, n numbers each of which is drawn from a uniform (rectangular) distribution on the interval 0 to 1. The random variable Y_k is defined to be the kth smallest number (so there are $k-1$ smaller numbers).

(i) Show that, for $0 \leqslant y \leqslant 1$,

$$\mathrm{P}\big(Y_k \leqslant y\big) = \sum_{m=k}^{n} \binom{n}{m} y^m \,(1-y)^{n-m} . \qquad (*)$$

(ii) Show that

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

and obtain a similar expression for $(n-m)\dbinom{n}{m}$.

Starting from $(*)$, show that the probability density function of Y_k is

$$n\binom{n-1}{k-1} y^{k-1}\,(1-y)^{n-k} .$$

Deduce an expression for $\displaystyle\int_0^1 y^{k-1}(1-y)^{n-k}\,\mathrm{d}y$.

(iii) Find $\mathrm{E}(Y_k)$ in terms of n and k.

13 The random variable X takes only non-negative integer values and has probability generating function $\mathrm{G}(t)$. Show that

$$\mathrm{P}(X = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6 \ \ldots) = \frac{1}{2}\big(\mathrm{G}\,(1) + \mathrm{G}\,(-1)\big).$$

You are now given that X has a Poisson distribution with mean λ. Show that

$$\mathrm{G}(t) = \mathrm{e}^{-\lambda(1-t)}\,.$$

(i) The random variable Y is defined by

$$\mathrm{P}(Y = r) = \begin{cases} k\mathrm{P}(X = r) & \text{if } r = 0,\ 2,\ 4,\ 6,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where k is an appropriate constant.

Show that the probability generating function of Y is $\dfrac{\cosh \lambda t}{\cosh \lambda}$.

Deduce that $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

(ii) The random variable Z is defined by

$$\mathrm{P}(Z = r) = \begin{cases} c\mathrm{P}(X = r) & \text{if } r = 0,\ 4,\ 8,\ 12,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where c is an appropriate constant.

Is $\mathrm{E}(Z) < \lambda$ for all positive values of λ?

Section A: Pure Mathematics

1 **(i)** Prove that, for any positive integers n and r,

$$\frac{1}{{}^{n+r}\mathrm{C}_{r+1}} = \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}\mathrm{C}_r} - \frac{1}{{}^{n+r}\mathrm{C}_r}\right).$$

Hence determine

$$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}\mathrm{C}_{r+1}}\,,$$

and deduce that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}\mathrm{C}_3} = \frac{1}{2}$.

(ii) Show that, for $n \geqslant 3$,

$$\frac{3!}{n^3} < \frac{1}{{}^{n+1}\mathrm{C}_3} \qquad \text{and} \qquad \frac{20}{{}^{n+1}\mathrm{C}_3} - \frac{1}{{}^{n+2}\mathrm{C}_5} < \frac{5!}{n^3}\,.$$

By summing these inequalities for $n \geqslant 3$, show that

$$\frac{115}{96} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{116}{96}\,.$$

Note: ${}^n\mathrm{C}_r$ is another notation for $\binom{n}{r}$.

2 The transformation R in the complex plane is a rotation (anticlockwise) by an angle θ about the point represented by the complex number a. The transformation S in the complex plane is a rotation (anticlockwise) by an angle ϕ about the point represented by the complex number b.

(i) The point P is represented by the complex number z. Show that the image of P under R is represented by

$$\mathrm{e}^{\mathrm{i}\theta}z + a(1 - \mathrm{e}^{\mathrm{i}\theta})\,.$$

(ii) Show that the transformation SR (equivalent to R followed by S) is a rotation about the point represented by c, where

$$c\,\sin\tfrac{1}{2}(\theta+\phi) = a\,\mathrm{e}^{\mathrm{i}\phi/2}\sin\tfrac{1}{2}\theta + b\,\mathrm{e}^{-\mathrm{i}\theta/2}\sin\tfrac{1}{2}\phi\,,$$

provided $\theta + \phi \neq 2n\pi$ for any integer n.

What is the transformation SR if $\theta + \phi = 2\pi$?

(iii) Under what circumstances is $RS = SR$?

3 Let α, β, γ and δ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0\,.$$

You are given that, for any such equation, $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0\,.$$

Determine A.

Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

(i) Find the value of $\alpha\beta+\gamma\delta$, given that it is the largest root of the corresponding cubic equation.

(ii) Hence, using the values of q and s, find the value of $(\alpha+\beta)(\gamma+\delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.

(iii) Using these results, and the values of p and r, solve the quartic equation.

4 For any function f satisfying $\mathrm{f}(x) > 0$, we define the *geometric mean*, F, by

$$\mathrm{F}(y) \;=\; \mathrm{e}^{\frac{1}{y}\int_0^y \ln \mathrm{f}(x)\,\mathrm{d}x} \qquad (y > 0)\,.$$

(i) The function f satisfies $\mathrm{f}(x) > 0$ and a is a positive number with $a \neq 1$. Prove that

$$\mathrm{F}(y) = \; a^{\frac{1}{y}\int_0^y \log_a \mathrm{f}(x)\,\mathrm{d}x} \,.$$

(ii) The functions f and g satisfy $\mathrm{f}(x) > 0$ and $\mathrm{g}(x) > 0$, and the function h is defined by $\mathrm{h}(x) = \mathrm{f}(x)\mathrm{g}(x)$. Their geometric means are F, G and H, respectively. Show that $\mathrm{H}(y) = \mathrm{F}(y)\mathrm{G}(y)$.

(iii) Prove that, for any positive number b, the geometric mean of b^x is $\sqrt{b^y}$.

(iv) Prove that, if $\mathrm{f}(x) > 0$ and the geometric mean of $\mathrm{f}(x)$ is $\sqrt{\mathrm{f}(y)}$, then $\mathrm{f}(x) = b^x$ for some positive number b.

5 **(i)** The point with cartesian coordinates (x, y) lies on a curve with polar equation $r = \mathrm{f}(\theta)$. Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\mathrm{f}(\theta)$, $\mathrm{f}'(\theta)$ and $\tan\theta$.

(ii) Two curves, with polar equations $r = \mathrm{f}(\theta)$ and $r = \mathrm{g}(\theta)$, meet at right angles. Show that where they meet

$$\mathrm{f}'(\theta)\mathrm{g}'(\theta) + \mathrm{f}(\theta)\mathrm{g}(\theta) = 0\,.$$

(iii) The curve C has polar equation $r = \mathrm{f}(\theta)$ and passes through the point given by $r = 4$, $\theta = -\frac{1}{2}\pi$. For each positive value of a, the curve with polar equation $r = a(1 + \sin\theta)$ meets C at right angles. Find $\mathrm{f}(\theta)$.

(iv) Sketch on a single diagram the three curves with polar equations $r = 1+\sin\theta$, $r = 4(1+\sin\theta)$ and $r = \mathrm{f}(\theta)$.

6 *In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.*

The function T is defined for $x > 0$ by

$$\mathrm{T}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,,$$

and $T_\infty = \displaystyle\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u$ (which has a finite value).

(i) By making an appropriate substitution in the integral for $\mathrm{T}(x)$, show that

$$\mathrm{T}(x) = \mathrm{T}_\infty - \mathrm{T}(x^{-1})\,.$$

(ii) Let $v = \dfrac{u+a}{1-au}$, where a is a constant. Verify that, for $u \neq a^{-1}$,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}\,.$$

Hence show that, for $a > 0$ and $x < \dfrac{1}{a}$,

$$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)\,.$$

Deduce that

$$\mathrm{T}(x^{-1}) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a^{-1})$$

and hence that, for $b > 0$ and $y > \dfrac{1}{b}$,

$$\mathrm{T}(y) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{y+b}{by-1}\right) - \mathrm{T}(b)\,.$$

(iii) Use the above results to show that $\mathrm{T}(\sqrt{3}) = \frac{2}{3}\mathrm{T}_\infty$ and $\mathrm{T}(\sqrt{2}-1) = \frac{1}{4}\mathrm{T}_\infty$.

7 Show that the point T with coordinates

$$\left(\frac{a(1-t^2)}{1+t^2}\,,\ \frac{2bt}{1+t^2}\right) \qquad (*)$$

(where a and b are non-zero) lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

(i) The line L is the tangent to the ellipse at T. The point (X, Y) lies on L, and $X^2 \neq a^2$. Show that

$$(a+X)bt^2 - 2aYt + b(a-X) = 0\,.$$

Deduce that if $a^2Y^2 > (a^2 - X^2)b^2$, then there are two distinct lines through (X, Y) that are tangents to the ellipse. Interpret this result geometrically. Show, by means of a sketch, that the result holds also if $X^2 = a^2$.

(ii) The distinct points P and Q are given by $(*)$, with $t = p$ and $t = q$, respectively. The tangents to the ellipse at P and Q meet at the point with coordinates (X, Y), where $X^2 \neq a^2$. Show that

$$(a+X)pq = a - X$$

and find an expression for $p+q$ in terms of a, b, X and Y.

Given that the tangents meet the y-axis at points $(0, y_1)$ and $(0, y_2)$, where $y_1 + y_2 = 2b$, show that

$$\frac{X^2}{a^2}+\frac{Y}{b}=1\,.$$

8 Prove that, for any numbers a_1, a_2, $\ldots$, and b_1, b_2, $\ldots$, and for $n \geqslant 1$,

$$\sum_{m=1}^{n} a_m(b_{m+1}-b_m) = a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m)\,.$$

(i) By setting $b_m = \sin mx$, show that

$$\sum_{m=1}^{n} \cos(m+\tfrac{1}{2})x = \tfrac{1}{2}\big(\sin(n+1)x - \sin x\big)\,\operatorname{cosec}\tfrac{1}{2}x\,.$$

Note: $\sin A - \sin B = 2\cos\big(\frac{A+B}{2}\big)\,\sin\big(\frac{A-B}{2}\big)$.

(ii) Show that

$$\sum_{m=1}^{n} m\sin mx = \big(p\sin(n+1)x + q\sin nx\big)\,\operatorname{cosec}^2\tfrac{1}{2}x\,,$$

where p and q are to be determined in terms of n.

Note: $2\sin A\sin B = \cos(A-B) - \cos(A+B)$;

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$.

Section B: Mechanics

9 Two particles A and B of masses m and $2m$, respectively, are connected by a light spring of natural length a and modulus of elasticity λ. They are placed on a smooth horizontal table with AB perpendicular to the edge of the table, and A is held on the edge of the table. Initially the spring is at its natural length.

(i) Particle A is released. At a time t later, particle A has dropped a distance y and particle B has moved a distance x from its initial position (where $x < a$). Show that $y + 2x = \frac{1}{2}gt^2$.

(ii) The value of λ is such that particle B reaches the edge of the table at a time T given by $T = \sqrt{6a/g}$. By considering the total energy of the system (without solving any differential equations), show that the speed of particle B at this time is $\sqrt{2ag/3}$.

10 A uniform rod PQ of mass m and length $3a$ is freely hinged at P.

(i) The rod is held horizontally and a particle of mass m is placed on top of the rod at a distance ℓ from P, where $\ell < 2a$. The coefficient of friction between the rod and the particle is μ.

The rod is then released. Show that, while the particle does not slip along the rod,

$$(3a^2 + \ell^2)\dot{\theta}^2 = g(3a + 2\ell)\sin\theta\,,$$

where θ is the angle through which the rod has turned, and the dot denotes the time derivative.

(ii) Hence, or otherwise, find an expression for $\ddot{\theta}$ and show that the normal reaction of the rod on the particle is non-zero when θ is acute.

(iii) Show further that, when the particle is on the point of slipping,

$$\tan\theta = \frac{\mu a(2a - \ell)}{2(\ell^2 + a\ell + a^2)}\,.$$

(iv) What happens at the moment the rod is released if, instead, $\ell > 2a$?

11 A railway truck, initially at rest, can move forwards without friction on a long straight horizontal track. On the truck, n guns are mounted parallel to the track and facing backwards, where $n > 1$. Each of the guns is loaded with a single projectile of mass m. The mass of the truck and guns (but not including the projectiles) is M.

When a gun is fired, the projectile leaves its muzzle horizontally with a speed $v - V$ relative to the ground, where V is the speed of the truck immediately before the gun is fired.

(i) All n guns are fired simultaneously. Find the speed, u, with which the truck moves, and show that the kinetic energy, K, which is gained by the system (truck, guns and projectiles) is given by

$$K = \tfrac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right).$$

(ii) Instead, the guns are fired one at a time. Let u_r be the speed of the truck when r guns have been fired, so that $u_0 = 0$. Show that, for $1 \leqslant r \leqslant n$,

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m} \qquad (*)$$

and hence that $u_n < u$.

(iii) Let K_r be the total kinetic energy of the system when r guns have been fired (one at a time), so that $K_0 = 0$. Using $(*)$, show that, for $1 \leqslant r \leqslant n$,

$$K_r - K_{r-1} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv(u_r - u_{r-1})$$

and hence show that

$$K_n = \tfrac{1}{2}nmv^2 + \tfrac{1}{2}mvu_n.$$

Deduce that $K_n < K$.

Section C: Probability and Statistics

12 The discrete random variables X and Y can each take the values $1, \ldots, n$ (where $n \geqslant 2$). Their joint probability distribution is given by

$$\mathrm{P}(X = x,\ Y = y) = k(x + y)\,,$$

where k is a constant.

(i) Show that

$$\mathrm{P}(X = x) = \frac{n + 1 + 2x}{2n(n + 1)}\,.$$

Hence determine whether X and Y are independent.

(ii) Show that the covariance of X and Y is negative.

13 The random variable X has mean μ and variance σ^2, and the function V is defined, for $-\infty < x < \infty$, by

$$\mathrm{V}(x) = \mathrm{E}\big((X - x)^2\big).$$

(i) Express $\mathrm{V}(x)$ in terms of x, μ and σ.

(ii) The random variable Y is defined by $Y = \mathrm{V}(X)$. Show that

$$\mathrm{E}(Y) = 2\sigma^2. \qquad (*)$$

(iii) Now suppose that X is uniformly distributed on the interval $0 \leqslant x \leqslant 1$. Find $\mathrm{V}(x)$. Find also the probability density function of Y and use it to verify that $(*)$ holds in this case.

Section A: Pure Mathematics

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2+2ax+b)^n}\,\mathrm{d}x\,,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b-a^2}\,\tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b-a^2}}\,.$$

(ii) Show that $2n(b-a^2)\,I_{n+1} = (2n-1)\,I_n$.

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1}\,.$$

2 The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P.

(iii) Let T be the point of intersection of OP and QR, where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P.

Show further that the distance from the x-axis to T is less than $\dfrac{a}{\sqrt{2}}$.

3 **(i)** Given that

$$\int \frac{x^3-2}{(x+1)^2}\,\mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\,\mathrm{e}^x + \text{constant}\,,$$

where $\mathrm{P}(x)$and $\mathrm{Q}(x)$ are polynomials, show that $\mathrm{Q}(x)$ has a factor of $x+1$.

Show also that the degree of $\mathrm{P}(x)$ is exactly one more than the degree of $\mathrm{Q}(x)$, and find $\mathrm{P}(x)$ in the case $\mathrm{Q}(x) = x+1$.

(ii) Show that there are no polynomials $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ such that

$$\int \frac{1}{x+1}\,\mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\,\mathrm{e}^x + \text{constant}\,.$$

You need consider only the case when $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ have no common factors.

4 **(i)** By considering $\dfrac{1}{1+x^r} - \dfrac{1}{1+x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})}\,.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{x}{1-x^2}\,.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 2\mathrm{e}^{-y}\operatorname{cosech}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y)\,,$$

for $y > 0$.

5 **(i)** By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer m.

(ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s, if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m, $P_{m+1\,,\,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1\,,\,2m+1} < 2^{2m}.$$

(iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

(iv) Prove that $\mathrm{P}_{1,n} < 4^n$ for $n \geqslant 2$.

6 Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\operatorname{arcosh}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \operatorname{artanh}\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.

7 Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1\,.$$

The points X_0, X_1, ... , X_{n-1} lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1\,,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n\,.$$

8 **(i)** The function f satisfies, for all x, the equation

$$\mathrm{f}(x) + (1-x)\mathrm{f}(-x) = x^2\,.$$

Show that $\mathrm{f}(-x) + (1+x)\mathrm{f}(x) = x^2$. Hence find $\mathrm{f}(x)$ in terms of x. You should verify that your function satisfies the original equation.

(ii) The function K is defined, for $x \neq 1$, by

$$\mathrm{K}(x) = \frac{x+1}{x-1}\,.$$

Show that, for $x \neq 1$, $\mathrm{K}(\mathrm{K}(x)) = x$.

The function g satisfies the equation

$$\mathrm{g}(x) + x\,\mathrm{g}\Big(\frac{x+1}{x-1}\Big) = x \qquad (x \neq 1)\,.$$

Show that, for $x \neq 1$, $\mathrm{g}(x) = \dfrac{2x}{x^2+1}$.

(iii) Find $\mathrm{h}(x)$, for $x \neq 0$, $x \neq 1$, given that

$$\mathrm{h}(x) + \mathrm{h}\Big(\frac{1}{1-x}\Big) = 1 - x - \frac{1}{1-x} \qquad (x \neq 0,\quad x \neq 1)\,.$$

Section B: Mechanics

9 Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}\,a$. Initially the particle is in equilibrium.

(i) Show that the extension in each spring is $\frac{2}{\sqrt{3}}\,a - l$.

(ii) The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}\; - l\right),$$

where x is the displacement of X from its equilibrium position.

(iii) Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}\,l)\lambda}}\;.$$

10 A smooth plane is inclined at an angle α to the horizontal. A particle P of mass m is attached to a fixed point A above the plane by a light inextensible string of length a. The particle rests in equilibrium on the plane, and the string makes an angle β with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of u.

(i) Show that the particle will not immediately leave the plane if $ag\cos(\alpha + \beta) > u^2 \tan\beta$.

(ii) Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6\tan\alpha\tan\beta < 1$.

11 A car of mass m travels along a straight horizontal road with its engine working at a constant rate P. The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed $4U$.

(i) Given that the resistance is proportional to the car's speed, show that the distance X_1 travelled by the car while it accelerates from speed U to speed $2U$, is given by

$$\lambda X_1 = 2\ln\tfrac{9}{5} - 1\,,$$

where $\lambda = P/(16mU^3)$.

(ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance X_2 travelled by the car while it accelerates from speed U to speed $2U$ is given by

$$\lambda X_2 = \tfrac{4}{3}\ln\tfrac{9}{8}\,.$$

(iii) Given that $3.17 < \ln 24 < 3.18$ and $1.60 < \ln 5 < 1.61$, determine which is the larger of X_1 and X_2.

Section C: Probability and Statistics

12 Let X be a random variable with mean μ and standard deviation σ. *Chebyshev's inequality*, which you may use without proof, is

$$\mathrm{P}\left(|X-\mu| > k\sigma\right) \leqslant \frac{1}{k^2}\,,$$

where k is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n \leqslant N \leqslant 24n$.

Use Chebyshev's inequality to show that

$$\alpha \geqslant 1 - \frac{1}{n}\,.$$

(ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geqslant \left(1 - \frac{1}{n}\right)\mathrm{e}^n\,.$$

13 Given a random variable X with mean μ and standard deviation σ, we define the *kurtosis*, κ, of X by

$$\kappa = \frac{\mathrm{E}\big((X-\mu)^4\big)}{\sigma^4} - 3\,.$$

Show that the random variable $X - a$, where a is a constant, has the same kurtosis as X.

(i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

(ii) Let Y_1, Y_2, ..., Y_n be n independent, identically distributed, random variables with mean 0, and let $T = \sum\limits_{r=1}^{n} Y_r$. Show that

$$\mathrm{E}(T^4) = \sum_{r=1}^{n} \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathrm{E}(Y_s^2)\mathrm{E}(Y_r^2)\,.$$

(iii) Let X_1, X_2, ..., X_n be n independent, identically distributed, random variables each with kurtosis κ. Show that the kurtosis of their sum is $\dfrac{\kappa}{n}$.

Section A: Pure Mathematics

1 **(i)** Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u\,,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}\,.$$

(ii) Let

$$J = \int_0^\infty \mathrm{f}\big((x - x^{-1})^2\big)\,\mathrm{d}x\,,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2}\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \frac{1}{2}\int_0^\infty (1+x^{-2})\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \int_0^\infty \mathrm{f}\big(u^2\big)\,\mathrm{d}u\,.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4 - x^2 + 1)^n}\,\mathrm{d}x\,,$$

where n is a positive integer.

2 If s_1, s_2, s_3, ... and t_1, t_2, t_3, ... are sequences of positive numbers, we write

$$(s_n) \leqslant (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leqslant t_n$ whenever $n \geqslant m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \leqslant (n^2)$.

(ii) If it is not the case that $(s_n) \leqslant (t_n)$, then it is the case that $(t_n) \leqslant (s_n)$.

(iii) If $(s_n) \leqslant (t_n)$ and $(t_n) \leqslant (u_n)$, then $(s_n) \leqslant (u_n)$.

(iv) $(n^2) \leqslant (2^n)$.

3 In this question, r and θ are polar coordinates with $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$, and a and b are positive constants.

Let L be a fixed line and let A be a fixed point not lying on L. Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a\sec\theta| = b\,, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify L, d and A). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

$$|r - a\sec\theta| = b\,.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec\theta \, \mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.]

4 **(i)** If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \quad S_2 = z_1^2 + z_2^2 + z_3^2, \quad S_3 = z_1^3 + z_2^3 + z_3^3\,,$$

where z_1, z_2 and z_3 are the roots of the equation $(*)$. Express a and b in terms of S_1 and S_2, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3\,.$$

(iii) The six real numbers r_k and θ_k $(k = 1,\ 2,\ 3)$, where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0\,, \quad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0\,, \quad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0\,.$$

Show that $\theta_k = 0$ for at least one value of k.

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

5 **(i)** In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set S to be the set of positive integers with the following property:

   n is in S if and only if $n\sqrt{2}$ is an integer.

3. Show that the set S contains at least one positive integer.
4. Define the integer k to be the smallest positive integer in S.
5. Show that $(\sqrt{2} - 1)k$ is in S.
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

(ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.

6 **(i)** Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leqslant 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{gathered} w + z - u = 0 \\ w^2 + z^2 - u^2 = -\tfrac{2}{3} \\ w^3 + z^3 - \lambda u = -\lambda \end{gathered}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

7 An operator D is defined, for any function f, by

$$\mathrm{Df}(x) = x\frac{\mathrm{df}(x)}{\mathrm{d}x}.$$

The notation D^n means that D is applied n times; for example

$$\mathrm{D}^2\mathrm{f}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{df}(x)}{\mathrm{d}x}\right).$$

Show that, for any constant a, $\mathrm{D}^2 x^a = a^2 x^a$.

(i) Show that if $\mathrm{P}(x)$ is a polynomial of degree r (where $r \geqslant 1$) then, for any positive integer n, $\mathrm{D}^n\mathrm{P}(x)$ is also a polynomial of degree r.

(ii) Show that if n and m are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n = 0.$$

8 **(i)** Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0\,.$$

Sketch a solution in the x-y plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2+y^2)\big)\ \frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2+y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A.

Section B: Mechanics

9 A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ. The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

(i) An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$ and the dot denotes differentiation with respect to t.

(ii) Find, in terms of k, a and v, the greatest value, x_0, attained by x. Find also the acceleration of P at $x = x_0$.

(iii) Obtain, in the form of an integral, an expression for the period of the motion. Show that in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1-u^4}}\,\mathrm{d}u\,.$$

10 A light rod of length $2a$ has a particle of mass m attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates (x, y), where the x-axis is horizontal (within the plane of motion) and y is the height above a horizontal table. Initially, the rod is vertical, and at time t later it is inclined at an angle θ to the vertical.

(i) Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}$$

and that

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

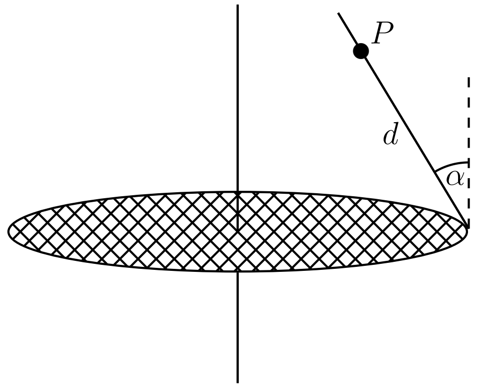
where the dots denote differentiation with respect to time t and T is the tension in the rod. Obtain the corresponding equations for the other particle.

(ii) Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.

(iii) Initially, the midpoint of the rod is a height h above the table, the velocity of the higher particle is $\begin{pmatrix} u \\ v \end{pmatrix}$, and the velocity of the lower particle is $\begin{pmatrix} 0 \\ v \end{pmatrix}$. Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva\,.$$

11 **(i)** A horizontal disc of radius r rotates about a vertical axis through its centre with angular speed ω. One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle P of mass m is attached to the rod at a distance d from the hinge. The rod makes a constant angle α with the upward vertical, as shown in the diagram, and $d\sin\alpha < r$.



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by P is parallel to the rod.

Show also that

$$r\cot\alpha = a + d\cos\alpha\,,$$

where $a = \dfrac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of m, g and α.

(ii) The disc and rod rotate as in part (i), but two particles (instead of P) are attached to the rod. The masses of the particles are m_1 and m_2 and they are attached to the rod at distances d_1 and d_2 from the hinge, respectively. The rod makes a constant angle β with the upward vertical and $d_1\sin\beta < d_2\sin\beta < r$. Show that β satisfies an equation of the form

$$r\cot\beta = a + b\cos\beta\,,$$

where b should be expressed in terms of d_1, d_2, m_1 and m_2.

Section C: Probability and Statistics

12 A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown n times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable S_n is the sum of the outcomes.

(i) The random variable R_n is the remainder when S_n is divided by 6. Write down the probability generating function, $\mathrm{G}(x)$, of R_1 and show that the probability generating function of R_2 is also $\mathrm{G}(x)$. Use a generating function to find the probability that S_n is divisible by 6.

(ii) The random variable T_n is the remainder when S_n is divided by 5. Write down the probability generating function, $\mathrm{G}_1(x)$, of T_1 and show that $\mathrm{G}_2(x)$, the probability generating function of T_2, is given by

$$\mathrm{G}_2(x) = \tfrac{1}{36}(x^2 + 7y)$$

where $y = 1 + x + x^2 + x^3 + x^4$.

Obtain the probability generating function of T_n and hence show that the probability that S_n is divisible by 5 is

$$\frac{1}{5}\left(1 - \frac{1}{6^n}\right)$$

if n is not divisible by 5. What is the corresponding probability if n is divisible by 5?

13 Each of the two independent random variables X and Y is uniformly distributed on the interval $[0, 1]$.

(i) By considering the lines $x + y = \text{constant}$ in the x-y plane, find the cumulative distribution function of $X + Y$.

(ii) Hence show that the probability density function f of $(X + Y)^{-1}$ is given by

$$\mathrm{f}(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \\ t^{-3} & \text{for } 1 \leqslant t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $\mathrm{E}\left(\dfrac{1}{X+Y}\right)$.

(iii) Find the cumulative distribution function of Y/X and use this result to find the probability density function of $\dfrac{X}{X+Y}$.

Write down $\mathrm{E}\left(\dfrac{X}{X+Y}\right)$ and verify your result by integration.

Section A: Pure Mathematics

1 Let a, b and c be real numbers such that $a+b+c=0$ and let

$$(1+ax)(1+bx)(1+cx)=1+qx^2+rx^3$$

for all real x. Show that $q=bc+ca+ab$ and $r=abc$.

(i) Show that the coefficient of x^n in the series expansion (in ascending powers of x) of $\ln(1+qx^2+rx^3)$ is $(-1)^{n+1}S_n$ where

$$S_n=\frac{a^n+b^n+c^n}{n}\,, \qquad (n\geqslant 1).$$

(ii) Find, in terms of q and r, the coefficients of x^2, x^3 and x^5 in the series expansion (in ascending powers of x) of $\ln(1+qx^2+rx^3)$ and hence show that $S_2S_3=S_5$.

(iii) Show that $S_2S_5=S_7$.

(iv) Give a proof of, or find a counterexample to, the claim that $S_2S_7=S_9$.

2 **(i)** Show, by means of the substitution $u=\cosh x$, that

$$\int\frac{\sinh x}{\cosh 2x}\,\mathrm{d}x=\frac{1}{2\sqrt2}\ln\left|\frac{\sqrt2\cosh x-1}{\sqrt2\cosh x+1}\right|+C\,.$$

(ii) Use a similar substitution to find an expression for

$$\int\frac{\cosh x}{\cosh 2x}\,\mathrm{d}x\,.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1\frac{1}{1+u^4}\,\mathrm{d}u=\frac{\pi+2\ln(\sqrt2+1)}{4\sqrt2}\,.$$

3 **(i)** The line L has equation $y = mx + c$, where $m > 0$ and $c > 0$. Show that, in the case $mc > a > 0$, the shortest distance between L and the parabola $y^2 = 4ax$ is

$$\frac{mc - a}{m\sqrt{m^2+1}}\,.$$

What is the shortest distance in the case that $mc \leqslant a$?

(ii) Find the shortest distance between the point $(p, 0)$, where $p > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the value of p/a. [*You may wish to use the parametric coordinates* $(at^2, 2at)$ *of points on the parabola.*]

Hence find the shortest distance between the circle $(x-p)^2 + y^2 = b^2$, where $p > 0$ and $b > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the values of p, a and b.

4 **(i)** Let

$$I = \int_0^1 \left((y')^2 - y^2\right) \mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2 \,\mathrm{d}x\,,$$

where y is a given function of x satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \geqslant 0$. Show further that $I = 0$ only if $y = 0$ for all x $(0 \leqslant x \leqslant 1)$.

(ii) Let

$$J = \int_0^1 \left((y')^2 - a^2y^2\right) \mathrm{d}x\,,$$

where a is a given positive constant and y is a given function of x, not identically zero, satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2 \,\mathrm{d}x\,,$$

where b is suitably chosen, show that $J \geqslant 0$. You should state the range of values of a, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function y (not everywhere zero) such that $J = 0$.

5 A quadrilateral drawn in the complex plane has vertices A, B, C and D, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers a, b, c and d. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a + c = b + d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathrm{i}(a - c) = b - d$.

Let $PQRS$ be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than $180°$. Squares with centres X, Y, Z and T are constructed externally to the quadrilateral on the sides PQ, QR, RS and SP, respectively.

(i) If P and Q are represented by the complex numbers p and q, respectively, show that X can be represented by

$$\tfrac{1}{2}\big(p(1 + \mathrm{i}) + q(1 - \mathrm{i})\big)\,.$$

(ii) Show that $XYZT$ is a square if and only if $PQRS$ is a parallelogram.

6 Starting from the result that

$$\mathrm{h}(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x \mathrm{h}(t)\,\mathrm{d}t > 0\,,$$

show that, if $\mathrm{f}''(t) > 0$ for $0 < t < x_0$ and $\mathrm{f}(0) = \mathrm{f}'(0) = 0$, then $\mathrm{f}(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

7 The four distinct points P_i ($i = 1, 2, 3, 4$) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let $\mathbf{p}_i$ be the position vector of the point P_i ($i = 1, 2, 3, 4$). Show that there exist numbers a_i, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i\mathbf{p}_i = \mathbf{0}\,. \tag{$*$}$$

(iii) Let a_i ($i = 1, 2, 3, 4$) be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\,.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.

8 The numbers $\mathrm{f}(r)$ satisfy $\mathrm{f}(r) > \mathrm{f}(r+1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer n,

$$k^n(k-1)\,\mathrm{f}(k^{n+1}) \leqslant \sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leqslant k^n(k-1)\,\mathrm{f}(k^n)$$

where k is an integer greater than 1.

(i) By taking $\mathrm{f}(r) = 1/r$, show that

$$\frac{N+1}{2} \leqslant \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leqslant N+1\,.$$

Deduce that the sum $\sum\limits_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $\mathrm{f}(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leqslant 1\tfrac{1}{3}\,.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n.

Section B: Mechanics

9 A particle of mass m is projected with velocity $\mathbf{u}$. It is acted upon by the force $m\mathbf{g}$ due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a positive constant.

(i) Given that, at time t after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + \mathrm{e}^{-kt}}{k^2}\,\mathbf{g} + \frac{1 - \mathrm{e}^{-kt}}{k}\,\mathbf{u}\,,$$

find an expression for $\mathbf{v}$ in terms of k, t, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

(ii) Let $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\,\mathbf{j}$, where $0 < \alpha < 90^\circ$, and let T be the time after projection at which $\mathbf{r}\,.\,\mathbf{j} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - \mathrm{e}^{-kT}} - 1\right) g\,.$$

(iii) Let β be the acute angle between $\mathbf{v}$ and $\mathbf{i}$ at time T. Show that

$$\tan\beta = \frac{(\mathrm{e}^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha\,.$$

(iv) Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.

10 Two particles X and Y, of equal mass m, lie on a smooth horizontal table and are connected by a light elastic spring of natural length a and modulus of elasticity λ. Two more springs, identical to the first, connect X to a point P on the table and Y to a point Q on the table. The distance between P and Q is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released.

(i) At time t, the particle X is a distance $a + x$ from P and the particle Y is a distance $a + y$ from Q. Show that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$.

(ii) Deduce that

$$x - y = A\cos\omega t + B\sin\omega t$$

where A and B are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$.

(iii) Show that Y will never return to its initial position.

11 A particle P of mass m is connected by two light inextensible strings to two fixed points A and B, with A vertically above B. The string AP has length x. The particle is rotating about the vertical through A and B with angular velocity ω, and both strings are taut. Angles PAB and PBA are α and β, respectively.

(i) Find the tensions T_A and T_B in the strings AP and BP (respectively), and hence show that $\omega^2 x\cos\alpha \geqslant g$.

(ii) Consider now the case that $\omega^2 x\cos\alpha = g$. Given that $AB = h$ and $BP = d$, where $h > d$, show that $h\cos\alpha \geqslant \sqrt{h^2 - d^2}$. Show further that

$$mg < T_A \leqslant \frac{mgh}{\sqrt{h^2 - d^2}}\,.$$

(iii) Describe the geometry of the strings when T_A attains its upper bound.

Section C: Probability and Statistics

12 The random variable X has probability density function $\mathrm{f}(x)$ (which you may assume is differentiable) and cumulative distribution function $\mathrm{F}(x)$ where $-\infty < x < \infty$. The random variable Y is defined by $Y = \mathrm{e}^X$. You may assume throughout this question that X and Y have unique modes.

(i) Find the median value y_m of Y in terms of the median value x_m of X.

(ii) Show that the probability density function of Y is $\mathrm{f}(\ln y)/y$, and deduce that the mode λ of Y satisfies $\mathrm{f}'(\ln\lambda) = \mathrm{f}(\ln\lambda)$.

(iii) Suppose now that $X \sim \mathrm{N}(\mu, \sigma^2)$, so that

$$\mathrm{f}(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\,.$$

Explain why

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-\mu-\sigma^2)^2/(2\sigma^2)}\,\mathrm{d}x = 1$$

and hence show that $\mathrm{E}(Y) = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}$.

(iv) Show that, when $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$\lambda < y_m < \mathrm{E}(Y)\,.$$

13 I play a game which has repeated rounds. Before the first round, my score is 0. Each round can have three outcomes:

1. my score is unchanged and the game ends;
2. my score is unchanged and I continue to the next round;
3. my score is increased by one and I continue to the next round.

The probabilities of these outcomes are a, b and c, respectively (the same in each round), where $a+b+c=1$ and $abc \neq 0$. The random variable N represents my score at the end of a randomly chosen game.

Let $\mathrm{G}(t)$ be the probability generating function of N.

(i) Suppose in the first round, the game ends. Show that the probability generating function conditional on this happening is 1.

(ii) Suppose in the first round, the game continues to the next round with no change in score. Show that the probability generating function conditional on this happening is $\mathrm{G}(t)$.

(iii) By comparing the coefficients of t^n, show that $\mathrm{G}(t) = a + b\mathrm{G}(t) + ct\mathrm{G}(t)$. Deduce that, for $n \geqslant 0$,

$$P(N=n) = \frac{ac^n}{(1-b)^{n+1}} \,.$$

(iv) Show further that, for $n \geqslant 0$,

$$P(N=n) = \frac{\mu^n}{(1+\mu)^{n+1}} \,,$$

where $\mu = \mathrm{E}(N)$.

Section A: Pure Mathematics

1 **(i)** Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

(ii) Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0<a<1).$$

(iii) Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

2 In this question, you may ignore questions of convergence.

(i) Let $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

(ii) and prove that, for any positive integer n,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0\,.$$

(iii) Hence obtain the Maclaurin series for $\dfrac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of x.

(iv) Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times 3^2}{7!} + \cdots + \frac{2^2\times 3^2\times\cdots\times n^2}{(2n+1)!} + \cdots\,.$$

3 The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

4 Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1\,.$

Write down the $(2n)$th roots of -1 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1 \right)\,.$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}\,.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right)\cos^2\left(\tfrac{3\pi}{2n}\right)\cos^2\left(\tfrac{5\pi}{2n}\right)\cdots\cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}\,.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

5 In this question, you may assume that, if a, b and c are positive integers such that a and b are coprime and a divides bc, then a divides c. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers p, q, n and N such that p and q are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer k and deduce the value of q.

Hence prove that, for any positive integers n and N, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers a, b, c and d such that a and b are coprime and c and d are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if p is a prime factor of d, then p is also a prime factor of b.

If p^m and p^n are the highest powers of the prime number p that divide d and b, respectively, express b in terms of a, m and n and hence show that $p^n \leqslant n$. Deduce the value of b. (You may assume that if $x > 0$ and $y \geqslant 2$ then $y^x > x$.)

Hence prove that, if r is a positive rational number such that r^r is rational, then r is a positive integer.

6 Let z and w be complex numbers. Use a diagram to show that $|z - w| \leqslant |z| + |w|$.

For any complex numbers z and w, E is defined by

$$E = zw^* + z^*w + 2|zw|\,.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leqslant \frac{|z| + |w|}{1 + |zw|}\,.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

7 **(i)** Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$ for $x \geqslant 0$ and deduce that $\cosh v(x) \leqslant \frac{5}{3}$ for $x \geqslant 0$.

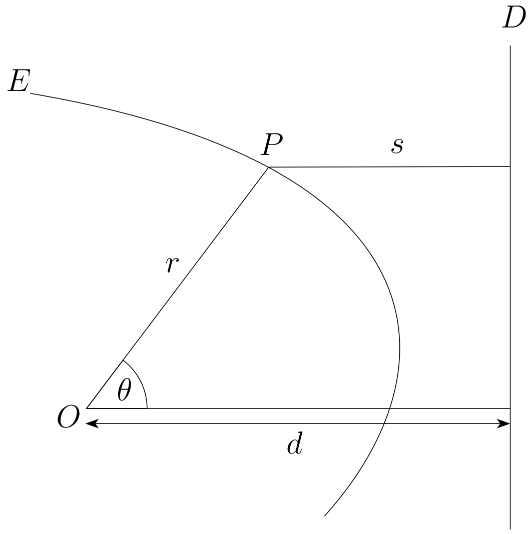
(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leqslant \frac{5}{4}$ for $x \geqslant 0$.

8 **(i)** Evaluate $\displaystyle\sum_{r=0}^{n-1} \mathrm{e}^{2\mathrm{i}(\alpha+r\pi/n)}$ where α is a fixed angle and $n \geqslant 2$.

(ii) The fixed point O is a distance d from a fixed line D. For any point P, let s be the distance from P to D and let r be the distance from P to O. Write down an expression for s in terms of d, r and the angle θ, where θ is as shown in the diagram below.



(iii) The curve E shown in the diagram is such that, for any point P on E, the relation $r = ks$ holds, where k is a fixed number with $0 < k < 1$.

Each of the n lines $L_1, \ldots, L_n$ passes through O and the angle between adjacent lines is $\frac{\pi}{n}$. The line L_j $(j = 1, \ldots, n)$ intersects E in two points forming a chord of length l_j. Show that, for $n \geqslant 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2-k^2)n}{4kd}.$$

Section B: Mechanics

9 **(i)** A sphere of radius R and uniform density ρ_{s} is floating in a large tank of liquid of uniform density ρ. Given that the centre of the sphere is a distance x above the level of the liquid, where $x < R$, show that the volume of liquid displaced is

$$\frac{\pi}{3}(2R^3 - 3R^2x + x^3)\,.$$

(ii) The sphere is acted upon by two forces only: its weight and an upward force equal in magnitude to the weight of the liquid it has displaced. Show that

$$4R^3\rho_{\text{s}}(g + \ddot{x}) = (2R^3 - 3R^2x + x^3)\rho g\,.$$

(iii) Given that the sphere is in equilibrium when $x = \frac{1}{2}R$, find ρ_{s} in terms of ρ. Find, in terms of R and g, the period of small oscillations about this equilibrium position.

10 **(i)** A uniform rod AB has mass M and length $2a$. The point P lies on the rod a distance $a - x$ from A. Show that the moment of inertia of the rod about an axis through P and perpendicular to the rod is

$$\tfrac{1}{3}M(a^2 + 3x^2)\,.$$

(ii) The rod is free to rotate, in a horizontal plane, about a fixed vertical axis through P. Initially the rod is at rest. The end B is struck by a particle of mass m moving horizontally with speed u in a direction perpendicular to the rod. The coefficient of restitution between the rod and the particle is e. Show that the angular velocity of the rod immediately after impact is

$$\frac{3mu(1+e)(a+x)}{M(a^2+3x^2)+3m(a+x)^2}\,.$$

(iii) In the case $m = 2M$, find the value of x for which the angular velocity is greatest and show that this angular velocity is $u(1+e)/a$.

11 An equilateral triangle, comprising three light rods each of length $\sqrt{3}a$, has a particle of mass m attached to each of its vertices. The triangle is suspended horizontally from a point vertically above its centre by three identical springs, so that the springs and rods form a tetrahedron. Each spring has natural length a and modulus of elasticity kmg, and is light.

(i) Show that when the springs make an angle θ with the horizontal the tension in each spring is

$$\frac{kmg(1-\cos\theta)}{\cos\theta}.$$

(ii) Given that the triangle is in equilibrium when $\theta = \frac{1}{6}\pi$, show that $k = 4\sqrt{3}+6$.

(iii) The triangle is released from rest from the position at which $\theta = \frac{1}{3}\pi$. Show that when it passes through the equilibrium position its speed V satisfies

$$V^2 = \frac{4ag}{3}(6+\sqrt{3}).$$

Section C: Probability and Statistics

12 A list consists only of letters A and B arranged in a row. In the list, there are a letter As and b letter Bs, where $a \geqslant 2$ and $b \geqslant 2$, and $a + b = n$. Each possible ordering of the letters is equally probable. The random variable X_1 is defined by

$$X_1 = \begin{cases} 1 & \text{if the first letter in the row is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variables X_k $(2 \leqslant k \leqslant n)$ are defined by

$$X_k = \begin{cases} 1 & \text{if the } (k-1)\text{th letter is } B \text{ and the } k\text{th is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variable S is defined by $S = \sum\limits_{i=1}^{n} X_i$.

(i) Find expressions for $\mathrm{E}(X_i)$, distinguishing between the cases $i = 1$ and $i \neq 1$, and show that $\mathrm{E}(S) = \dfrac{a(b+1)}{n}$.

(ii) Show that:

(a) for $j \geqslant 3$, $\mathrm{E}(X_1 X_j) = \dfrac{a(a-1)b}{n(n-1)(n-2)}$;

(b) $\displaystyle\sum_{i=2}^{n-2} \left(\sum_{j=i+2}^{n} \mathrm{E}(X_i X_j) \right) = \frac{a(a-1)b(b-1)}{2n(n-1)}$;

(c) $\mathrm{Var}\,(S) = \dfrac{a(a-1)b(b+1)}{n^2(n-1)}$.

13 **(i)** The continuous random variable X satisfies $0 \leqslant X \leqslant 1$, and has probability density function $\mathrm{f}(x)$ and cumulative distribution function $\mathrm{F}(x)$. The greatest value of $\mathrm{f}(x)$ is M, so that $0 \leqslant \mathrm{f}(x) \leqslant M$.

(a) Show that $0 \leqslant \mathrm{F}(x) \leqslant Mx$ for $0 \leqslant x \leqslant 1$.

(b) For any function $\mathrm{g}(x)$, show that

$$\int_0^1 2\mathrm{g}(x)\mathrm{F}(x)\mathrm{f}(x)\mathrm{d}x = \mathrm{g}(1) - \int_0^1 \mathrm{g}'(x)\big(\mathrm{F}(x)\big)^2\mathrm{d}x\,.$$

(i) The continuous random variable Y satisfies $0 \leqslant Y \leqslant 1$, and has probability density function $k\mathrm{F}(y)\mathrm{f}(y)$, where f and F are as above.

(a) Determine the value of the constant k.

(b) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leqslant \mathrm{E}(Y^n) \leqslant 2M\mu_{n+1}\,,$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \geqslant 0$.

(c) Hence show that, for $n \geqslant 1$,

$$\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}\,.$$

Section A: Pure Mathematics

1 Given that $z = y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).$$

(i) Use the above result to show that the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

2 In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}),$$

where n is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = -\sum_{r=0}^{\infty}\ln\left(1+x^{2^r}\right),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii) Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$

3 It is given that the two curves

$$y = 4 - x^2 \quad \text{and} \quad mx = k - y^2\,,$$

where $m > 0$, touch exactly once.

(i) In each of the following four cases, sketch the two curves on a single diagram, noting the coordinates of any intersections with the axes:

(a) $k < 0$;

(b) $0 < k < 16$, $k/m < 2$;

(c) $k > 16$, $k/m > 2$;

(d) $k > 16$, $k/m < 2$.

(ii) Now set $m = 12$.
Show that the x-coordinate of any point at which the two curves meet satisfies

$$x^4 - 8x^2 + 12x + 16 - k = 0\,.$$

Let a be the value of x at the point where the curves touch. Show that a satisfies

$$a^3 - 4a + 3 = 0$$

and hence find the three possible values of a.

Derive also the equation

$$k = -4a^2 + 9a + 16\,.$$

Which of the four sketches in part (i) arise?

4 **(i)** Show that

$$\sum_{n=1}^{\infty} \frac{n+1}{n!} = 2\mathrm{e} - 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} = 5\mathrm{e} - 1\,.$$

Sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!}$.

(ii) Sum the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)}$, giving your answer in terms of natural logarithms.

5 **(i)** The point with coordinates (a, b), where a and b are rational numbers, is called:

an *integer rational point* if both a and b are integers;

a *non-integer rational point* if neither a nor b is an integer.

(a) Write down an integer rational point and a non-integer rational point on the circle $x^2 + y^2 = 1$.

(b) Write down an integer rational point on the circle $x^2 + y^2 = 2$. Simplify

$$(\cos\theta + \sqrt{m}\sin\theta)^2 + (\sin\theta - \sqrt{m}\cos\theta)^2$$

and hence obtain a non-integer rational point on the circle $x^2 + y^2 = 2$.

(ii) The point with coordinates $(p + \sqrt{2}\,q\,,\ r + \sqrt{2}\,s)$, where p, q, r and s are rational numbers, is called:

an *integer* 2*-rational point* if all of p, q, r and s are integers;

a *non-integer* 2*-rational point* if none of p, q, r and s is an integer.

(a) Write down an integer 2-rational point, and obtain a non-integer 2-rational point, on the circle $x^2 + y^2 = 3$.

(b) Obtain a non-integer 2-rational point on the circle $x^2 + y^2 = 11$.

(c) Obtain a non-integer 2-rational point on the hyperbola $x^2 - y^2 = 7$.

6 Let $x + \mathrm{i}y$ be a root of the quadratic equation $z^2 + pz + 1 = 0$, where p is a real number.

(i) Show that $x^2 - y^2 + px + 1 = 0$ and $(2x + p)y = 0$.

(ii) Show further that

$$\text{either} \quad p = -2x \quad \text{or} \quad p = -(x^2 + 1)/x \quad \text{with} \quad x \neq 0\,.$$

(iii) Hence show that the set of points in the Argand diagram that can (as p varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

(iv) Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0$$

and the root locus of the equation

$$pz^2 + p^2 z + 2 = 0\,.$$

7 A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z)\,, \qquad \dot{z} = -\dot{y} - 3z\,,$$

where the dot denotes differentiation with respect to t.

Obtain a second order differential equation for y and hence derive the solution

$$y = Ae^{-t} + Be^{-6t}\,, \qquad z = \tfrac{1}{2}Ae^{-t} - 2Be^{-6t}\,,$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \leqslant t \leqslant 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t-n)\,,$$

provided c takes a particular value that you should find.

8 The sequence F_0, F_1, F_2, $\ldots$ is defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geqslant 0$,

$$F_{n+2} = F_{n+1} + F_n\,.$$

(i) Show that $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4\,.$

(ii) Find the values of $F_nF_{n+3} - F_{n+1}F_{n+2}$ in the two cases that arise.

(iii) Prove that, for $r = 1, 2, 3, \ldots,$

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right)\,.$$

Section B: Mechanics

9 A pulley consists of a disc of radius r with centre O and a light thin axle through O perpendicular to the plane of the disc. The disc is non-uniform, its mass is M and its centre of mass is at O. The axle is fixed and horizontal.

Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are connected by a light inextensible string which passes over the pulley. The contact between the string and the pulley is rough enough to prevent the string sliding. The pulley turns and the vertical force on the axle is found, by measurement, to be $P + Mg$.

(i) The moment of inertia of the pulley about its axle is calculated assuming that the pulley rotates without friction about its axle. Show that the calculated value is

$$\frac{((m_1 + m_2)P - 4m_1m_2g)r^2}{(m_1 + m_2)g - P}\,. \qquad (*)$$

(ii) Instead, the moment of inertia of the pulley about its axle is calculated assuming that a couple of magnitude C due to friction acts on the axle of the pulley. Determine whether this calculated value is greater or smaller than $(*)$.

Show that $C < (m_1 - m_2)rg$.

10 A small ring of mass m is free to slide without friction on a hoop of radius a. The hoop is fixed in a vertical plane. The ring is connected by a light elastic string of natural length a to the highest point of the hoop. The ring is initially at rest at the lowest point of the hoop and is then slightly displaced. In the subsequent motion the angle of the string to the downward vertical is ϕ.

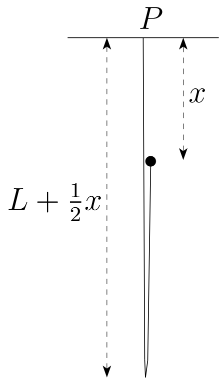
(i) Given that the ring first comes to rest just as the string becomes slack, find an expression for the modulus of elasticity of the string in terms of m and g.

(ii) Show that, throughout the motion, the magnitude R of the reaction between the ring and the hoop is given by

$$R = (12\cos^2\phi - 15\cos\phi + 5)mg$$

and that R is non-zero throughout the motion.

11 One end of a thin heavy uniform inextensible perfectly flexible rope of length $2L$ and mass $2M$ is attached to a fixed point P. A particle of mass m is attached to the other end. Initially, the particle is held at P and the rope hangs vertically in a loop below P. The particle is then released so that it and a section of the rope (of decreasing length) fall vertically as shown in the diagram.



You may assume that each point on the moving section of the rope falls at the same speed as the particle.

(i) Given that energy is conserved, show that, when the particle has fallen a distance x (where $x < 2L$), its speed v is given by

$$v^2 = \frac{2gx\big(mL + ML - \frac{1}{4}Mx\big)}{mL + ML - \frac{1}{2}Mx}\,.$$

(ii) Hence show that the acceleration of the particle is

$$g + \frac{Mgx\big(mL + ML - \frac{1}{4}Mx\big)}{2\big(mL + ML - \frac{1}{2}Mx\big)^2}\,.$$

(iii) Deduce that the acceleration of the particle after it is released is greater than g.

Section C: Probability and Statistics

12 **(i)** A point P lies in an equilateral triangle ABC of height 1. The perpendicular distances from P to the sides AB, BC and CA are x_1, x_2 and x_3, respectively. By considering the areas of triangles with one vertex at P, show that $x_1 + x_2 + x_3 = 1$.

Suppose now that P is placed at random in the equilateral triangle (so that the probability of it lying in any given region of the triangle is proportional to the area of that region). The perpendicular distances from P to the sides AB, BC and CA are random variables X_1, X_2 and X_3, respectively. In the case $X_1 = \min(X_1, X_2, X_3)$, give a sketch showing the region of the triangle in which P lies.

Let $X = \min(X_1, X_2, X_3)$. Show that the probability density function for X is given by

$$\mathrm{f}(x) = \begin{cases} 6(1-3x) & 0 \leqslant x \leqslant \frac{1}{3}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Find the expected value of X.

(ii) A point is chosen at random in a regular tetrahedron of height 1. Find the expected value of the distance from the point to the closest face.
[The volume of a tetrahedron is $\frac{1}{3}$ × area of base × height and its centroid is a distance $\frac{1}{4}$ × height from the base.]

13 **(i)** The random variable Z has a Normal distribution with mean 0 and variance 1. Show that the expectation of Z given that $a < Z < b$ is

$$\frac{\exp(-\frac{1}{2}a^2) - \exp(-\frac{1}{2}b^2)}{\sqrt{2\pi}\,\big(\Phi(b) - \Phi(a)\big)},$$

where Φ denotes the cumulative distribution function for Z.

(ii) The random variable X has a Normal distribution with mean μ and variance σ^2. Show that

$$\mathrm{E}(X \mid X > 0) = \mu + \sigma \mathrm{E}(Z \mid Z > -\mu/\sigma).$$

Hence, or otherwise, show that the expectation, m, of $|X|$ is given by

$$m = \mu\big(1 - 2\Phi(-\mu/\sigma)\big) + \sigma\sqrt{2/\pi}\ \exp(-\tfrac{1}{2}\mu^2/\sigma^2)\,.$$

Obtain an expression for the variance of $|X|$ in terms of μ, σ and m.

Section A: Pure Mathematics

1 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right) u = 0\,.$$

(ii) Show that substituting $y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

2 The polynomial $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,,$$

where $n \geqslant 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $\mathrm{f}(x) = 0$ has a rational root p/q, where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}\mathrm{f}(p/q)$, find the value of q and deduce that any rational root of the equation $\mathrm{f}(x) = 0$ must be an integer.

(i) Show that the nth root of 2 is irrational for $n \geqslant 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geqslant 2$.

3 **(i)** Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad (p \neq 0,\ q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3\,,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β.

(ii) Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2 - 2^{\frac{1}{3}})}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$.

(iii) Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r.

4 The following result applies to any function f which is continuous, has positive gradient and satisfies $\mathrm{f}(0) = 0$:

$$ab \leqslant \int_0^a \mathrm{f}(x)\,\mathrm{d}x + \int_0^b \mathrm{f}^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geqslant 0$ and $b \geqslant 0$.

(i) By considering the graph of $y = \mathrm{f}(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $\mathrm{f}(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leqslant a \leqslant \frac{1}{2}\pi$ and $0 \leqslant b \leqslant 1$,

$$ab \leqslant b \arcsin b + \sqrt{1 - b^2} - \cos a\,.$$

Deduce that, for $t \geqslant 1$,

$$\arcsin(t^{-1}) \geqslant t - \sqrt{t^2 - 1}\,.$$

5 A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ.

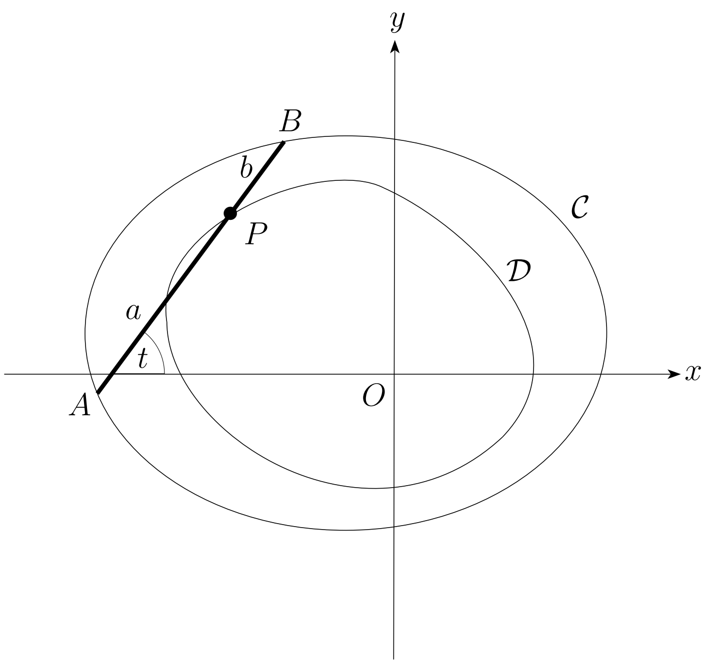
(i) Starting with the expression

$$\tfrac{1}{2}\int r^2 \, \mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t \, . \qquad (*)$$

(ii) The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

(iii) The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right) \cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right) \sin t \right) \mathrm{d}t \, .$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

6 The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\, \mathrm{d}t\,, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\, \mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\, \mathrm{d}v\,, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\, \mathrm{d}x\,.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

7 Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a}\right)^n\,,$$

where n is a positive integer and a is any given positive integer.

(i) In the case when n is even, show by induction that T_n can be written in the form

$$A_n + B_n\sqrt{a(a+1)}\,,$$

where A_n and B_n are integers (depending on a and n) and $A_n^2 = a(a+1)B_n^2 + 1$.

(ii) In the case when n is odd, show by considering $(\sqrt{a+1} + \sqrt{a})T_m$ where m is even, or otherwise, that T_n can be written in the form

$$C_n\sqrt{a+1} + D_n\sqrt{a}\,,$$

where C_n and D_n are integers (depending on a and n) and $(a+1)C_n^2 = aD_n^2 + 1\,.$

(iii) Deduce that, for each n, T_n can be written as the sum of the square roots of two consecutive integers.

8 The complex numbers z and w are related by

$$w = \frac{1 + \mathrm{i}z}{\mathrm{i} + z}\,.$$

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real. Express u and v in terms of x and y.

(i) By setting $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$ with one point omitted.

(ii) Find the locus of w when the locus of z is the line segment $y = 0$, $-1 < x < 1\,.$

(iii) Find the locus of w when the locus of z is the line segment $x = 0$, $-1 < y < 1\,.$

(iv) Find the locus of w when the locus of z is the line $y = 1$, $-\infty < x < \infty\,.$

Section B: Mechanics

9 Particles P and Q have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius a which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at O. Initially, the particles are in equilibrium.

Equilibrium is slightly disturbed and Q begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle θ between OQ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g\,.$$

(i) Given that Q loses contact with the cylinder first, show that it does so when $\theta = \beta$, where β satisfies

$$15\cos\beta + 6\sin\beta = 10.$$

(ii) Show also that while P and Q are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin\theta + \cos\theta)$.

10 Particles P and Q, each of mass m, lie initially at rest a distance a apart on a smooth horizontal plane. They are connected by a light elastic string of natural length a and modulus of elasticity $\frac{1}{2}ma\omega^2$, where ω is a constant.

(i) Then P receives an impulse which gives it a velocity u directly away from Q. Show that when the string next returns to length a, the particles have travelled a distance $\frac{1}{2}\pi u/\omega$, and find the speed of each particle.

(ii) Find also the total time between the impulse and the subsequent collision of the particles.

11 A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc.

(i) Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin\phi = 2a \sin\tfrac{1}{2}\theta\,.$$

(ii) Show further that the magnitude of the couple is

$$\frac{mga^2 \sin\theta}{\sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}}\,.$$

(iii) The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta}\,.$$

Section C: Probability and Statistics

12 The random variable N takes positive integer values and has pgf (probability generating function) $\mathrm{G}(t)$. The random variables X_i, where $i = 1, 2, 3, \ldots$, are independently and identically distributed, each with pgf $\mathrm{H}(t)$. The random variables X_i are also independent of N. The random variable Y is defined by

$$Y = \sum_{i=1}^{N} X_i \,.$$

(i) Given that the pgf of Y is $\mathrm{G}(\mathrm{H}(t))$, show that

$$\mathrm{E}(Y) = \mathrm{E}(N)\mathrm{E}(X_i) \qquad \text{and} \qquad \mathrm{Var}\,(Y) = \mathrm{Var}\,(N)\big(\mathrm{E}(X_i)\big)^2 + \mathrm{E}(N)\mathrm{Var}\,(X_i)\,.$$

(ii) A fair coin is tossed until a head occurs. The total number of tosses is N. The coin is then tossed a further N times and the total number of heads in these N tosses is Y. Find in this particular case the pgf of Y, $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

13 In this question, the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, so for example $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

(i) A bag contains n balls, of which b are black. A sample of k balls is drawn, one after another, at random *with* replacement. The random variable X denotes the number of black balls in the sample. By considering

$$\frac{\mathrm{P}(X = r+1)}{\mathrm{P}(X = r)}\,,$$

show that, in the case that it is unique, the most probable number of black balls in the sample is

$$\left\lfloor \frac{(k+1)b}{n} \right\rfloor .$$

Under what circumstances is the answer not unique?

(ii) A bag contains n balls, of which b are black. A sample of k balls (where $k \leqslant b$) is drawn, one after another, at random *without* replacement. Find, in the case that it is unique, the most probable number of black balls in the sample.

Under what circumstances is the answer not unique?

Section A: Pure Mathematics

1 Let x_1, x_2, ..., x_n and x_{n+1} be any fixed real numbers. The numbers A and B are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k\,, \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2\,,$$

and the numbers C and D are defined by

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k\,, \quad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2\,.$$

(i) Express C in terms of A, x_{n+1} and n.

(ii) Show that $B = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k^2 - A^2\,.$

(iii) Express D in terms of B, A, x_{n+1} and n.

Hence show that $(n+1)D \geqslant nB$ for all values of x_{n+1}, but that $D < B$ if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}\,.$$

2 In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,\mathrm{d}x = \frac{a}{2\sinh a}\,.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,\mathrm{d}x$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,\mathrm{d}x\,.$$

3 For any given positive integer n, a number a (which may be complex) is said to be a *primitive nth root of unity* if $a^n = 1$ and there is no integer m such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive nth roots of unity, the coefficient of the highest power of x is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

(i) Find $\mathrm{C}_1(x)$, $\mathrm{C}_2(x)$, $\mathrm{C}_3(x)$, $\mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of n for which $\mathrm{C}_n(x) = x^4 + 1$.

(iii) Given that p is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers q, r and s such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.

4 **(i)** The number α is a common root of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ (that is, α satisfies both equations). Given that $a \neq c$, show that

$$\alpha = -\frac{b-d}{a-c}.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

Does this result still hold if the condition $a \neq c$ is not imposed?

(ii) Show that the equations $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0.$$

Hence, or otherwise, find the values of b for which the equations $2x^2 + 5x + 2b = 0$ and $2x^3 + 7x^2 + 5x + 1 = 0$ have at least one common root.

5 The vertices A, B, C and D of a square have coordinates $(0,0)$, $(a,0)$, (a,a) and $(0,a)$, respectively. The points P and Q have coordinates $(an,0)$ and $(0,am)$ respectively, where $0 < m < n < 1$. The line CP produced meets DA produced at R and the line CQ produced meets BA produced at S. The line PQ produced meets the line RS produced at T.

(i) Show that TA is perpendicular to AC.

(ii) Explain how, given a square of area a^2, a square of area $2a^2$ may be constructed using only a straight-edge.

[**Note**: a straight-edge is a ruler with no markings on it; no measurements (and no use of compasses) are allowed in the construction.]

6 The points P, Q and R lie on a sphere of unit radius centred at the origin, O, which is fixed. Initially, P is at $P_0(1,0,0)$, Q is at $Q_0(0,1,0)$ and R is at $R_0(0,0,1)$.

(i) The sphere is then rotated about the z-axis, so that the line OP turns directly towards the positive y-axis through an angle ϕ. The position of P after this rotation is denoted by P_1. Write down the coordinates of P_1.

(ii) The sphere is now rotated about the line in the x-y plane perpendicular to OP_1, so that the line OP turns directly towards the positive z-axis through an angle λ. The position of P after this rotation is denoted by P_2. Find the coordinates of P_2. Find also the coordinates of the points Q_2 and R_2, which are the positions of Q and R after the two rotations.

(iii) The sphere is now rotated for a third time, so that P returns from P_2 to its original position P_0. During the rotation, P remains in the plane containing P_0, P_2 and O. Show that the angle of this rotation, θ, satisfies

$$\cos\theta = \cos\phi\cos\lambda\,,$$

and find a vector in the direction of the axis about which this rotation takes place.

7 **(i)** Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

(ii) Obtain a similar equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(iii) Conjecture and prove a relation between $\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

(iv) Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

(v) State the degree of the polynomial.

8 Given that $\mathrm{P}(x) = \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$, write down an expression for

$$\int \frac{\mathrm{P}(x)}{\big(\mathrm{Q}(x)\big)^2}\,\mathrm{d}x\,.$$

(i) By choosing the function $\mathrm{R}(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2-4x-3}{(1+2x+3x^2)^2}\,\mathrm{d}x\,.$$

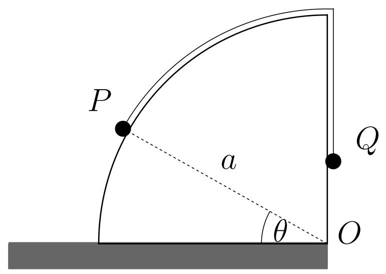
Show that the choice of $\mathrm{R}(x)$ is not unique and, by comparing the two functions $\mathrm{R}(x)$ corresponding to two different values of a, explain how the different choices are related.

(ii) Find the general solution of

$$(1+\cos x+2\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x\,.$$

Section B: Mechanics

9



The diagram shows two particles, P and Q, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre O, which is at the edge of the table, and radius a. The angle between OP and the table is θ. The masses of P and Q are m and M, respectively, where $m < M$.

Initially, P is held at rest on the table and in contact with the block, Q is vertically above O, and the string is taut. Then P is released. Given that, in the subsequent motion, P remains in contact with the block as θ increases from 0 to $\frac{1}{2}\pi$, find an expression, in terms of m, M, θ and g, for the normal reaction of the block on P and show that

$$\frac{m}{M} \geqslant \frac{\pi - 1}{3}\,.$$

10 A small bead B, of mass m, slides without friction on a fixed horizontal ring of radius a. The centre of the ring is at O. The bead is attached by a light elastic string to a fixed point P in the plane of the ring such that $OP = b$, where $b > a$. The natural length of the elastic string is c, where $c < b - a$, and its modulus of elasticity is λ.

(i) Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta + \phi)\,,$$

where $\theta = \angle BPO$ and $\phi = \angle BOP$.

(ii) Given that θ and ϕ are small, show that $a(\theta + \phi) \approx b\theta$.

(iii) Hence find the period of small oscillations about the equilibrium position $\theta = \phi = 0$.

11 A bullet of mass m is fired horizontally with speed u into a wooden block of mass M at rest on a horizontal surface. The coefficient of friction between the block and the surface is μ. While the bullet is moving through the block, it experiences a constant force of resistance to its motion of magnitude R, where $R > (M+m)\mu g$. The bullet moves horizontally in the block and does not emerge from the other side of the block.

(i) Show that the magnitude, a, of the deceleration of the bullet relative to the block while the bullet is moving through the block is given by

$$a = \frac{R}{m} + \frac{R-(M+m)\mu g}{M}\,.$$

(ii) Show that the common speed, v, of the block and bullet when the bullet stops moving through the block satisfies

$$av = \frac{Ru-(M+m)\mu g u}{M}\,.$$

(iii) Obtain an expression, in terms of u, v and a, for the distance moved by the block while the bullet is moving through the block.

(iv) Show that the total distance moved by the block is

$$\frac{muv}{2(M+m)\mu g}\,.$$

Describe briefly what happens if $R < (M+m)\mu g$.

Section C: Probability and Statistics

12 **(i)** The infinite series S is given by

$$S = 1 + (1+d)r + (1+2d)r^2 + \cdots + (1+nd)r^n + \cdots ,$$

for $|r| < 1$. By considering $S - rS$, or otherwise, prove that

$$S = \frac{1}{1-r} + \frac{rd}{(1-r)^2} .$$

(ii) Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is a; the probability that an arrow shot by Boadicea hits the target is b. Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is $1/a$.

(iii) Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is α and the probability that Boadicea wins is β. Show that

$$\alpha = \frac{a}{1 - a'b'} ,$$

where $a' = 1 - a$ and $b' = 1 - b$, and find β.

(iv) Show that the expected number of shots in the contest is $\dfrac{\alpha}{a} + \dfrac{\beta}{b}$.

13 In this question, $\mathrm{Corr}(U,V)$ denotes the product moment correlation coefficient between the random variables U and V, defined by

$$\mathrm{Corr}(U,V) \equiv \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}\,(U)\mathrm{Var}\,(V)}}\,.$$

(i) The independent random variables Z_1, Z_2 and Z_3 each have expectation 0 and variance 1. What is the value of $\mathrm{Corr}(Z_1, Z_2)$?

(ii) Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12} Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}} Z_2\,,$$

where ρ_{12} is a given constant with $-1 < \rho_{12} < 1$. Find $\mathrm{E}(Y_2)$, $\mathrm{Var}\,(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

(iii) Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where a, b and c are real constants and $c \geqslant 0$. Given that $\mathrm{E}(Y_3) = 0$, $\mathrm{Var}\,(Y_3) = 1$, $\mathrm{Corr}(Y_1, Y_3) = \rho_{13}$ and $\mathrm{Corr}(Y_2, Y_3) = \rho_{23}$, express a, b and c in terms of ρ_{23}, ρ_{13} and ρ_{12}.

(iv) Given constants μ_i and σ_i, for $i = 1$, 2 and 3, give expressions in terms of the Y_i for random variables X_i such that $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}\,(X_i) = \sigma_i^2$ and $\mathrm{Corr}(X_i, X_j) = \rho_{ij}$.

Section A: Pure Mathematics

1 The points S, T, U and V have coordinates (s, ms), (t, mt), (u, nu) and (v, nv), respectively. The lines SV and UT meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively.

(i) Show that

$$p = \frac{(m-n)sv}{ms - nv},$$

and write down a similar expression for q.

(ii) Given that S and T lie on the circle $x^2 + (y-c)^2 = r^2$, find a quadratic equation satisfied by s and by t, and hence determine st and $s + t$ in terms of m, c and r.

(iii) Given that S, T, U and V lie on the above circle, show that $p + q = 0$.

2 **(i)** Let $y = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients a_n are independent of x and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and write down a similar expression for y''.

Write out explicitly each of the three series as far as the term containing a_3.

(ii) It is given that y satisfies the differential equation

$$xy'' - y' + 4x^3 y = 0.$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$.

Show that, for $n \geqslant 4$,

$$a_n = -\frac{4}{n(n-2)}\, a_{n-4},$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$.

Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

3 The function $\mathrm{f}(t)$ is defined, for $t \neq 0$, by

$$\mathrm{f}(t) = \frac{t}{\mathrm{e}^t - 1}\,.$$

(i) By expanding e^t, show that $\lim_{t\to 0} \mathrm{f}(t) = 1$. Find $\mathrm{f}'(t)$ and evaluate $\lim_{t\to 0} \mathrm{f}'(t)$.

(ii) Show that $\mathrm{f}(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $\mathrm{g}(t)$ is said to be *even* if $\mathrm{g}(t) \equiv \mathrm{g}(-t)$.]

(iii) Show with the aid of a sketch that $\mathrm{e}^t(1-t) \leqslant 1$ and deduce that $\mathrm{f}'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $\mathrm{f}(t)$.

4 For any given (suitable) function f, the *Laplace transform* of f is the function F defined by

$$\mathrm{F}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\mathrm{d}t \qquad (s > 0)\,.$$

(i) Show that the Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$, where $b > 0$, is $\mathrm{F}(s+b)$.

(ii) Show that the Laplace transform of $\mathrm{f}(at)$, where $a > 0$, is $a^{-1}\mathrm{F}(\frac{s}{a})$.

(iii) Show that the Laplace transform of $\mathrm{f}'(t)$ is $s\mathrm{F}(s) - \mathrm{f}(0)$.

(iv) In the case $\mathrm{f}(t) = \sin t$, show that $\mathrm{F}(s) = \dfrac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $\mathrm{e}^{-pt}\cos qt$, where $p > 0$ and $q > 0$.

5 The numbers x, y and z satisfy

$$x + y + z = 1$$
$$x^2 + y^2 + z^2 = 2$$
$$x^3 + y^3 + z^3 = 3\,.$$

(i) Show that

$$yz + zx + xy = -\frac{1}{2}\,.$$

(ii) Show also that $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$, and hence that

$$xyz = \frac{1}{6}\,.$$

(iii) Let $S_n = x^n + y^n + z^n$. Use the above results to find numbers a, b and c such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2}\,,$$

holds for all n.

6 **(i)** Show that $\left|\mathrm{e}^{\imath\beta} - \mathrm{e}^{\imath\alpha}\right| = 2\sin\frac{1}{2}(\beta-\alpha)$ for $0 < \alpha < \beta < 2\pi$.

(ii) Hence show that

$$\left|\mathrm{e}^{\imath\alpha} - \mathrm{e}^{\imath\beta}\right|\,\left|\mathrm{e}^{\imath\gamma} - \mathrm{e}^{\imath\delta}\right| + \left|\mathrm{e}^{\imath\beta} - \mathrm{e}^{\imath\gamma}\right|\,\left|\mathrm{e}^{\imath\alpha} - \mathrm{e}^{\imath\delta}\right| = \left|\mathrm{e}^{\imath\alpha} - \mathrm{e}^{\imath\gamma}\right|\,\left|\mathrm{e}^{\imath\beta} - \mathrm{e}^{\imath\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

(iii) Interpret this result as a theorem about cyclic quadrilaterals.

7 **(i)** The functions $\mathrm{f}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{f}_0(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad \mathrm{f}_{n+1}(x) = \frac{\mathrm{d}\mathrm{f}_n(x)}{\mathrm{d}x}\,.$$

Prove, for $n \geqslant 1$, that

$$(1+x^2)\mathrm{f}_{n+1}(x) + 2(n+1)x\mathrm{f}_n(x) + n(n+1)\mathrm{f}_{n-1}(x) = 0\,.$$

(ii) The functions $\mathrm{P}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{P}_n(x) = (1+x^2)^{n+1}\mathrm{f}_n(x)\,.$$

Find expressions for $\mathrm{P}_0(x)$, $\mathrm{P}_1(x)$ and $\mathrm{P}_2(x)$.

Prove, for $n \geqslant 0$, that

$$\mathrm{P}_{n+1}(x) - (1+x^2)\frac{\mathrm{d}\mathrm{P}_n(x)}{\mathrm{d}x} + 2(n+1)x\mathrm{P}_n(x) = 0\,,$$

and that $\mathrm{P}_n(x)$ is a polynomial of degree n.

8 Let m be a positive integer and let n be a non-negative integer.

(i) Use the result $\lim\limits_{t\to\infty} \mathrm{e}^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m(\ln x)^n = 0\,.$$

By writing x^x as $\mathrm{e}^{x\ln x}$ show that

$$\lim_{x\to 0} x^x = 1\,.$$

(ii) Let $I_n = \displaystyle\int_0^1 x^m(\ln x)^n \mathrm{d}x$. Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate I_n.

(iii) Show that

$$\int_0^1 x^x \mathrm{d}x = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots\,.$$

Section B: Mechanics

9 A particle is projected under gravity from a point P and passes through a point Q. The angles of the trajectory with the positive horizontal direction at P and at Q are θ and ϕ, respectively. The angle of elevation of Q from P is α.

(i) Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

(ii) It is given that there is a second trajectory from P to Q with the same speed of projection. The angles of this trajectory with the positive horizontal direction at P and at Q are θ' and ϕ', respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta + \theta') = -\cot\alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

10 A light spring is fixed at its lower end and its axis is vertical. When a certain particle P rests on the top of the spring, the compression is d. When, instead, P is dropped onto the top of the spring from a height h above it, the compression at time t after P hits the top of the spring is x.

(i) Obtain a second-order differential equation relating x and t for $0 \leqslant t \leqslant T$, where T is the time at which P first loses contact with the spring.

(ii) Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t)\,,$$

where the constants A, B, C and ω are to be given in terms of d, g and h as appropriate.

(iii) Show that

$$T = \sqrt{d/g}\,\left(2\pi - 2\arctan\sqrt{2h/d}\,\right)\,.$$

11 A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude Mf acting in the direction of its motion. When it entered the cloud, the comet had mass M and speed V. After a time t, it has travelled a distance x through the cloud, its mass is $M(1+bx)$, where b is a positive constant, and its speed is v.

(i) In the case when $f = 0$, write down an equation relating V, x, v and b. Hence find an expression for x in terms of b, V and t.

(ii) In the case when f is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft+V}{1+bx}\,.$$

Hence find an expression for x in terms of b, V, f and t.

Show that it is possible, if b, V and f are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.

Section C: Probability and Statistics

12 **(i)** Albert tosses a fair coin k times, where k is a given positive integer. The number of heads he gets is X_1. He then tosses the coin X_1 times, getting X_2 heads. He then tosses the coin X_2 times, getting X_3 heads. The random variables X_4, X_5, ... are defined similarly. Write down $\mathrm{E}(X_1)$.

By considering $\mathrm{E}(X_2 \mid X_1 = x_1)$, or otherwise, show that $\mathrm{E}(X_2) = \frac{1}{4}k$.

Find $\displaystyle\sum_{i=1}^{\infty} \mathrm{E}(X_i)$.

(ii) Bertha has k fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is Y_1. She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is Y_2. The random variables Y_3, Y_4, ... , Y_k are defined similarly, and $Y = \sum_{i=1}^{k} Y_i$.

Obtain the probability generating function of Y, and use it to find $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

13 **(i)** The point P lies on the circumference of a circle of unit radius and centre O. The angle, θ, between OP and the positive x-axis is a random variable, uniformly distributed on the interval $0 \leqslant \theta < 2\pi$. The cartesian coordinates of P with respect to O are (X, Y). Find the probability density function for X, and calculate $\mathrm{Var}\,(X)$.

Show that X and Y are uncorrelated and discuss briefly whether they are independent.

(ii) The points P_i $(i = 1, 2, \ldots, n)$ are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates (X_i, Y_i). The point $\overline{P}$ has coordinates $(\overline{X}, \overline{Y})$, where $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$. Show that $\overline{X}$ and $\overline{Y}$ are uncorrelated.

Show that, for large n, $\mathrm{P}\left(|\overline{X}| \leqslant \sqrt{\dfrac{2}{n}}\right) \approx 0.95$.

Section A: Pure Mathematics

1 Find all values of a, b, x and y that satisfy the simultaneous equations

$$\begin{aligned} a+b &= 1 \\ ax+by &= \tfrac{1}{3} \\ ax^2+by^2 &= \tfrac{1}{5} \\ ax^3+by^3 &= \tfrac{1}{7}\,. \end{aligned}$$

[Hint: you may wish to start by multiplying the second equation by $x+y$. **]**

2 Let $S_k(n) \equiv \sum_{r=0}^{n} r^k$, where k is a positive integer, so that

$$S_1(n) \equiv \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad S_2(n) \equiv \tfrac{1}{6}n(n+1)(2n+1)\,.$$

(i) By considering $\sum_{r=0}^{n} \left[(r+1)^k - r^k\right]$, show that

$$kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2}S_{k-2}(n) - \binom{k}{3}S_{k-3}(n) - \cdots - \binom{k}{k-1}S_1(n)\,. \quad (*)$$

Obtain simplified expressions for $S_3(n)$ and $S_4(n)$.

(ii) Explain, using $(*)$, why $S_k(n)$ is a polynomial of degree $k+1$ in n.

(iii) Show that in this polynomial the constant term is zero and the sum of the coefficients is 1.

3 The point $P(a\cos\theta\,,\,b\sin\theta)$, where $a>b>0$, lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

The point $S(-ea\,,\,0)$, where $b^2=a^2(1-e^2)$, is a focus of the ellipse. The point N is the foot of the perpendicular from the origin, O, to the tangent to the ellipse at P. The lines SP and ON intersect at T.

(i) Show that the y-coordinate of T is

$$\frac{b\sin\theta}{1+e\cos\theta}\,.$$

(ii) Show that T lies on the circle with centre S and radius a.

4 **(i)** Show, with the aid of a sketch, that $y>\tanh(y/2)$ for $y>0$ and deduce that

$$\operatorname{arcosh} x>\frac{x-1}{\sqrt{x^2-1}}\quad\text{for}\quad x>1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\operatorname{arcosh} x>2\dfrac{x-1}{\sqrt{x^2-1}}$ for $x>1$.

(iii) Show that $\operatorname{arcosh} x>3\dfrac{\sqrt{x^2-1}}{x+2}$ for $x>1$.

[**Note:** $\operatorname{arcosh} x$ is another notation for $\cosh^{-1}x$.]

5 The functions $\mathrm{T}_n(x)$, for $n=0$, 1, 2, ..., satisfy the recurrence relation

$$\mathrm{T}_{n+1}(x)-2x\mathrm{T}_n(x)+\mathrm{T}_{n-1}(x)=0 \qquad (n\geqslant 1). \qquad (*)$$

(i) Show by induction that

$$\left(\mathrm{T}_n(x)\right)^2-\mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x)=\mathrm{f}(x)\,,$$

where $\mathrm{f}(x)=\left(\mathrm{T}_1(x)\right)^2-\mathrm{T}_0(x)\mathrm{T}_2(x)\,.$

(ii) In the case $\mathrm{f}(x)\equiv 0$, determine (with proof) an expression for $\mathrm{T}_n(x)$ in terms of $\mathrm{T}_0(x)$ (assumed to be non-zero) and $\mathrm{r}(x)$, where $\mathrm{r}(x)=\mathrm{T}_1(x)/\mathrm{T}_0(x)$.

(iii) Find the two possible expressions for $\mathrm{r}(x)$ in terms of x.

6 In this question, p denotes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Given that

$$y = p^2 + 2xp\,,$$

show by differentiating with respect to x that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Hence show that $x = -\frac{2}{3}p + Ap^{-2}$, where A is an arbitrary constant.

Find y in terms of x if $p = -3$ when $x = 2$.

(ii) Given instead that

$$y = 2xp + p\ln p\,,$$

and that $p = 1$ when $x = -\frac{1}{4}$, show that $x = -\frac{1}{2}\ln p - \frac{1}{4}$ and find y in terms of x.

7 The points A, B and C in the Argand diagram are the vertices of an equilateral triangle described anticlockwise. Show that the complex numbers a, b and c representing A, B and C satisfy

$$2c = (a + b) + \mathrm{i}\sqrt{3}(b - a).$$

Find a similar relation in the case that A, B and C are the vertices of an equilateral triangle described clockwise.

(i) The quadrilateral $DEFG$ lies in the Argand diagram. Show that points P, Q, R and S can be chosen so that PDE, QEF, RFG and SGD are equilateral triangles and $PQRS$ is a parallelogram.

(ii) The triangle LMN lies in the Argand diagram. Show that the centroids U, V and W of the equilateral triangles drawn externally on the sides of LMN are the vertices of an equilateral triangle.

[**Note:** The *centroid* of a triangle with vertices represented by the complex numbers x, y and z is the point represented by $\frac{1}{3}(x + y + z)$.]

8 **(i)** The coefficients in the series

$$S = \tfrac{1}{3}x + \tfrac{1}{6}x^2 + \tfrac{1}{12}x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+1} + pa_r = 0$. Write down the value of p.

By considering $(1 + px)S$, find an expression for the sum to infinity of S (assuming that it exists). Find also an expression for the sum of the first $n + 1$ terms of S.

(ii) The coefficients in the series

$$T = 2 + 8x + 18x^2 + 37x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+2} + pa_{r+1} + qa_r = 0$. Find an expression for the sum to infinity of T (assuming that it exists). By expressing T in partial fractions, or otherwise, find an expression for the sum of the first $n + 1$ terms of T.

Section B: Mechanics

9 A particle of mass m is initially at rest on a rough horizontal surface. The particle experiences a force $mg\sin\pi t$, where t is time, acting in a fixed horizontal direction. The coefficient of friction between the particle and the surface is μ. Given that the particle starts to move first at $t = T_0$, state the relation between T_0 and μ.

(i) For $\mu = \mu_0$, the particle comes to rest for the first time at $t = 1$. Sketch the acceleration-time graph for $0 \leqslant t \leqslant 1$. Show that

$$1 + \left(1 - \mu_0^2\right)^{\frac{1}{2}} - \mu_0\pi + \mu_0 \arcsin \mu_0 = 0\,.$$

(ii) For $\mu = \mu_0$ sketch the acceleration-time graph for $0 \leqslant t \leqslant 3$. Describe the motion of the particle in this case and in the case $\mu = 0$.

[**Note:** $\arcsin x$ is another notation for $\sin^{-1} x$.]

10 A long string consists of n short light strings joined together, each of natural length ℓ and modulus of elasticity λ. It hangs vertically at rest, suspended from one end. Each of the short strings has a particle of mass m attached to its lower end. The short strings are numbered 1 to n, the nth short string being at the top.

(i) By considering the tension in the rth short string, determine the length of the long string.

(ii) Find also the elastic energy stored in the long string.

(iii) A uniform heavy rope of mass M and natural length L_0 has modulus of elasticity λ. The rope hangs vertically at rest, suspended from one end. Show that the length, L, of the rope is given by

$$L = L_0\left(1 + \frac{Mg}{2\lambda}\right),$$

and find an expression in terms of L, L_0 and λ for the elastic energy stored in the rope.

11 A circular wheel of radius r has moment of inertia I about its axle, which is fixed in a horizontal position. A light string is wrapped around the circumference of the wheel and a particle of mass m hangs from the free end. The system is released from rest and the particle descends. The string does not slip on the wheel.

As the particle descends, the wheel turns through n_1 revolutions, and the string then detaches from the wheel. At this moment, the angular speed of the wheel is ω_0. The wheel then turns through a further n_2 revolutions, in time T, before coming to rest. The couple on the wheel due to resistance is constant.

Show that

$$\frac{1}{2}\omega_0 T = 2\pi n_2$$

and

$$I = \frac{mgrn_1T^2 - 4\pi mr^2n_2^2}{4\pi n_2(n_1 + n_2)} \, .$$

Section C: Probability and Statistics

12 Let X be a random variable with a Laplace distribution, so that its probability density function is given by

$$\mathrm{f}(x) = \frac{1}{2}\mathrm{e}^{-|x|}\,, \qquad -\infty < x < \infty\,. \tag{$*$}$$

(i) Sketch $\mathrm{f}(x)$. Show that its moment generating function $\mathrm{M}_X(\theta)$ is given by $\mathrm{M}_X(\theta) = (1-\theta^2)^{-1}$ and hence find the variance of X.

(ii) A frog is jumping up and down, attempting to land on the same spot each time. In fact, in each of n successive jumps he always lands on a fixed straight line but when he lands from the ith jump ($i = 1\,,2\,,\ldots\,,n$) his displacement from the point from which he jumped is X_i cm, where X_i has the distribution ($*$). His displacement from his starting point after n jumps is Y cm (so that $Y = \sum\limits_{i=1}^{n} X_i$). Each jump is independent of the others.

Obtain the moment generating function for $Y/\sqrt{2n}$ and, by considering its logarithm, show that this moment generating function tends to $\exp(\frac{1}{2}\theta^2)$ as $n \to \infty$.

(iii) Given that $\exp(\frac{1}{2}\theta^2)$ is the moment generating function of the standard Normal random variable, estimate the least number of jumps such that there is a 5% chance that the frog lands 25 cm or more from his starting point.

13 A box contains n pieces of string, each of which has two ends. I select two string ends at random and tie them together. This creates either a ring (if the two ends are from the same string) or a longer piece of string. I repeat the process of tying together string ends chosen at random until there are none left.

(i) Find the expected number of rings created at the first step and hence obtain an expression for the expected number of rings created by the end of the process.

(ii) Find also an expression for the variance of the number of rings created.

(iii) Given that $\ln 20 \approx 3$ and that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$ for large n, determine approximately the expected number of rings created in the case $n = 40\,000$.

Section A: Pure Mathematics

1 *In this question, do not consider the special cases in which the denominators of any of your expressions are zero.*

(i) Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc.

(ii) Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$.

(iii) The four real numbers θ_1, θ_2, θ_3 and θ_4 lie in the range $0 \leqslant \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0\,,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

2 **(i)** Show that $1.3.5.7.\ \ldots\ .(2n-1) = \dfrac{(2n)!}{2^n n!}$ and that, for $|x| < \frac{1}{4}$,

$$\frac{1}{\sqrt{1-4x}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}\, x^n\,.$$

(ii) By differentiating the above result, deduce that

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n-1)!} \left(\frac{6}{25}\right)^n = 60\,.$$

(iii) Show that

$$\sum_{n=1}^{\infty} \frac{2^{n+1}(2n)!}{3^{2n}(n+1)!\,n!} = 1\,.$$

3 A sequence of numbers, F_1, F_2, ..., is defined by $F_1 = 1$, $F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geqslant 3.$$

(i) Write down the values of F_3, F_4, ... , F_8.

(ii) Prove that $F_{2k+3}F_{2k+1} - F_{2k+2}^2 = -F_{2k+2}F_{2k} + F_{2k+1}^2$.

(iii) Prove by induction or otherwise that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$ and deduce that $F_{2n}^2 + 1$ is divisible by F_{2n+1} .

(iv) Prove that $F_{2n-1}^2 + 1$ is divisible by F_{2n+1} .

4 **(i)** A curve is given parametrically by

$$x = a\big(\cos t + \ln\tan\tfrac{1}{2}t\big)\,,$$
$$y = a\sin t\,,$$

where $0 < t < \frac{1}{2}\pi$ and a is a positive constant. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan t$ and sketch the curve.

(ii) Let P be the point with parameter t and let Q be the point where the tangent to the curve at P meets the x-axis. Show that $PQ = a$.

(iii) The *radius of curvature*, ρ, at P is defined by

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}\,,$$

where the dots denote differentiation with respect to t. Show that $\rho = a\cot t$.

(iv) The point C lies on the normal to the curve at P, a distance ρ from P and above the curve. Show that CQ is parallel to the y-axis.

5 Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$.

(i) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2},$$

(ii) and find an expression in terms of r and θ for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

(iii) Find, with proof, a similar formula for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

6 The distinct points P, Q, R and S in the Argand diagram lie on a circle of radius a centred at the origin and are represented by the complex numbers p, q, r and s, respectively. Show that

$$pq = -a^2 \frac{p-q}{p^* - q^*}.$$

Deduce that, if the chords PQ and RS are perpendicular, then $pq + rs = 0$.

The distinct points A_1, A_2, …, A_n (where $n \geqslant 3$) lie on a circle. The points B_1, B_2, …, B_n lie on the same circle and are chosen so that the chords B_1B_2, B_2B_3, …, B_nB_1 are perpendicular, respectively, to the chords A_1A_2, A_2A_3, …, A_nA_1. Show that, for $n = 3$, there are only two choices of B_1 for which this is possible. What is the corresponding result for $n = 4$? State the corresponding results for values of n greater than 4.

7 The functions $\mathrm{s}(x)$ $(0 \leqslant x < 1)$ and $\mathrm{t}(x)$ $(x \geqslant 0)$, and the real number p, are defined by

$$\mathrm{s}(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\,, \quad \mathrm{t}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,, \quad p = 2\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u\,.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number π.

(i) Use the substitution $u = v^{-1}$ to show that $\mathrm{t}(x) = \displaystyle\int_{1/x}^\infty \frac{1}{1+v^2}\,\mathrm{d}v$. Hence evaluate $\mathrm{t}(1/x) + \mathrm{t}(x)$ in terms of p and deduce that $2\mathrm{t}(1) = \frac{1}{2}p$.

(ii) Let $y = \dfrac{u}{\sqrt{1+u^2}}$. Express u in terms of y, and show that $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $\mathrm{t}(x)$, show that

$$\mathrm{t}(x) = \mathrm{s}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $\mathrm{s}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}p$.

(iii) Let $z = \dfrac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u}$. Show that $\mathrm{t}(\frac{1}{\sqrt{3}}) = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\,\mathrm{d}z$, and hence that $3\mathrm{t}(\frac{1}{\sqrt{3}}) = \frac{1}{2}p$.

8 **(i)** Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

Section B: Mechanics

9 Two small beads, A and B, each of mass m, are threaded on a smooth horizontal circular hoop of radius a and centre O. The angle θ is the acute angle determined by $2\theta = \angle AOB$.

The beads are connected by a light straight spring. The energy stored in the spring is

$$mk^2a^2(\theta - \alpha)^2,$$

where k and α are constants satisfying $k > 0$ and $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

The spring is held in compression with $\theta = \beta$ and then released. Find the period of oscillations in the two cases that arise according to the value of β and state the value of β for which oscillations do not occur.

10 A particle is projected from a point on a plane that is inclined at an angle ϕ to the horizontal. The position of the particle at time t after it is projected is (x, y), where $(0, 0)$ is the point of projection, x measures distance up the line of greatest slope and y measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \pi/2$.

(i) Write down expressions for x and y.

(ii) The particle bounces on the plane and returns along the same path to the point of projection. Show that

$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where R is the range along the plane.

(iii) Show further that

$$\frac{2V^2}{gR} = 3\sin\phi + \operatorname{cosec}\phi$$

and deduce that the largest possible value of R is $V^2/(\sqrt{3}\,g)$.

11 **(i)** A wheel consists of a thin light circular rim attached by light spokes of length a to a small hub of mass m. The wheel rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the wheel is vertical throughout the motion. The speed of the wheel is u, where $u^2 < ag$.

Show that, after the wheel reaches the edge of the table and while it is still in contact with the table, the frictional force on the wheel is zero. Show also that the hub will fall a vertical distance $(ag - u^2)/(3g)$ before the rim loses contact with the table.

(ii) Two particles, each of mass $m/2$, are attached to a light circular hoop of radius a, at the ends of a diameter. The hoop rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the hoop is vertical throughout the motion. When the centre of the hoop is vertically above the edge of the table it has speed u, where $u^2 < ag$, and one particle is vertically above the other.

Show that, after the hoop reaches the edge of the table and while it is still in contact with the table, the frictional force on the hoop is non-zero and deduce that the hoop will slip before it loses contact with the table.

Section C: Probability and Statistics

12 **(i)** I choose a number from the integers $1, 2, \ldots, (2n-1)$ and the outcome is the random variable N. Calculate $\mathrm{E}(N)$ and $\mathrm{E}(N^2)$.

(ii) I then repeat a certain experiment N times, the outcome of the ith experiment being the random variable X_i $(1 \leqslant i \leqslant N)$. For each i, the random variable X_i has mean μ and variance σ^2, and X_i is independent of X_j for $i \neq j$ and also independent of N. The random variable Y is defined by $Y = \sum\limits_{i=1}^{N} X_i$.

(iii) Show that $\mathrm{E}(Y) = n\mu$ and that $\mathrm{Cov}(Y, N) = \frac{1}{3}n(n-1)\mu$.

(iv) Find $\mathrm{Var}\,(Y)$ in terms of n, σ^2 and μ.

13 A frog jumps towards a large pond. Each jump takes the frog either 1 m or 2 m nearer to the pond. The probability of a 1 m jump is p and the probability of a 2 m jump is q, where $p + q = 1$, the occurence of long and short jumps being independent.

(i) Let $p_n(j)$ be the probability that the frog, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, lands in the pond for the first time on its jth jump. Show that $p_2(2) = p$.

(ii) Let u_n be the expected number of jumps, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, required to land in the pond for the first time. Write down the value of u_1. By finding first the relevant values of $p_n(m)$, calculate u_2 and show that $u_3 = 3 - 2q + q^2$.

(iii) Given that u_n can be expressed in the form $u_n = A(-q)^{n-1} + B + Cn$, where A, B and C are constants (independent of n), show that $C = (1+q)^{-1}$ and find A and B in terms of q. Hence show that, for large n, $u_n \approx \dfrac{n}{p + 2q}$ and explain carefully why this result is to be expected.

14 **(i)** My favourite dartboard is a disc of unit radius and centre O. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart n times (where $n > 1$). Find the expected area of the smallest circle, with centre O, that encloses all the n holes made by my dart.

Find also the expected area of the smallest circle, with centre O, that encloses all the $(n-1)$ holes nearest to O.

(ii) My other dartboard is a square of side 2 units, with centre Q. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart n times (where $n > 1$). Find the expected area of the smallest square, with centre Q, that encloses all the n holes made by my dart.

(iii) Determine, without detailed calculations, whether the expected area of the smallest circle, with centre Q, on my square dartboard that encloses all the n holes made by my darts is larger or smaller than that for my circular dartboard.

Section A: Pure Mathematics

1 Sketch the curve with cartesian equation

$$y = \frac{2x(x^2-5)}{x^2-4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

(i) $3x(x^2-5) = (x^2-4)(x+3)\,;$

(ii) $4x(x^2-5) = (x^2-4)(5x-2)\,;$

(iii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)\,.$

2 Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha}\,\mathrm{d}\theta \quad \text{and} \quad J = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$$

where $0 < \alpha < \frac{1}{4}\pi\,.$

(i) Show that $I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha}\,\mathrm{d}\theta$ and hence that $2I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta\,.$

(ii) Find J.

(iii) By considering $I\sin^2 2\alpha + J\cos^2 2\alpha$, or otherwise, show that $I = \frac{1}{2}\pi\sec^2\alpha$.

(iv) Evaluate I in the case $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

3 **(i)** Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \cot x = \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n$$

for $0 < x < \frac{1}{2}\pi$. Explain why $a_n = 0$ for even n.

Prove the identity

$$\cot x - \tan x \equiv 2\cot 2x$$

and show that

$$a_n = (1 - 2^{n+1})b_n \,.$$

(ii) Let $\operatorname{cosec} x = \dfrac{1}{x} + \displaystyle\sum_{n=0}^{\infty} c_n x^n$ for $0 < x < \frac{1}{2}\pi$. By considering $\cot x + \tan x$, or otherwise, show that

$$c_n = (2^{-n} - 1)b_n \,.$$

(iii) Show that

$$\left(1 + x\sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x\sum_{n=0}^{\infty} c_n x^n\right)^2 \,.$$

Deduce from this and the previous results that $a_1 = 1$, and find a_3.

4 The function f satisfies the identity

$$\mathrm{f}(x) + \mathrm{f}(y) \equiv \mathrm{f}(x+y) \qquad (*)$$

for all x and y. Show that $2\mathrm{f}(x) \equiv \mathrm{f}(2x)$ and deduce that $\mathrm{f}''(0) = 0$. By considering the Maclaurin series for $\mathrm{f}(x)$, find the most general function that satisfies $(*)$.
[*Do not consider issues of existence or convergence of Maclaurin series in this question.*]

(i) By considering the function G, defined by $\ln\big(\mathrm{g}(x)\big) = \mathrm{G}(x)$, find the most general function that, for all x and y, satisfies the identity

$$\mathrm{g}(x)\mathrm{g}(y) \equiv \mathrm{g}(x+y)\,.$$

(ii) By considering the function H, defined by $\mathrm{h}(\mathrm{e}^u) = \mathrm{H}(u)$, find the most general function that satisfies, for all positive x and y, the identity

$$\mathrm{h}(x) + \mathrm{h}(y) \equiv \mathrm{h}(xy)\,.$$

(iii) Find the most general function t that, for all x and y, satisfies the identity

$$\mathrm{t}(x) + \mathrm{t}(y) \equiv \mathrm{t}(z)\,,$$

where $z = \dfrac{x+y}{1-xy}$.

5 **(i)** Show that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0\,.$$

(ii) Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

represent the vertices of an equilateral triangle if and only if $a^2 = 3b$.

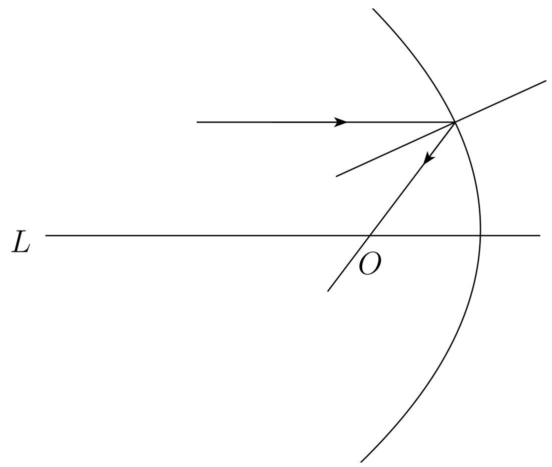
(iii) Under the transformation $z = pw + q$, where p and q are given complex numbers with $p \neq 0$, the equation $(*)$ becomes

$$w^3 + Aw^2 + Bw + C = 0\,. \qquad (**)$$

Show that if the roots of equation $(*)$ represent the vertices of an equilateral triangle, then the roots of equation $(**)$ also represent the vertices of an equilateral triangle.

6 **(i)** Show that in polar coordinates the gradient of any curve at the point (r, θ) is

$$\frac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}\,.$$



(ii) A mirror is designed so that if an incident ray of light is parallel to a fixed line L the reflected ray passes through a fixed point O on L. Prove that the mirror intersects any plane containing L in a parabola. You should assume that the angle between the incident ray and the normal to the mirror is the same as the angle between the reflected ray and the normal.

7 **(i)** Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = \mathrm{f}(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

8 $\triangle$ is an operation that takes polynomials in x to polynomials in x; that is, given any polynomial $\mathrm{h}(x)$, there is a polynomial called $\triangle\mathrm{h}(x)$ which is obtained from $\mathrm{h}(x)$ using the rules that define $\triangle$. These rules are as follows:

(i) $\triangle x = 1$;

(ii) $\triangle\big(\mathrm{f}(x)+\mathrm{g}(x)\big) = \triangle\mathrm{f}(x) + \triangle\mathrm{g}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$;

(iii) $\triangle\big(\lambda\mathrm{f}(x)\big) = \lambda\triangle\mathrm{f}(x)$ for any constant λ and any polynomial $\mathrm{f}(x)$;

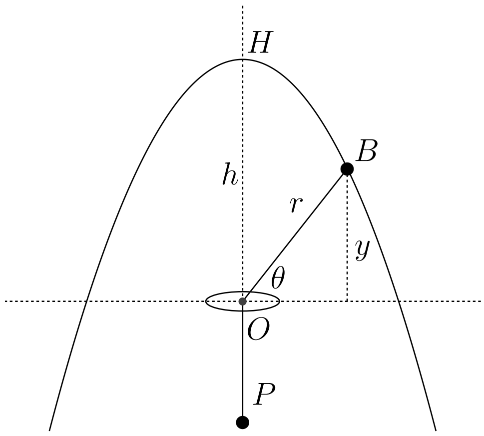
(iv) $\triangle\big(\mathrm{f}(x)\mathrm{g}(x)\big) = \mathrm{f}(x)\triangle\mathrm{g}(x) + \mathrm{g}(x)\triangle\mathrm{f}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

Using these rules show that, if $\mathrm{f}(x)$ is a polynomial of degree zero (that is, a constant), then $\triangle\mathrm{f}(x) = 0$. Calculate $\triangle x^2$ and $\triangle x^3$.

Prove that $\triangle\mathrm{h}(x) \equiv \dfrac{\mathrm{dh}(x)}{\mathrm{d}x}$ for any polynomial $\mathrm{h}(x)$. You should make it clear whenever you use one of the above rules in your proof.

Section B: Mechanics

9 A long, light, inextensible string passes through a small, smooth ring fixed at the point O. One end of the string is attached to a particle P of mass m which hangs freely below O. The other end is attached to a bead, B, also of mass m, which is threaded on a smooth rigid wire fixed in the same vertical plane as O. The distance OB is r, the distance OH is h and the height of the bead above the horizontal plane through O is y, as shown in the diagram.



(i) The shape of the wire is such that the system can be in static equilibrium for all positions of the bead. By considering potential energy, show that the equation of the wire is $y + r = 2h$.

(ii) The bead is initially at H. It is then projected along the wire with initial speed V. Show that, in the subsequent motion,

$$\dot{\theta} = -\frac{h\dot{r}}{r\sqrt{rh - h^2}}$$

where θ is given by $\theta = \arcsin(y/r)$.

(iii) Hence show that the speed of the particle P is $V\left(\frac{r-h}{2r-h}\right)^{\frac{1}{2}}$.

[*Note that* $\arcsin\theta$ *is another notation for* $\sin^{-1}\theta$.]

10 A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2 (where $k > 0$). Along one diameter is a smooth narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is a distance a from the axis and is moving with speed V along the groove, towards the axis, where $k^2V^2 = \Omega^2a^2(k^2 + a^2)$.

(i) Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\Omega^2r^2(k^2 + a^2)^2}{k^2(k^2 + r^2)} .$$

(ii) Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}} ,$$

where θ is the angle through which the disc has turned by time t.

(iii) By making the substitution $u = k/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Deduce that the particle never reaches the axis.

11 A lift of mass M and its counterweight of mass M are connected by a light inextensible cable which passes over a fixed frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is h. Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than h. A small tile of mass m becomes detached from the ceiling of the lift and falls to the floor of the lift.

(i) Show that the speed of the tile just before the impact is

$$\sqrt{\frac{(2M - m)gh}{M}} .$$

(ii) The coefficient of restitution between the tile and the floor of the lift is e. Given that the magnitude of the impulsive force on the lift due to tension in the cable is equal to the magnitude of the impulsive force on the counterweight due to tension in the cable, show that the loss of energy of the system due to the impact is $mgh(1 - e^2)$. Comment on this result.

Section C: Probability and Statistics

12 Fifty times a year, 1024 tourists disembark from a cruise liner at a port. From there they must travel to the city centre either by bus or by taxi. Tourists are equally likely to be directed to the bus station or to the taxi rank. Each bus of the bus company holds 32 passengers, and the company currently runs 15 buses. The company makes a profit of £1 for each passenger carried. It carries as many passengers as it can, with any excess being (eventually) transported by taxi. Show that the largest annual licence fee, in pounds, that the company should consider paying to be allowed to run an extra bus is approximately

$$1600\Phi(2) - \frac{800}{\sqrt{2\pi}}\left(1 - \mathrm{e}^{-2}\right),$$

where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}t^2}\mathrm{d}t$.

[*You should not consider continuity corrections.*]

13 Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. The area of the triangle whose vertices are these two points and the midpoint of the diameter is denoted by the random variable A. Show that the expected value of A is $(2+\pi)^{-1}$.

14 For any random variables X_1 and X_2, state the relationship between $\mathrm{E}(aX_1 + bX_2)$ and $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$, where a and b are constants. If X_1 and X_2 are independent, state the relationship between $\mathrm{E}(X_1X_2)$ and $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$.

(i) An industrial process produces rectangular plates. The length and the breadth of the plates are modelled by independent random variables X_1 and X_2 with non-zero means μ_1 and μ_2 and non-zero standard deviations σ_1 and σ_2, respectively. Using the results in the paragraph above, and without quoting a formula for $\mathrm{Var}\,(aX_1 + bX_2)$, find the means and standard deviations of the perimeter P and area A of the plates. Show that P and A are not independent.

(ii) The random variable Z is defined by $Z = P - \alpha A$, where α is a constant. Show that Z and A are not independent if

$$\alpha \neq \frac{2(\mu_1\sigma_2^2 + \mu_2\sigma_1^2)}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2}.$$

(iii) Given that X_1 and X_2 can each take values 1 and 3 only, and that they each take these values with probability $\frac{1}{2}$, show that Z and A are not independent for any value of α.

Section A: Pure Mathematics

1 **(i)** Show that $\sin A = \cos B$ if and only if $A = (4n+1)\dfrac{\pi}{2} \pm B$ for some integer n.

(ii) Show also that $\left|\sin x \pm \cos x\right| \leqslant \sqrt{2}$ for all values of x and deduce that there are no solutions to the equation $\sin(\sin x) = \cos(\cos x)$.

(iii) Sketch, on the same axes, the graphs of $y = \sin(\sin x)$ and $y = \cos(\cos x)$. Sketch, not on the previous axes, the graph of $y = \sin(2\sin x)$.

2 Find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{xy}{x^2+a^2}$, where $a \neq 0$, and show that it can be written in the form $y^2(x^2+a^2) = c^2$, where c is an arbitrary constant. Sketch this curve.

Find an expression for $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)$ and show that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 2\left(1 - \frac{c^2}{(x^2+a^2)^2}\right) + \frac{8c^2x^2}{(x^2+a^2)^3} .$$

(i) Show that, if $0 < c < a^2$, the points on the curve whose distance from the origin is least are $\left(0\,,\ \pm\dfrac{c}{a}\right)$.

(ii) If $c > a^2$, determine the points on the curve whose distance from the origin is least.

3 **(i)** Let $\mathrm{f}(x) = x^2+px+q$ and $\mathrm{g}(x) = x^2+rx+s$. Find an expression for $\mathrm{f}(\mathrm{g}(x))$ and hence find a necessary and sufficient condition on a, b and c for it to be possible to write the quartic expression $x^4+ax^3+bx^2+cx+d$ in the form $\mathrm{f}(\mathrm{g}(x))$, for some choice of values of p, q, r and s.

(ii) Show further that this condition holds if and only if it is possible to write the quartic expression $x^4+ax^3+bx^2+cx+d$ in the form $(x^2+vx+w)^2-k$, for some choice of values of v, w and k.

(iii) Find the roots of the quartic equation $x^4-4x^3+10x^2-12x+4=0$.

4 The sequence u_n $(n = 1, 2, \ldots)$ satisfies the recurrence relation

$$u_{n+2} = \frac{u_{n+1}}{u_n}(ku_n - u_{n+1})$$

where k is a constant.

If $u_1 = a$ and $u_2 = b$, where a and b are non-zero and $b \neq ka$, prove by induction that

$$u_{2n} = \left(\frac{b}{a}\right)u_{2n-1}$$

$$u_{2n+1} = cu_{2n}$$

for $n \geqslant 1$, where c is a constant to be found in terms of k, a and b. Hence express u_{2n} and u_{2n-1} in terms of a, b, c and n.

Find conditions on a, b and k in the three cases:

(i) the sequence u_n is geometric;

(ii) u_n has period 2;

(iii) the sequence u_n has period 4.

5 **(i)** Let P be the point on the curve $y = ax^2 + bx + c$ (where a is non-zero) at which the gradient is m. Show that the equation of the tangent at P is

$$y - mx = c - \frac{(m-b)^2}{4a}\ .$$

(ii) Show that the curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ (where a_1 and a_2 are non-zero) have a common tangent with gradient m if and only if

$$(a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + 4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2 = 0\ .$$

(iii) Show that, in the case $a_1 \neq a_2$, the two curves have exactly one common tangent if and only if they touch each other. In the case $a_1 = a_2$, find a necessary and sufficient condition for the two curves to have exactly one common tangent.

6 In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln(y + \sqrt{y^2-1}).$$

[**Note:** $\mathrm{arcosh}y$ is another notation for $\cosh^{-1} y$]

(i) Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geqslant b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = (c + \sqrt{c^2 - b^3})^{\frac{1}{3}}$.

(ii) Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation

$$x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0\,,$$

and find these roots in terms of u, b and ω, where $\omega = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$.

(iii) Solve completely the equation $x^3 - 6x = 6\,.$

7 Show that if $\displaystyle\int \frac{1}{u\,\mathrm{f}(u)}\,\mathrm{d}u = \mathrm{F}(u) + c$, then $\displaystyle\int \frac{m}{x\,\mathrm{f}(x^m)}\,\mathrm{d}x = \mathrm{F}(x^m) + c$, where $m \neq 0$.

Find:

(i) $\displaystyle\int \frac{1}{x^n - x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int \frac{1}{\sqrt{x^n + x^2}}\,\mathrm{d}x\,.$

8 In this question, a and c are distinct non-zero complex numbers. The complex conjugate of any complex number z is denoted by z^*.

(i) Show that

$$|a-c|^2 = aa^* + cc^* - ac^* - ca^*$$

and hence prove that the triangle OAC in the Argand diagram, whose vertices are represented by 0, a and c respectively, is right angled at A if and only if $2aa^* = ac^* + ca^*$.

(ii) Points P and P' in the Argand diagram are represented by the complex numbers ab and $\dfrac{a}{b^*}$, where b is a non-zero complex number. A circle in the Argand diagram has centre C and passes through the point A, and is such that OA is a tangent to the circle. Show that the point P lies on the circle if and only if the point P' lies on the circle.

(iii) Conversely, show that if the points represented by the complex numbers ab and $\dfrac{a}{b^*}$, for some non-zero complex number b with $bb^* \neq 1$, both lie on a circle centre C in the Argand diagram which passes through A, then OA is a tangent to the circle.

Section B: Mechanics

9 Two particles, A and B, move without friction along a horizontal line which is perpendicular to a vertical wall. The coefficient of restitution between the two particles is e and the coefficient of restitution between particle B and the wall is also e, where $0 < e < 1$. The mass of particle A is $4em$ (with $m > 0$), and the mass of particle B is $(1-e)^2 m$.

(i) Initially, A is moving towards the wall with speed $(1-e)v$ (where $v > 0$) and B is moving away from the wall and towards A with speed $2ev$. The two particles collide at a distance d from the wall. Find the speeds of A and B after the collision.

(ii) When B strikes the wall, it rebounds along the same line. Show that a second collision will take place, at a distance de from the wall.

(iii) Deduce that further collisions will take place. Find the distance from the wall at which the nth collision takes place, and show that the times between successive collisions are equal.

10 Two thin discs, each of radius r and mass m, are held on a rough horizontal surface with their centres a distance $6r$ apart. A thin light elastic band, of natural length $2\pi r$ and modulus $\dfrac{\pi mg}{12}$, is wrapped once round the discs, its straight sections being parallel. The contact between the elastic band and the discs is smooth. The coefficient of static friction between each disc and the horizontal surface is μ, and each disc experiences a force due to friction equal to μmg when it is sliding.

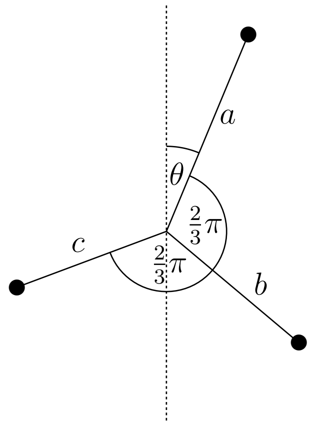
The discs are released simultaneously. If the discs collide, they rebound and a half of their total kinetic energy is lost in the collision.

(i) Show that the discs start sliding, but come to rest before colliding, if and only if $\frac{2}{3} < \mu < 1$.

(ii) Show that, if the discs collide at least once, their total kinetic energy just before the first collision is $\frac{4}{3}mgr(2-3\mu)$.

(iii) Show that if $\frac{4}{9} > \mu^2 > \frac{5}{27}$ the discs come to rest exactly once after the first collision.

11 A horizontal spindle rotates freely in a fixed bearing. Three light rods are each attached by one end to the spindle so that they rotate in a vertical plane. A particle of mass m is fixed to the other end of each of the three rods. The rods have lengths a, b and c, with $a > b > c$ and the angle between any pair of rods is $\frac{2}{3}\pi$. The angle between the rod of length a and the vertical is θ, as shown in the diagram.



(i) Find an expression for the energy of the system and show that, if the system is in equilibrium, then

$$\tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c}\,.$$

(ii) Deduce that there are exactly two equilibrium positions and determine which of the two equilibrium positions is stable.

(iii) Show that, for the system to make complete revolutions, it must pass through its position of stable equilibrium with an angular velocity of at least

$$\sqrt{\frac{4gR}{a^2+b^2+c^2}}\,,$$

where $2R^2 = (a-b)^2 + (b-c)^2 + (c-a)^2$.

Section C: Probability and Statistics

12 Five independent timers time a runner as she runs four laps of a track. Four of the timers measure the individual lap times, the results of the measurements being the random variables T_1 to T_4, each of which has variance σ^2 and expectation equal to the true time for the lap. The fifth timer measures the total time for the race, the result of the measurement being the random variable T which has variance σ^2 and expectation equal to the true race time (which is equal to the sum of the four true lap times).

(i) Find a random variable X of the form $aT + b(T_1 + T_2 + T_3 + T_4)$, where a and b are constants independent of the true lap times, with the two properties:

(1) whatever the true lap times, the expectation of X is equal to the true race time;

(2) the variance of X is as small as possible.

(ii) Find also a random variable Y of the form $cT + d(T_1 + T_2 + T_3 + T_4)$, where c and d are constants independent of the true lap times, with the property that, whatever the true lap times, the expectation of Y^2 is equal to σ^2.

(iii) In one particular race, T takes the value 220 seconds and $(T_1 + T_2 + T_3 + T_4)$ takes the value 220.5 seconds. Use the random variables X and Y to estimate an interval in which the true race time lies.

13 A pack of cards consists of $n+1$ cards, which are printed with the integers from 0 to n. A game consists of drawing cards repeatedly at random from the pack until the card printed with 0 is drawn, at which point the game ends. After each draw, the player receives $\pounds 1$ if the card drawn shows any of the integers from 1 to w inclusive but receives nothing if the card drawn shows any of the integers from $w+1$ to n inclusive.

(i) In one version of the game, each card drawn is replaced immediately and randomly in the pack. Explain clearly why the probability that the player wins a total of exactly $\pounds 3$ is equal to the probability of the following event occurring: out of the first four cards drawn which show numbers in the range 0 to w, the numbers on the first three are non-zero and the number on the fourth is zero. Hence show that the probability that the player wins a total of exactly $\pounds 3$ is equal to $\dfrac{w^3}{(w+1)^4}$.

Write down the probability that the player wins a total of exactly $\pounds r$ and hence find the expected total win.

(ii) In another version of the game, each card drawn is removed from the pack. Show that the expected total win in this version is half of the expected total win in the other version.

14 In this question, you may use the result

$$\int_0^\infty \frac{t^m}{(t+k)^{n+2}}\,\mathrm{d}t = \frac{m!\,(n-m)!}{(n+1)!\,k^{n-m+1}}\,,$$

where m and n are positive integers with $n \geqslant m$, and where $k > 0$.

(i) The random variable V has density function

$$\mathrm{f}(x) = \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}} \qquad (0 \leqslant x < \infty)\,,$$

where a is a positive integer.

(ii) Show that $C = \dfrac{(2a+1)!}{a!\,a!}$.

(iii) Show, by means of a suitable substitution, that

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,\mathrm{d}u$$

and deduce that the median value of V is k. Find the expected value of V.

(iv) The random variable V represents the speed of a randomly chosen gas molecule. The time taken for such a particle to travel a fixed distance s is given by the random variable $T = \dfrac{s}{V}$.

Show that

$$\mathrm{P}(T < t) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\,\mathrm{d}x \qquad (*)$$

and hence find the density function of T. You may find it helpful to make the substitution $u = \dfrac{s}{x}$ in the integral $(*)$.

(v) Hence show that the product of the median time and the median speed is equal to the distance s, but that the product of the expected time and the expected speed is greater than s.

Section A: Pure Mathematics

1 **(i)** Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x\,.$$

(ii) Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\,.$$

(iii) By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u\,.$$

2 The equation of a curve is $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x - 4 - \frac{16\,(2x+1)^2}{x^2\,(x-4)}\,.$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $\mathrm{f}(x) = 0$ has a double root.

(iii) Sketch the curve.

3 **(i)** Given that $f''(x) > 0$ when $a \leqslant x \leqslant b$, explain with the aid of a sketch why

$$(b-a)\,f\Big(\frac{a+b}{2}\Big) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}\ .$$

(ii) By choosing suitable a, b and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1.

(iii) Deduce that

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right).$$

(iv) Show that

$$\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots\right) < \frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty}\frac{1}{n^2} < \frac{7}{4}\ .$$

4 The triangle OAB is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle C_0 has its centre at the midpoint of the base AB of the triangle, and the sides OA and OB of the triangle are both tangent to the semi-circle. $C_1, C_2, C_3, \ldots$ are circles such that C_n is tangent to C_{n-1} and to sides OA and OB of the triangle.

(i) Let r_n be the radius of C_n. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1-\sin\alpha}{1+\sin\alpha}\ .$$

(ii) Let S be the total area of the semi-circle C_0 and the circles C_1, C_2, C_3, $\ldots$. Show that

$$S = \frac{1+\sin^2\alpha}{4\sin\alpha}\,\pi r_0^2\ .$$

(iii) Show that there are values of α for which S is more than four fifths of the area of triangle OAB.

5 Show that if $\cos(x-\alpha) = \cos\beta$ then either $\tan x = \tan(\alpha+\beta)$ or $\tan x = \tan(\alpha-\beta)$. By choosing suitable values of x, α and β, give an example to show that if $\tan x = \tan(\alpha+\beta)$, then $\cos(x-\alpha)$ need not equal $\cos\beta$.

Let ω be the acute angle such that $\tan\omega = \frac{4}{3}$.

(i) For $0 \leqslant x \leqslant 2\pi$, solve the equation

$$\cos x - 7\sin x = 5$$

giving both solutions in terms of ω.

(ii) For $0 \leqslant x \leqslant 2\pi$, solve the equation

$$2\cos x + 11\sin x = 10$$

showing that one solution is twice the other and giving both in terms of ω.

6 Given a sequence w_0, w_1, w_2, ..., the sequence F_1, F_2, ... is defined by

$$F_n = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1}\,.$$

Show that $F_n - F_{n-1} = (w_n - w_{n-2})\,(w_n + w_{n-2} - 4w_{n-1})$ for $n \geqslant 2$.

(i) The sequence u_0, u_1, u_2, ... has $u_0 = 1$, and $u_1 = 2$ and satisfies

$$u_n = 4u_{n-1} - u_{n-2} \quad (n \geqslant 2)\,.$$

Prove that $u_n^2 + u_{n-1}^2 = 4u_n u_{n-1} - 3$ for $n \geqslant 1$.

(ii) A sequence v_0, v_1, v_2, ... has $v_0 = 1$ and satisfies

$$v_n^2 + v_{n-1}^2 = 4v_n v_{n-1} - 3 \quad (n \geqslant 1). \qquad (*)$$

(a) Find v_1 and prove that, for each $n \geqslant 2$, either $v_n = 4v_{n-1} - v_{n-2}$ or $v_n = v_{n-2}$.

(b) Show that the sequence, with period 2, defined by

$$v_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 2 & \text{for } n \text{ odd} \end{cases}$$

satisfies $(*)$.

(c) Find a sequence v_n with period 4 which has $v_0 = 1$, and satisfies $(*)$.

7 For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

(i) By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \leqslant t \leqslant 1$ show that $I_{n+1} < \frac{1}{2} I_n$.

(ii) Show also that $I_{n+1} = -\dfrac{1}{n\,2^n} + I_n$.

(iii) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

(iv) Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

8 **(i)** Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)y + \frac{\mathrm{g}(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$.

(ii) Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}\,.$$

(iii) Determine the solution curves of this equation which pass through $(1\,,1)$, $(2\,,2)$ and $(4\,,4)$ and sketch graphs of all three curves on the same axes.

Section B: Mechanics

9 A circular hoop of radius a is free to rotate about a fixed horizontal axis passing through a point P on its circumference. The plane of the hoop is perpendicular to this axis. The hoop hangs in equilibrium with its centre, O, vertically below P. The point A on the hoop is vertically below O, so that POA is a diameter of the hoop.

A mouse M runs at constant speed u round the rough inner surface of the lower part of the hoop.

(i) Show that the mouse can choose its speed so that the hoop remains in equilibrium with diameter POA vertical.

(ii) Describe what happens to the hoop when the mouse passes the point at which angle $AOM = 2\arctan\mu$, where μ is the coefficient of friction between mouse and hoop.

10 A particle P of mass m is attached to points A and B, where A is a distance $9a$ vertically above B, by elastic strings, each of which has modulus of elasticity $6mg$. The string AP has natural length $6a$ and the string BP has natural length $2a$. Let x be the distance AP.

(i) The system is released from rest with P on the vertical line AB and $x = 6a$. Show that the acceleration $\ddot{x}$ of P is $\dfrac{4g}{a}(7a - x)$ for $6a < x < 7a$ and $\dfrac{g}{a}(7a - x)$ for $7a < x < 9a$.

(ii) Find the time taken for the particle to reach B.

11 Particles P, of mass 2, and Q, of mass 1, move along a line. Their distances from a fixed point are x_1 and x_2, respectively where $x_2 > x_1$. Each particle is subject to a repulsive force from the other of magnitude $\dfrac{2}{z^3}$, where $z = x_2 - x_1$.

(i) Initially, $x_1 = 0$, $x_2 = 1$, Q is at rest and P moves towards Q with speed 1. Show that z obeys the equation $\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \dfrac{3}{z^3}$.

(ii) By first writing $\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2} = v\dfrac{\mathrm{d}v}{\mathrm{d}z}$, where $v = \dfrac{\mathrm{d}z}{\mathrm{d}t}$, show that $z = \sqrt{4t^2 - 2t + 1}$.

(iii) By considering the equation satisfied by $2x_1 + x_2$, find x_1 and x_2 in terms of t.

Section C: Probability and Statistics

12 A team of m players, numbered from 1 to m, puts on a set of a m shirts, similarly numbered from 1 to m. The players change in a hurry, so that the shirts are assigned to them randomly, one to each player.

(i) Let C_i be the random variable that takes the value 1 if player i is wearing shirt i, and 0 otherwise. Show that $\mathrm{E}\,(C_1) = \frac{1}{m}$ and find $\mathrm{Var}\ (C_1)$ and $\mathrm{Cov}\,(C_1\,,\ C_2)\,.$

(ii) Let $N = C_1 + C_2 + \cdots + C_m$ be the random variable whose value is the number of players who are wearing the correct shirt. Show that $\mathrm{E}\,(N) = \mathrm{Var}\ (N) = 1\,.$

(iii) Explain why a Normal approximation to N is not likely to be appropriate for any m, but that a Poisson approximation might be reasonable.

(iv) In the case $m = 4$, find, by listing equally likely possibilities or otherwise, the probability that no player is wearing the correct shirt and verify that an appropriate Poisson approximation to N gives this probability with a relative error of about 2%. [Use $\mathrm{e} \approx 2\frac{72}{100}$.]

13 A men's endurance competition has an unlimited number of rounds. In each round, a competitor has, independently, a probability p of making it through the round; otherwise, he fails the round. Once a competitor fails a round, he drops out of the competition; before he drops out, he takes part in every round. The grand prize is awarded to any competitor who makes it through a round which all the other remaining competitors fail; if all the remaining competitors fail at the same round the grand prize is not awarded.

If the competition begins with three competitors, find the probability that:

(i) all three drop out in the same round;

(ii) two of them drop out in round r (with $r \geqslant 2$) and the third in an earlier round;

(iii) the grand prize is awarded.

14 *In this question, $\Phi(z)$ is the cumulative distribution function of a standard normal random variable.*

A random variable is known to have a Normal distribution with mean μ and standard deviation either σ_0 or σ_1, where $\sigma_0 < \sigma_1$. The mean, $\overline{X}$, of a random sample of n values of X is to be used to test the hypothesis $\mathrm{H}_0 : \sigma = \sigma_0$ against the alternative $\mathrm{H}_1 : \sigma = \sigma_1$.

(i) Explain carefully why it is appropriate to use a two sided test of the form: accept H_0 if $\mu - c < \overline{X} < \mu + c$, otherwise accept H_1.

(ii) Given that the probability of accepting H_1 when H_0 is true is α, determine c in terms of n, σ_0 and z_α, where z_α is defined by $\Phi(z_\alpha) = 1 - \frac{1}{2}\alpha$.

(iii) The probability of accepting H_0 when H_1 is true is denoted by β. Show that β is independent of n.

(iv) Given that $\Phi(1.960) \approx 0.975$ and that $\Phi(0.063) \approx 0.525$, determine, approximately, the minimum value of $\dfrac{\sigma_1}{\sigma_0}$ if α and β are both to be less than 0.05.

Section A: Pure Mathematics

1 Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right) = \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{arcosh}\left(\dfrac{x+b}{x+a}\right)$.

Hence, or otherwise, integrate, for $x > -1$,

(i) $\displaystyle\int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x$,

(ii) $\displaystyle\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x$.

[You may use the results $\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{arcosh}\, x = \dfrac{1}{\sqrt{x^2-1}}$.]

2 Show that ${}^{2r}\mathrm{C}_r = \dfrac{1\times 3\times\cdots\times(2r-1)}{r!}\times 2^r$, for $r \geqslant 1$.

(i) Give the first four terms of the binomial series for $(1-p)^{-\frac{1}{2}}$.

By choosing a suitable value for p in this series, or otherwise, show that

$$\sum_{r=0}^{\infty} \frac{{}^{2r}\mathrm{C}_r}{8^r} = \sqrt{2}\,.$$

(ii) Show that

$$\sum_{r=0}^{\infty} \frac{(2r+1)\ {}^{2r}\mathrm{C}_r}{5^r} = \left(\sqrt{5}\right)^3 .$$

[**Note:** ${}^{n}\mathrm{C}_r$ is an alternative notation for $\dbinom{n}{r}$ for $r \geqslant 1$, and ${}^{0}\mathrm{C}_0 = 1$.]

3 **(i)** If m is a positive integer, show that $(1+x)^m + (1-x)^m \neq 0$ for any real x.

(ii) The function f is defined by

$$\mathrm{f}(x) = \frac{(1+x)^m - (1-x)^m}{(1+x)^m + (1-x)^m}\,.$$

Find and simplify an expression for $\mathrm{f}'(x)$.

(iii) In the case $m = 5$, sketch the curves $y = \mathrm{f}(x)$ and $y = \dfrac{1}{\mathrm{f}(x)}$.

4 A curve is defined parametrically by

$$x = t^2\,, \quad y = t(1+t^2)\,.$$

(i) The tangent at the point with parameter t, where $t \neq 0$, meets the curve again at the point with parameter T, where $T \neq t$. Show that

$$T = \frac{1-t^2}{2t} \quad \text{and} \quad 3t^2 \neq 1\,.$$

(ii) Given a point P_0 on the curve, with parameter t_0, a sequence of points $P_0,\ P_1,\ P_2, \ldots$ on the curve is constructed such that the tangent at P_i meets the curve again at P_{i+1}. If $t_0 = \tan \frac{7}{18}\pi$, show that $P_3 = P_0$ but $P_1 \neq P_0$.

(iii) Find a second value of t_0, with $t_0 > 0$, for which $P_3 = P_0$ but $P_1 \neq P_0$.

5 **(i)** Find the coordinates of the turning point on the curve $y = x^2 - 2bx + c$.

(ii) Sketch the curve in the case that the equation $x^2 - 2bx + c = 0$ has two distinct real roots.

(iii) Use your sketch to determine necessary and sufficient conditions on b and c for the equation $x^2 - 2bx + c = 0$ to have two distinct real roots.

(iv) Determine necessary and sufficient conditions on b and c for this equation to have two distinct positive roots.

(v) Find the coordinates of the turning points on the curve $y = x^3 - 3b^2x + c$ (with $b > 0$) and hence determine necessary and sufficient conditions on b and c for the equation $x^3 - 3b^2x + c = 0$ to have three distinct real roots.

(vi) Determine necessary and sufficient conditions on a, b and c for the equation $(x-a)^3 - 3b^2(x-a) + c = 0$ to have three distinct positive roots.

(vii) Show that the equation $2x^3 - 9x^2 + 7x - 1 = 0$ has three distinct positive roots.

6 **(i)** Show that

$$2\sin\tfrac{1}{2}\theta\,\cos r\theta = \sin\left(r + \tfrac{1}{2}\right)\theta - \sin\left(r - \tfrac{1}{2}\right)\theta\,.$$

(ii) Hence, or otherwise, find all solutions of the equation

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0\,,$$

where a and b are positive integers with $a < b - 1$.

7 In the x–y plane, the point A has coordinates $(a\,,0)$ and the point B has coordinates $(0\,,b)$, where a and b are positive. The point P , which is distinct from A and B, has coordinates (s,t) . X and Y are the feet of the perpendiculars from P to the x–axis and y–axis respectively, and N is the foot of the perpendicular from P to the line AB .

(i) Show that the coordinates $(x\,,y)$ of N are given by

$$x = \frac{ab^2 - a(bt - as)}{a^2+b^2}\ , \quad y = \frac{a^2b + b(bt-as)}{a^2+b^2}\ .$$

(ii) Show that, if $\left(\dfrac{t-b}{s}\right)\left(\dfrac{t}{s-a}\right) = -1$, then N lies on the line XY .

(iii) Give a geometrical interpretation of this result.

8 **(i)** Show that the gradient at a point $(x\,,\,y)$ on the curve

$$(y+2x)^3\,(y-4x) = c\ ,$$

where c is a constant, is given by
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16x - y}{2y - 5x}\ .$$

(ii) By considering the derivative with respect to x of $(y+ax)^n\,(y+bx)$, or otherwise, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{10x - 4y}{3x - y}\ .$$

Section B: Mechanics

9 A particle P of mass m is constrained to move on a vertical circle of smooth wire with centre O and of radius a. L is the lowest point of the circle and H the highest and $\angle LOP = \theta$. The particle is attached to H by an elastic string of natural length a and modulus of elasticity αmg, where $\alpha > 1$.

(i) Show that, if $\alpha > 2$, there is an equilibrium position with $0 < \theta < \pi$.

(ii) Given that $\alpha = 2 + \sqrt{2}$, and that $\theta = \frac{1}{2}\pi + \phi$, show that

$$\ddot{\phi} \approx -\frac{g(\sqrt{2}+1)}{2a}\,\phi$$

when ϕ is small.

(iii) For this value of α, explain briefly what happens to the particle if it is given a small displacement when $\theta = \frac{1}{2}\pi$.

10 A particle moves along the x-axis in such a way that its acceleration is $kx\dot{x}$ where k is a positive constant. When $t = 0$, $x = d$ (where $d > 0$) and $\dot{x} = U$.

(i) Find x as a function of t in the case $U = kd^2$ and show that x tends to infinity as t tends to $\dfrac{\pi}{2dk}$.

(ii) If $U < 0$, find x as a function of t and show that it tends to a limit, which you should state in terms of d and U, as t tends to infinity.

11 Point B is a distance d due south of point A on a horizontal plane. Particle P is at rest at B at $t = 0$, when it begins to move with constant acceleration a in a straight line with fixed bearing β. Particle Q is projected from point A at $t = 0$ and moves in a straight line with constant speed v.

(i) Show that if the direction of projection of Q can be chosen so that Q strikes P, then

$$v^2 \geqslant ad\,(1 - \cos\beta)\ .$$

(ii) Show further that if $v^2 > ad(1 - \cos\beta)$ then the direction of projection of Q can be chosen so that Q strikes P before P has moved a distance d.

Section C: Probability and Statistics

12 Brief interruptions to my work occur on average every ten minutes and the number of interruptions in any given time period has a Poisson distribution.

(i) Given that an interruption has just occurred, find the probability that I will have less than t minutes to work before the next interruption. If the random variable T is the time I have to work before the next interruption, find the probability density function of T.

(ii) I need an uninterrupted half hour to finish an important paper. Show that the expected number of interruptions before my first uninterrupted period of half an hour or more is $\mathrm{e}^3 - 1$.

(iii) Find also the expected length of time between interruptions that are less than half an hour apart.

(iv) Hence write down the expected wait before my first uninterrupted period of half an hour or more.

13 In a rabbit warren, underground chambers A, B, C and D are at the vertices of a square, and burrows join A to B, B to C, C to D and D to A. Each of the chambers also has a tunnel to the surface. A rabbit finding itself in any chamber runs along one of the two burrows to a neighbouring chamber, or leaves the burrow through the tunnel to the surface. Each of these three possibilities is equally likely.

Let p_A, p_B, p_C and p_D be the probabilities of a rabbit leaving the burrow through the tunnel from chamber A, given that it is currently in chamber A, B, C or D, respectively.

(i) Explain why $p_A = \frac{1}{3} + \frac{1}{3}p_B + \frac{1}{3}p_D$.

(ii) Determine p_A.

(iii) Find the probability that a rabbit which starts in chamber A does not visit chamber C, given that it eventually leaves the burrow through the tunnel in chamber A.

14 **(i)** Write down the probability generating function for the score on a standard, fair six-faced die whose faces are labelled $1, 2, 3, 4, 5, 6$.

(ii) Hence show that the probability generating function for the sum of the scores on two standard, fair six-faced dice, rolled independently, can be written as

$$\frac{1}{36}t^2\left(1+t\right)^2\left(1-t+t^2\right)^2\left(1+t+t^2\right)^2 \ .$$

(iii) Write down, in factorised form, the probability generating functions for the scores on two fair six-faced dice whose faces are labelled with the numbers $1, 2, 2, 3, 3, 4$ and $1, 3, 4, 5, 6, 8,$ and hence show that when these dice are rolled independently, the probability of any given sum of the scores is the same as for the two standard fair six-faced dice.

(iv) Standard, fair four-faced dice are tetrahedra whose faces are labelled $1, 2, 3, 4,$ the score being taken from the face which is not visible after throwing, and each score being equally likely. Find all the ways in which two fair four-faced dice can have their faces labelled with positive integers if the probability of any given sum of the scores is to be the same as for the two standard fair four-faced dice.

Section A: Pure Mathematics

1 **(i)** Find the area of the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$. What happens to this area as a tends to infinity?

(ii) Find the volume of the solid obtained when the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$, is rotated through 2π radians about the x-axis. What happens to this volume as a tends to infinity?

2 **(i)** Prove that $\arctan a + \arctan b = \arctan\left(\dfrac{a+b}{1-ab}\right)$ when $0 < a < 1$ and $0 < b < 1$.

(ii) Prove by induction that, for $n \geqslant 1$,

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{r^2+r+1}\right) = \arctan\left(\frac{n}{n+2}\right)$$

and hence find

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2+r+1}\right).$$

(iii) Hence prove that

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2-r+1}\right) = \frac{\pi}{2}.$$

3 Let

$$\mathrm{f}(x) = a\sqrt{x} - \sqrt{x-b},$$

where $x \geqslant b > 0$ and $a > 1$. Sketch the graph of $\mathrm{f}(x)$. Hence show that the equation $\mathrm{f}(x) = c$, where $c > 0$, has no solution when $c^2 < b\,(a^2-1)$. Find conditions on c^2 in terms of a and b for the equation to have exactly one or exactly two solutions.

Solve the equations

(i) $3\sqrt{x} - \sqrt{x-2} = 4$,

(ii) $3\sqrt{x} - \sqrt{x-3} = 5$.

4 **(i)** Show that if x and y are positive and $x^3 + x^2 = y^3 - y^2$ then $x < y$.

(ii) Show further that if $0 < x \leqslant y - 1$, then $x^3 + x^2 < y^3 - y^2$.

(iii) Prove that there does not exist a pair of *positive* integers such that the difference of their cubes is equal to the sum of their squares.

(iv) Find all the pairs of integers such that the difference of their cubes is equal to the sum of their squares.

5 **(i)** Give a condition that must be satisfied by p, q and r for it to be possible to write the quadratic polynomial $px^2 + qx + r$ in the form $p\,(x + h)^2$, for some h.

(ii) Obtain an equation, which you need not simplify, that must be satisfied by t if it is possible to write

$$\left(x^2 + \tfrac{1}{2}bx + t\right)^2 - \left(x^4 + bx^3 + cx^2 + dx + e\right)$$

in the form $k\,(x + h)^2$, for some k and h.

(iii) Hence, or otherwise, write $x^4 + 6x^3 + 9x^2 - 2x - 7$ as a product of two quadratic factors.

6 Find all the solution curves of the differential equation

$$y^4 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^4 = \left(y^2 - 1\right)^2$$

that pass through either of the points

(i) $\left(0,\, \tfrac{1}{2}\sqrt{3}\right)$,

(ii) $\left(0,\, \tfrac{1}{2}\sqrt{5}\right)$.

Show also that $y = 1$ and $y = -1$ are solutions of the differential equation. Sketch all these solution curves on a single set of axes.

7 **(i)** Given that α and β are acute angles, show that $\alpha+\beta = \frac{1}{2}\pi$ if and only if $\cos^2\alpha + \cos^2\beta = 1$.

(ii) In the x–y plane, the point A has coordinates $(0, s)$ and the point C has coordinates $(s, 0)$, where $s > 0$. The point B lies in the first quadrant ($x > 0$, $y > 0$). The lengths of AB, OB and CB are respectively a, b and c.

Show that

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4s^2b^2$$

and hence that

$$(2s^2 - a^2 - c^2)^2 + (2b^2 - a^2 - c^2)^2 = 4a^2c^2 \ .$$

(iii) Deduce that

$$(a-c)^2 \leqslant 2b^2 \leqslant (a+c)^2 \ .$$

8 Four complex numbers u_1, u_2, u_3 and u_4 have unit modulus, and arguments θ_1, θ_2, θ_3 and θ_4, respectively, with $-\pi < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \pi$.

(i) Show that

$$\arg(u_1 - u_2) = \tfrac{1}{2}(\theta_1 + \theta_2 - \pi) + 2n\pi$$

where $n = 0$ or 1.

(ii) Deduce that

$$\arg\left((u_1 - u_2)(u_4 - u_3)\right) = \arg\left((u_1 - u_4)(u_3 - u_2)\right) + 2n\pi$$

for some integer n.

(iii) Prove that

$$|\,(u_1 - u_2)(u_4 - u_3)\,| + |\,(u_1 - u_4)(u_3 - u_2)\,| = |\,(u_1 - u_3)(u_4 - u_2)\,| \ .$$

Section B: Mechanics

9 A tall container made of light material of negligible thickness has the form of a prism, with a square base of area a^2. It contains a volume ka^3 of fluid of uniform density. The container is held so that it stands on a rough plane, which is inclined at angle θ to the horizontal, with two of the edges of the base of the container horizontal.

(i) In the case $k > \frac{1}{2}\tan\theta$, show that the centre of mass of the fluid is at a distance x from the lower side of the container and at a distance y from the base of the container, where

$$\frac{x}{a} = \frac{1}{2} - \frac{\tan\theta}{12k}\ , \ \frac{y}{a} = \frac{k}{2} + \frac{\tan^2\theta}{24k}\ .$$

(ii) Determine the corresponding coordinates in the case $k < \frac{1}{2}\tan\theta$.

(iii) The container is now released. Given that $k < \frac{1}{2}$, show that the container will topple if $\theta > 45°$.

10 A light hollow cylinder of radius a can rotate freely about its axis of symmetry, which is fixed and horizontal. A particle of mass m is fixed to the cylinder, and a second particle, also of mass m, moves on the rough inside surface of the cylinder. Initially, the cylinder is at rest, with the fixed particle on the same horizontal level as its axis and the second particle at rest vertically below this axis. The system is then released.

(i) Show that, if θ is the angle through which the cylinder has rotated, then

$$\ddot{\theta} = \frac{g}{2a}\left(\cos\theta - \sin\theta\right)\ ,$$

provided that the second particle does not slip.

(ii) Given that the coefficient of friction is $(3+\sqrt{3})/6$, show that the second particle starts to slip when the cylinder has rotated through $60°$.

11 A particle moves on a smooth triangular horizontal surface AOB with angle $AOB = 30°$. The surface is bounded by two vertical walls OA and OB and the coefficient of restitution between the particle and the walls is e, where $e < 1$. The particle, which is initially at point P on the surface and moving with velocity u_1, strikes the wall OA at M_1, with angle $PM_1A = \theta$, and rebounds, with velocity v_1, to strike the wall OB at N_1, with angle $M_1N_1B = \theta$. Find e and $\dfrac{v_1}{u_1}$ in terms of θ.

The motion continues, with the particle striking side OA at M_2, M_3, $\ldots$ and striking side OB at N_2, N_3, $\ldots$. Show that, if $\theta < 60°$, the particle reaches O in a finite time.

Section C: Probability and Statistics

12 In a game, a player tosses a biased coin repeatedly until two successive tails occur, when the game terminates. For each head which occurs the player wins $\pounds 1$.

(i) If E is the expected number of tosses of the coin in the course of a game, and p is the probability of a head, explain why

$$E = p\,(1+E) + (1-p)\,p\,(2+E) + 2\,(1-p)^2\ ,$$

and hence determine E in terms of p.

(ii) Find also, in terms of p, the expected winnings in the course of a game.

(iii) A second game is played, with the same rules, except that the player continues to toss the coin until r successive tails occur. Show that the expected number of tosses in the course of a game is given by the expression $\dfrac{1-q^r}{pq^r}$, where $q = 1-p$.

13 **(i)** A continuous random variable is said to have an exponential distribution with parameter λ if its density function is $\mathrm{f}(t) = \lambda \mathrm{e}^{-\lambda t}\ \ (0 \leqslant t < \infty)$. If X_1 and X_2, which are independent random variables, have exponential distributions with parameters λ_1 and λ_2 respectively, find an expression for the probability that either X_1 or X_2 (or both) is less than x.

(ii) Prove that if X is the random variable whose value is the lesser of the values of X_1 and X_2, then X also has an exponential distribution.

(iii) Route A and Route B buses run from my house to my college. The time between buses on each route has an exponential distribution and the mean time between buses is 15 minutes for Route A and 30 minutes for Route B. The timings of the buses on the two routes are independent. If I emerge from my house one day to see a Route A bus and a Route B bus just leaving the stop, show that the median wait for the next bus to my college will be approximately 7 minutes.

14 Prove that, for any two discrete random variables X and Y,

$$\mathrm{Var}\,(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y),$$

where $\mathrm{Var}(X)$ is the variance of X and $\mathrm{Cov}(X,Y)$ is the covariance of X and Y.

When a Grandmaster plays a sequence of m games of chess, she is, independently, equally likely to win, lose or draw each game. If the values of the random variables W, L and D are the numbers of her wins, losses and draws respectively, justify briefly the following claims:

(i) $W+L+D$ has variance 0;

(ii) $W+L$ has a binomial distribution.

Find the value of $\dfrac{\mathrm{Cov}(W,L)}{\sqrt{\mathrm{Var}(W)\mathrm{Var}(L)}}$.

Section A: Pure Mathematics

1 **(i)** Given that $y = \ln(x + \sqrt{x^2+1})$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{x^2+1}}$.

(ii) Prove by induction that, for $n \geqslant 0$,

$$\left(x^2+1\right) y^{(n+2)} + (2n+1)\, xy^{(n+1)} + n^2 y^{(n)} = 0 \ ,$$

where $y^{(n)} = \dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ and $y^{(0)} = y$.

(iii) Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for y begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

2 **(i)** Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$.

(ii) Show that the area of the region defined by the inequalities $y^2 \geqslant x^2 - 8$ and $x^2 \geqslant 25y^2 - 16$ is $(72/5)\ln 2$.

3 Consider the equation

$$x^2 - bx + c = 0 \ ,$$

where b and c are real numbers.

(i) Show that the roots of the equation are real and positive if and only if $b > 0$ and $b^2 \geqslant 4c > 0$, and sketch the region of the b-c plane in which these conditions hold.

(ii) Sketch the region of the b-c plane in which the roots of the equation are real and less than 1 in magnitude.

4 In this question, the function $\sin^{-1}$ is defined to have domain $-1 \leqslant x \leqslant 1$ and range $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and the function $\tan^{-1}$ is defined to have the real numbers as its domain and range $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

(i) Let

$$\mathrm{g}(x) = \frac{2x}{1+x^2}\,, -\infty < x < \infty\,.$$

Sketch the graph of $\mathrm{g}(x)$ and state the range of g.

(ii) Let

$$\mathrm{f}\,(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)\,, \qquad -\infty < x < \infty\,.$$

Show that $\mathrm{f}(x) = 2\tan^{-1} x$ for $-1 \leqslant x \leqslant 1$ and $\mathrm{f}(x) = \pi - 2\tan^{-1} x$ for $x \geqslant 1$.

Sketch the graph of $\mathrm{f}(x)$.

5 **(i)** Show that the equation $x^3 + px + q = 0$ has exactly one real solution if $p \geqslant 0$.

(ii) A parabola C is given parametrically by

$$x = at^2, \quad y = 2at \qquad (a > 0)\;.$$

Find an equation which must be satisfied by t at points on C at which the normal passes through the point $(h,\ k)$. Hence show that, if $h \leqslant 2a$, exactly one normal to C will pass through $(h,\ k)$.

(iii) Find, in Cartesian form, the equation of the locus of the points from which exactly two normals can be drawn to C. Sketch the locus.

6 The plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

meets the co-ordinate axes at the points A, B and C. The point M has coordinates $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$ and O is the origin.

(i) Show that OM meets the plane at the centroid $\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right)$ of triangle ABC.

(ii) Show also that the perpendiculars to the plane from O and from M meet the plane at the orthocentre and at the circumcentre of triangle ABC respectively.

(iii) Hence prove that the centroid of a triangle lies on the line segment joining its orthocentre and circumcentre, and that it divides this line segment in the ratio $2:1$.

[The *orthocentre* of a triangle is the point at which the three altitudes intersect; the *circumcentre* of a triangle is the point equidistant from the three vertices.]

7 **(i)** Sketch the graph of the function $\ln x - \frac{1}{2}x^2$.

(ii) Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy}{x^2-1}$$

describes a family of parabolas each of which passes through the points $(1\,,\ 0)$ and $(-1\,,\ 0)$ and has its vertex on the y–axis.

(iii) Hence find the equation of the curve that passes through the point $(1\,,\ 1)$ and intersects each of the above parabolas orthogonally.

(iv) Sketch this curve.

[Two curves intersect *orthogonally* if their tangents at the point of intersection are perpendicular.]

8 **(i)** Prove that the equations

$$|z-(1+\mathrm{i})|^2 = 2 \qquad (*)$$

and

$$|z-(1-\mathrm{i})|^2 = 2\,|z-1|^2$$

describe the same locus in the complex z–plane. Sketch this locus.

(ii) Prove that the equation

$$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4} \qquad (**)$$

describes part of this same locus, and show on your sketch which part.

(iii) The complex number w is related to z by

$$w = \frac{2}{z}\,.$$

Determine the locus produced in the complex w–plane if z satisfies $(*)$. Sketch this locus and indicate the part of this locus that corresponds to $(**)$.

Section B: Mechanics

9 B_1 and B_2 are parallel, thin, horizontal fixed beams. B_1 is a vertical distance $d\sin\alpha$ above B_2, and a horizontal distance $d\cos\alpha$ from B_2, where $0 < \alpha < \pi/2$. A long heavy plank is held so that it rests on the two beams, perpendicular to each, with its centre of gravity at B_1. The coefficients of friction between the plank and B_1 and B_2 are μ_1 and μ_2, respectively, where $\mu_1 < \mu_2$ and $\mu_1 + \mu_2 = 2\tan\alpha$.

The plank is released and slips over the beams experiencing a force of resistance from each beam equal to the limiting frictional force (i.e. the product of the appropriate coefficient of friction and the normal reaction). Show that it will come to rest with its centre of gravity over B_2 in a time

$$\pi\left(\frac{d}{g(\mu_2-\mu_1)\cos\alpha}\right)^{\frac{1}{2}} .$$

10 Three ships A, B and C move with velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{u}$ respectively. The velocities of A and B relative to C are equal in magnitude and perpendicular.

(i) Write down conditions that $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy and show that

$$\left|\mathbf{u}-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right|^2 = \left|\tfrac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)\right|^2$$

and

$$\left(\mathbf{u}-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right)\cdot(\mathbf{v}_1-\mathbf{v}_2) = 0 .$$

(ii) Explain why these equations determine, for given $\mathbf{v}_1$ and $\mathbf{v}_2$, two possible velocities for C, provided $\mathbf{v}_1 \neq \mathbf{v}_2$.

(iii) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are equal in magnitude and perpendicular, show that if $\mathbf{u} \neq \mathbf{0}$ then $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$.

11 A uniform cylinder of radius a rotates freely about its axis, which is fixed and horizontal. The moment of inertia of the cylinder about its axis is I. A light string is wrapped around the cylinder and supports a mass m which hangs freely. A particle of mass M is fixed to the surface of the cylinder. The system is held at rest with the particle vertically below the axis of the cylinder, and then released. Find, in terms of I, a, M, m, g and θ, the angular velocity of the cylinder when it has rotated through angle θ.

Show that the cylinder will rotate without coming to a halt if $m/M > \sin\alpha$, where α satisifes $\alpha = \tan\frac{1}{2}\alpha$ and $0 < \alpha < \pi$.

Section C: Probability and Statistics

12 A bag contains b black balls and w white balls. Balls are drawn at random from the bag and when a white ball is drawn it is put aside.

(i) If the black balls drawn are also put aside, find an expression for the expected number of black balls that have been drawn when the last white ball is removed.

(ii) If instead the black balls drawn are put back into the bag, prove that the expected number of times a black ball has been drawn when the first white ball is removed is b/w. Hence write down, in the form of a sum, an expression for the expected number of times a black ball has been drawn when the last white ball is removed.

13 In a game for two players, a fair coin is tossed repeatedly. Each player is assigned a sequence of heads and tails and the player whose sequence appears first wins. Four players, A, B, C and D take turns to play the game. Each time they play, A is assigned the sequence TTH (i.e. Tail then Tail then Head), B is assigned THH, C is assigned HHT and D is assigned HTT.

(i) A and B play the game. Let p_{HH}, p_{HT}, p_{TH} and p_{TT} be the probabilities of A winning the game given that the first two tosses of the coin show HH, HT, TH and TT, respectively. Explain why $p_{\text{TT}} = 1$, and why $p_{\text{HT}} = \frac{1}{2}\, p_{\text{TH}} + \frac{1}{2}\, p_{\text{TT}}$. Show that $p_{\text{HH}} = p_{\text{HT}} = \frac{2}{3}$ and that $p_{\text{TH}} = \frac{1}{3}$. Deduce that the probability that A wins the game is $\frac{2}{3}$.

(ii) B and C play the game. Find the probability that B wins.

(iii) Show that if C plays D, then C is more likely to win than D, but that if D plays A, then D is more likely to win than A.

14 **(i)** A random variable X is distributed uniformly on $[\,0\,,\,a\,]$. Show that the variance of X is $\frac{1}{12}a^2$.

(ii) A sample, X_1 and X_2, of two independent values of the random variable is drawn, and the variance V of the sample is determined. Show that $V = \frac{1}{4}\left(X_1 - X_2\right)^2$, and hence prove that $2V$ is an unbiased estimator of the variance of X.

(iii) Find an exact expression for the probability that the value of V is less than $\frac{1}{12}a^2$ and estimate the value of this probability correct to one significant figure.

Section A: Pure Mathematics

1 **(i)** Sketch on the same axes the two curves C_1 and C_2, given by

$$\begin{aligned} C_1: \quad xy &= 1\,, \\ C_2: \quad x^2 - y^2 &= 2\,. \end{aligned}$$

(ii) The curves intersect at P and Q. Given that the coordinates of P are (a, b) (which you need not evaluate), write down the coordinates of Q in terms of a and b.

(iii) The tangent to C_1 through P meets the tangent to C_2 through Q at the point M, and the tangent to C_2 through P meets the tangent to C_1 through Q at N. Show that the coordinates of M are $(-b, a)$ and write down the coordinates of N.

(iv) Show that $PMQN$ is a square.

2 **(i)** Use the substitution $x = 2 - \cos\theta$ to evaluate the integral

$$\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{\frac{1}{2}} \mathrm{d}x.$$

(ii) Show that, for $a < b$,

$$\int_{p}^{q} \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} \mathrm{d}x = \frac{(b-a)(\pi + 3\sqrt{3} - 6)}{12},$$

where $p = (3a+b)/4$ and $q = (a+b)/2$.

3 **(i)** Given that $\alpha = \mathrm{e}^{\mathrm{i}\pi/3}$, prove that $1 + \alpha^2 = \alpha$.

(ii) A triangle in the Argand plane has vertices A, B, and C represented by the complex numbers p, $q\alpha^2$ and $-r\alpha$ respectively, where p, q and r are positive real numbers. Sketch the triangle ABC.

(iii) Three equilateral triangles ABL, BCM and CAN (each lettered clockwise) are erected on sides AB, BC and CA respectively. Show that the complex number representing N is $(1-\alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing L and M.

(iv) Show that lines LC, MA and NB all meet at the origin, and that these three line segments have the common length $p + q + r$.

4 The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = \frac{x(x-2)(x-a)}{x^2-1}.$$

(i) Prove algebraically that the line $y = x + c$ intersects the curve $y = \mathrm{f}(x)$ if $|a| \geqslant 1$, but there are values of c for which there are no points of intersection if $|a| < 1$.

(ii) Find the equation of the oblique asymptote of the curve $y = \mathrm{f}(x)$. Sketch the graph in the two cases (i) $a < -1$; and (ii) $-1 < a < -\frac{1}{2}$. (You need not calculate the turning points.)

5 Given two non-zero vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ define $\Delta(\mathbf{a}, \mathbf{b})$ by $\Delta(\mathbf{a}, \mathbf{b}) = a_1 b_2 - a_2 b_1$.

Let A, B and C be points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, no two of which are parallel. Let P, Q and R be points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, none of which are parallel.

(i) Show that there exists a 2×2 matrix $\mathbf{M}$ such that P and Q are the images of A and B under the transformation represented by $\mathbf{M}$.

(ii) Show that $\Delta(\mathbf{a}, \mathbf{b})\,\mathbf{c} + \boldsymbol{\Delta}(\mathbf{c}, \mathbf{a})\,\mathbf{b} + \boldsymbol{\Delta}(\mathbf{b}, \mathbf{c})\,\mathbf{a} = \mathbf{0}$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points P, Q, and R to be the images of points A, B and C under the transformation represented by some 2×2 matrix $\mathbf{M}$ is that

$$\Delta(\mathbf{a}, \mathbf{b}) : \Delta(\mathbf{b}, \mathbf{c}) : \Delta(\mathbf{c}, \mathbf{a}) = \Delta(\mathbf{p}, \mathbf{q}) : \Delta(\mathbf{q}, \mathbf{r}) : \Delta(\mathbf{r}, \mathbf{p})\,.$$

6 **(i)** Given that

$$x^4 + px^2 + qx + r = (x^2 - ax + b)(x^2 + ax + c),$$

express p, q and r in terms of a, b and c.

(ii) Show also that a^2 is a root of the cubic equation

$$u^3 + 2pu^2 + (p^2 - 4r)u - q^2 = 0.$$

(iii) Explain why this equation always has a non-negative root, and verify that $u = 9$ is a root in the case $p = -1$, $q = -6$, $r = 15$.

(iv) Hence, or otherwise, express

$$y^4 - 8y^3 + 23y^2 - 34y + 39$$

as a product of two quadratic factors.

7 **(i)** Given that

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots ,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < \mathrm{e}$$

for any positive integer n.

(ii) The product $\mathrm{P}(n)$ is defined, for any positive integer n, by

$$\mathrm{P}(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \ldots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} ,$$

to show that $\mathrm{P}(n) < \mathrm{e}$ for all n .

(iii) Explain briefly why $\mathrm{P}(n)$ tends to a limit as $n \to \infty$. Show that this limit, L, satisfies $2 < L \leqslant \mathrm{e}$.

8 The sequence a_n is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \qquad (n \geqslant 2).$$

(i) Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \qquad (n \geqslant 2).$$

(ii) Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \qquad (n \geqslant 1),$$

where $\alpha = \dfrac{1 + \sqrt{5}}{2}$.

Section B: Mechanics

9 Two small discs of masses m and μm lie on a smooth horizontal surface. The disc of mass μm is at rest, and the disc of mass m is projected towards it with velocity $\mathbf{u}$. After the collision, the disc of mass μm moves in the direction given by unit vector $\mathbf{n}$. The collision is perfectly elastic.

(i) Show that the speed of the disc of mass μm after the collision is $\dfrac{2\mathbf{u}\cdot\mathbf{n}}{1+\mu}$.

(ii) Given that the two discs have equal kinetic energy after the collision, find an expression for the cosine of the angle between $\mathbf{n}$ and $\mathbf{u}$ and show that $3-\sqrt{8}\leqslant \mu \leqslant 3+\sqrt{8}$.

10 A sphere of radius a and weight W rests on horizontal ground. A thin uniform beam of weight $3\sqrt{3}\,W$ and length $2a$ is freely hinged to the ground at X, which is a distance $\sqrt{3}\,a$ from the point of contact of the sphere with the ground. The beam rests on the sphere, lying in the same vertical plane as the centre of the sphere. The coefficients of friction between the beam and the sphere and between the sphere and the ground are μ_1 and μ_2 respectively.

Given that the sphere is on the point of slipping at its contacts with both the ground and the beam, find the values of μ_1 and μ_2.

11 A thin beam is fixed at a height $2a$ above a horizontal plane. A uniform straight rod ACB of length $9a$ and mass m is supported by the beam at C. Initially, the rod is held so that it is horizontal and perpendicular to the beam. The distance AC is $3a$, and the coefficient of friction between the beam and the rod is μ.

The rod is now released. Find the minimum value of μ for which B strikes the horizontal plane before slipping takes place at C.

Section C: Probability and Statistics

12 In a lottery, any one of N numbers, where N is large, is chosen at random and independently for each player by machine. Each week there are $2N$ players and one winning number is drawn. Write down an exact expression for the probability that there are three or fewer winners in a week, given that you hold a winning ticket that week.

(i) Using the fact that

$$\left(1-\frac{a}{n}\right)^n \approx \mathrm{e}^{-a}$$

for n much larger than a, or otherwise, show that this probability is approximately $\frac{2}{3}$.

(ii) Discuss briefly whether this probability would increase or decrease if the numbers were chosen by the players.

(iii) Show that the expected number of winners in a week, given that you hold a winning ticket that week, is $3 - N^{-1}$.

13 **(i)** A set of n dice is rolled repeatedly. For each die the probability of showing a six is p. Show that the probability that the first of the dice to show a six does so on the rth roll is

$$q^{nr}(q^{-n}-1)$$

where $q = 1 - p$.

(ii) Determine, and simplify, an expression for the probability generating function for this distribution, in terms of q and n. The first of the dice to show a six does so on the Rth roll. Find the expected value of R and show that, in the case $n = 2$, $p = 1/6$, this value is $36/11$.

(iii) Show that the probability that the last of the dice to show a six does so on the rth roll is

$$\big(1-q^r\big)^n - \big(1-q^{r-1}\big)^n.$$

(iv) Find, for the case $n = 2$, the probability generating function. The last of the dice to show a six does so on the Sth roll. Find the expected value of S and evaluate this when $p = 1/6$.

14 The random variable X takes only the values x_1 and x_2 (where $x_1 \neq x_2$), and the random variable Y takes only the values y_1 and y_2 (where $y_1 \neq y_2$). Their joint distribution is given by

$$\mathrm{P}(X = x_1, Y = y_1) = a \ ; \ \ \mathrm{P}(X = x_1, Y = y_2) = q - a \ ; \ \ \mathrm{P}(X = x_2, Y = y_1) = p - a \ .$$

(i) Show that if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ then

$$(a - pq)(x_1 - x_2)(y_1 - y_2) = 0.$$

(ii) Hence show that two random variables each taking only two distinct values are independent if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

(iii) Give a joint distribution for two random variables A and B, each taking the three values -1, 0 and 1 with probability $\frac{1}{3}$, which have $\mathrm{E}(AB) = \mathrm{E}(A)\mathrm{E}(B)$, but which are not independent.

Section A: Pure Mathematics

1 Consider the cubic equation

$$x^3 - px^2 + qx - r = 0 \ ,$$

where $p \neq 0$ and $r \neq 0$.

(i) If the three roots can be written in the form ak^{-1}, a and ak for some constants a and k, show that one root is q/p and that $q^3 - rp^3 = 0$.

(ii) If $r = q^3/p^3$, show that q/p is a root and that the product of the other two roots is $(q/p)^2$. Deduce that the roots are in geometric progression.

(iii) Find a necessary and sufficient condition involving p, q and r for the roots to be in arithmetic progression.

2 **(i)** Let $\mathrm{f}(x) = (1+x^2)\mathrm{e}^x$. Show that $\mathrm{f}'(x) \geqslant 0$ and sketch the graph of $\mathrm{f}(x)$. Hence, or otherwise, show that the equation

$$(1 + x^2)\mathrm{e}^x = k,$$

where k is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leqslant 0$.

(ii) Determine the number of real roots of the equation

$$(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$$

in the cases (a) $0 < k \leqslant 2/\pi$ and (b) $2/\pi < k < 1$.

3 **(i)** Justify, by means of a sketch, the formula

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n} \mathrm{f}(1+m/n)\right\} = \int_1^2 \mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Show that

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\} = \ln 2\,.$$

(iii) Evaluate

$$\lim_{n\to\infty}\left\{\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+n^2}\right\}\,.$$

4 A polyhedron is a solid bounded by F plane faces, which meet in E edges and V vertices. You may assume *Euler's formula*, that $V-E+F=2$.

(i) In a regular polyhedron the faces are equal regular m-sided polygons, n of which meet at each vertex. Show that

$$F=\frac{4n}{h}\,,$$

where $h=4-(n-2)(m-2)$.

(ii) By considering the possible values of h, or otherwise, prove that there are only five regular polyhedral, and find V, E and F for each.

5 The sequence u_0, u_1, u_2, ... is defined by

$$u_0=1,\quad u_1=1,\qquad u_{n+1}=u_n+u_{n-1}\quad \text{for}\ \ n\geqslant 1\,.$$

(i) Prove that

$$u_{n+2}^2+u_{n-1}^2=2(u_{n+1}^2+u_n^2)\,.$$

(ii) Using induction, or otherwise, prove the following result:

$$u_{2n}=u_n^2+u_{n-1}^2 \quad \text{and} \quad u_{2n+1}=u_{n+1}^2-u_{n-1}^2$$

for any positive integer n.

6 **(i)** A closed curve is given by the equation

$$x^{2/n} + y^{2/n} = a^{2/n} \qquad (*)$$

where n is an odd integer and a is a positive constant. Find a parametrization $x = x(t)$, $y = y(t)$ which describes the curve anticlockwise as t ranges from 0 to 2π.

(ii) Sketch the curve in the case $n = 3$, justifying the main features of your sketch.

(iii) The area A enclosed by such a curve is given by the formula

$$A = \frac{1}{2}\int_0^{2\pi} \left[x(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} - y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} \right] \mathrm{d}t\,.$$

Use this result to find the area enclosed by $(*)$ for $n = 3$.

7 Let a be a non-zero real number and define a binary operation on the set of real numbers by

$$x * y = x + y + axy\,.$$

(i) Show that the operation $*$ is associative.

(ii) Show that $(G, *)$ is a group, where G is the set of all real numbers except for one number which you should identify.

(iii) Find a subgroup of $(G, *)$ which has exactly 2 elements.

8 The function $y(x)$ is defined for $x \geqslant 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at } x = 0.$$

When x is in the range $2(n-1)\pi < x < 2n\pi$, where n is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + n^2y = 0.$$

Both y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0,\ 1,\ 2,\ \ldots$.

(i) Find $y(x)$ for $0 \leqslant x \leqslant 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leqslant x \leqslant 4\pi$, and find $y(x)$ for all $x \geqslant 0$.

(iii) Show that

$$\int_0^\infty y^2\,\mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

Section B: Mechanics

9 The gravitational force between two point particles of masses m and m' is mutually attractive and has magnitude

$$\frac{Gmm'}{r^2},$$

where G is a constant and r is the distance between them.

A particle of unit mass lies on the axis of a thin uniform circular ring of radius r and mass m, at a distance x from its centre.

(i) Explain why the net force on the particle is directed towards the centre of the ring and show that its magnitude is

$$\frac{Gmx}{(x^2+r^2)^{3/2}}.$$

(ii) The particle now lies inside a thin hollow spherical shell of uniform density, mass M and radius a, at a distance b from its centre. Show that the particle experiences no gravitational force due to the shell.

10 **(i)** A chain of mass m and length l is composed of n small smooth links. It is suspended vertically over a horizontal table with its end just touching the table, and released so that it collapses inelastically onto the table. Calculate the change in momentum of the $(k+1)$th link from the bottom of the chain as it falls onto the table.

(ii) Write down an expression for the total impulse sustained by the table in this way from the whole chain. By approximating the sum by an integral, show that this total impulse is approximately

$$\tfrac{2}{3}m\sqrt{2gl}$$

when n is large.

11 **(i)** Calculate the moment of inertia of a uniform thin circular hoop of mass m and radius a about an axis perpendicular to the plane of the hoop through a point on its circumference.

(ii) The hoop, which is rough, rolls with speed v on a rough horizontal table straight towards the edge and rolls over the edge without initially losing contact with the edge. Show that the hoop will lose contact with the edge when it has rotated about the edge of the table through an angle θ, where

$$\cos\theta = \frac{1}{2} + \frac{v^2}{2ag}.$$

Section C: Probability and Statistics

12 In the game of endless cricket the scores X and Y of the two sides are such that

$$\mathrm{P}(X = j,\ Y = k) = \mathrm{e}^{-1}\frac{(j+k)\lambda^{j+k}}{j!k!},$$

for some positive constant λ, where $j, k = 0,\ 1,\ 2,\ \ldots$.

(i) Find $\mathrm{P}(X + Y = n)$ for each $n > 0$.

(ii) Show that $2\lambda\mathrm{e}^{2\lambda - 1} = 1$.

(iii) Show that $2x\mathrm{e}^{2x-1}$ is an increasing function of x for $x > 0$ and deduce that the equation in (ii) has at most one solution and hence determine λ.

(iv) Calculate the expectation $\mathrm{E}(2^{X+Y})$.

13 The cakes in our canteen each contain exactly four currants, each currant being randomly placed in the cake. I take a proportion X of a cake where X is a random variable with density function

$$\mathrm{f}(x) = Ax$$

for $0 \leqslant x \leqslant 1$ where A is a constant.

(i) What is the expected number of currants in my portion?

(ii) If I find all four currants in my portion, what is the probability that I took more than half the cake?

14 In the basic version of Horizons (H1) the player has a maximum of n turns, where $n \geqslant 1$. At each turn, she has a probability p of success, where $0 < p < 1$. If her first success is at the rth turn, where $1 \leqslant r \leqslant n$, she collects r pounds and then withdraws from the game. Otherwise, her winnings are nil.

(i) Show that in H1, her expected winnings are

$$p^{-1}\left[1 + nq^{n+1} - (n+1)q^n\right] \quad \text{pounds,}$$

where $q = 1 - p$.

(ii) The rules of H2 are the same as those of H1, except that n is randomly selected from a Poisson distribution with parameter λ. If $n = 0$ her winnings are nil. Otherwise she plays H1 with the selected n. Show that in H2, her expected winnings are

$$\frac{1}{p}\left(1 - \mathrm{e}^{-\lambda p}\right) - \lambda q \mathrm{e}^{-\lambda p} \quad \text{pounds.}$$

Section A: Pure Mathematics

1 Let

$$f(x) = \sin^2 x + 2\cos x + 1$$

for $0 \leqslant x \leqslant 2\pi$.

(i) Sketch the curve $y = f(x)$, giving the coordinates of the stationary points.

(ii) Now let

$$g(x) = \frac{af(x) + b}{cf(x) + d} \qquad ad \neq bc\,,\ d \neq -3c\,,\ d \neq c\,.$$

Show that the stationary points of $y = g(x)$ occur at the same values of x as those of $y = f(x)$, and find the corresponding values of $g(x)$.

(iii) Explain why, if $d/c < -3$ or $d/c > 1$, $|g(x)|$ cannot be arbitrarily large.

2 Let

$$I(a, b) = \int_0^1 t^a (1-t)^b \, dt \qquad (a \geqslant 0,\ b \geqslant 0).$$

(i) Show that $I(a, b) = I(b, a)$,

(ii) Show that $I(a, b) = I(a + 1, b) + I(a, b + 1)$.

(iii) Show that $(a + 1)I(a, b) = bI(a + 1, b - 1)$ when a and b are positive and hence calculate $I(a, b)$ when a and b are positive integers.

3 The value V_N of a bond after N days is determined by the equation

$$V_{N+1} = (1+c)V_N - d \qquad (c > 0,\ d > 0),$$

where c and d are given constants.

(i) By looking for solutions of the form $V_T = Ak^T + B$ for some constants A, B and k, or otherwise, find V_N in terms of V_0.

(ii) What is the solution for $c = 0$? Show that this is the limit (for fixed N) as $c \to 0$ of your solution for $c > 0$.

4 **(i)** Show that the equation (in plane polar coordinates) $r = \cos\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, represents a circle.

(ii) Sketch the curve $r = \cos 2\theta$ for $0 \leqslant \theta \leqslant 2\pi$, and describe the curves $r = \cos 2n\theta$, where n is an integer. Show that the area enclosed by such a curve is independent of n.

(iii) Sketch also the curve $r = \cos 3\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

5 The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r \,,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

(i) Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

(ii) Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

(iii) For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M})\ =\ \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.

6 **(i)** Show that four vertices of a cube, no two of which are adjacent, form the vertices of a regular tetrahedron. Hence, or otherwise, find the volume of a regular tetrahedron whose edges are of unit length.

(ii) Find the volume of a regular octahedron whose edges are of unit length.

(iii) Show that the centres of the faces of a cube form the vertices of a regular octahedron. Show that its volume is half that of the tetrahedron whose vertices are the vertices of the cube.

[*A regular tetrahedron (octahedron) has four (eight) faces, all equilateral triangles.*]

7 **(i)** Sketch the graph of $\mathrm{f}(s) = \mathrm{e}^s(s-3)+3$ for $0 \leqslant s < \infty$. Taking $\mathrm{e} \approx 2.7$, find the smallest positive integer, m, such that $\mathrm{f}(m) > 0$.

(ii) Now let

$$\mathrm{b}(x) = \frac{x^3}{\mathrm{e}^{x/T}-1}$$

where T is a positive constant. Show that $\mathrm{b}(x)$ has a single turning point in $0 < x < \infty$. By considering the behaviour for small x and for large x, sketch $\mathrm{b}(x)$ for $0 \leqslant x < \infty$.

(iii) Let

$$\int_0^\infty \mathrm{b}(x)\,\mathrm{d}x = B,$$

which may be assumed to be finite. Show that $B = KT^n$ where K is a constant, and n is an integer which you should determine.

(iv) Given that $B \approx 2\displaystyle\int_0^{Tm} \mathrm{b}(x)\,\mathrm{d}x$, use your graph of $\mathrm{b}(x)$ to find a rough estimate for K.

8 **(i)** Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r}\cdot\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}\cdot\mathbf{b} - (\mathbf{b}\cdot\mathbf{m})^2\,.$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m}\cdot\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}\cdot\mathbf{d} = 2a^2\,.$$

Section B: Mechanics

9 **(i)** A uniform right circular cone of mass m has base of radius a and perpendicular height h from base to apex. Show that its moment of inertia about its axis is $\frac{3}{10}ma^2$, and calculate its moment of inertia about an axis through its apex parallel to its base.
[*Any theorems used should be stated clearly.*]

(ii) The cone is now suspended from its apex and allowed to perform small oscillations. Show that their period is

$$2\pi\sqrt{\frac{4h^2+a^2}{5gh}}\,.$$

[*You may assume that the centre of mass of the cone is a distance* $\frac{3}{4}h$ *from its apex.*]

10 Two identical spherical balls, moving on a horizontal, smooth table, collide in such a way that both momentum and kinetic energy are conserved. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the balls before the collision and let $\mathbf{v}_1'$ and $\mathbf{v}_2'$ be the velocities of the balls after the collision, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are two-dimensional vectors.

(i) Write down the equations for conservation of momentum and kinetic energy in terms of these vectors.

(ii) Hence show that their relative speed is also conserved.

(iii) Show that, if one ball is initially at rest but after the collision both balls are moving, their final velocities are perpendicular.

(iv) Now suppose that one ball is initially at rest, and the second is moving with speed V. After a collision in which they lose a proportion k of their original kinetic energy ($0 \leqslant k \leqslant 1$), the direction of motion of the second ball has changed by an angle θ. Find a quadratic equation satisfied by the final speed of the second ball, with coefficients depending on k, V and θ. Hence show that $k \leqslant \frac{1}{2}$.

11 Consider a simple pendulum of length l and angular displacement θ, which is **not** assumed to be small.

(i) Show that

$$\frac{1}{2}l\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = g(\cos\theta - \cos\gamma)\,,$$

where γ is the maximum value of θ. Show also that the period P is given by

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\gamma}\left(\sin^2(\gamma/2) - \sin^2(\theta/2)\right)^{-\frac{1}{2}}\,\mathrm{d}\theta\,.$$

(ii) By using the substitution $\sin(\theta/2) = \sin(\gamma/2)\sin\phi$, and then finding an approximate expression for the integrand using the binomial expansion, show that for small values of γ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right)\,.$$

Section C: Probability and Statistics

12 The mountain villages A, B, C and D lie at the vertices of a tetrahedron, and each pair of villages is joined by a road. After a snowfall the probability that any road is blocked is p, and is independent of the conditions of any other road. The probability that, after a snowfall, it is possible to travel from any village to any other village by some route is P. Show that

$$P = 1 - p^3(6p^3 - 12p^2 + 3p + 4).$$

13 Write down the probability of obtaining k heads in n tosses of a fair coin. Now suppose that k is known but n is unknown. A *maximum likelihood estimator* (MLE) of n is defined to be a value (which must be an integer) of n which maximizes the probability of k heads.

(i) A friend has thrown a fair coin a number of times. She tells you that she has observed one head. Show that in this case there are *two* MLEs of the number of tosses she has made.

(ii) She now tells you that in a repeat of the exercise she has observed k heads. Find the two MLEs of the number of tosses she has made.

(iii) She next uses a coin biased with probability p (known) of showing a head, and again tells you that she has observed k heads. Find the MLEs of the number of tosses made. What is the condition for the MLE to be unique?

14 A hostile naval power possesses a large, unknown number N of submarines. Interception of radio signals yields a small number n of their identification numbers X_i $(i = 1, 2, ..., n)$, which are taken to be independent and uniformly distributed over the continuous range from 0 to N.

(i) Show that Z_1 and Z_2, defined by

$$Z_1 = \frac{n+1}{n}\max\{X_1, X_2, ..., X_n\} \qquad \text{and} \qquad Z_2 = \frac{2}{n}\sum_{i=1}^{n} X_i \ ,$$

both have means equal to N.

(ii) Calculate the variance of Z_1 and of Z_2. Which estimator do you prefer, and why?

Section A: Pure Mathematics

1 **(i)** By considering the series expansion of $(x^2+5x+4)\,\mathrm{e}^x$ show that

$$10\,\mathrm{e} = 4 + \frac{3^2}{1!} + \frac{4^2}{2!} + \frac{5^2}{3!} + \cdots \,.$$

(ii) Show that

$$5\,\mathrm{e} = 1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots \,.$$

(iii) Evaluate

$$1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \cdots \,.$$

2 **(i)** Let

$$\mathrm{f}(t) = \frac{\ln t}{t} \quad \text{for } t > 0.$$

Sketch the graph of $\mathrm{f}(t)$ and find its maximum value. How many positive values of t correspond to a given value of $\mathrm{f}(t)$?

(ii) Find how many positive values of y satisfy $x^y = y^x$ for a given positive value of x. Sketch the set of points (x, y) which satisfy $x^y = y^x$ with $x, y > 0$.

3 **(i)** By considering the solutions of the equation $z^n - 1 = 0$, or otherwise, show that

$$(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = 1 + z + z^2 + \dots + z^{n-1},$$

where z is any complex number and $\omega = \mathrm{e}^{2\pi i/n}$.

(ii) Let $A_1, A_2, A_3, \dots, A_n$ be points equally spaced around a circle of radius r centred at O (so that they are the vertices of a regular n-sided polygon).

Show that

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \dots + \overrightarrow{OA_n} = \mathbf{0}.$$

(iii) Deduce, or prove otherwise, that

$$\sum_{k=1}^{n} |A_1A_k|^2 = 2r^2n.$$

4 In this question, you may assume that if $k_1, \dots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean.

Suppose that a, b, c and d are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^3 - 4ax^2 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots.

(i) Show that pqr, qrs, rsp and spq are distinct, where p, q, r and s are the roots of the polynomial f.

(ii) By considering the relationship between the coefficients of f and its roots, show that $c > d$.

(iii) Explain why the polynomial $\mathrm{f}'(x)$ must have three distinct roots.

(iv) By differentiating f, show that $b > c$.

(v) Show that $a > b$.

5 Find the ratio, over one revolution, of the distance moved by a wheel rolling on a flat surface to the distance traced out by a point on its circumference.

6 Suppose that y_n satisfies the equations

$$(1-x^2)\frac{\mathrm{d}^2y_n}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_n}{\mathrm{d}x} + n^2y_n = 0,$$

$$y_n(1) = 1, \quad y_n(x) = (-1)^n y_n(-x).$$

(i) If $x = \cos\theta$, show that

$$\frac{\mathrm{d}^2y_n}{\mathrm{d}\theta^2} + n^2y_n = 0,$$

and hence obtain y_n as a function of θ.

(ii) Deduce that for $|x| \leqslant 1$

$$y_0 = 1, \quad y_1 = x,$$

$$y_{n+1} - 2xy_n + y_{n-1} = 0.$$

7 For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots;$$

$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots.$$

(i) Evaluate b_n.

(ii) Show that $0 < a_n < 1/n$.

(iii) Deduce that $a_n = n!\mathrm{e} - [n!\mathrm{e}]$ (where $[x]$ is the integer part of x).

(iv) Hence show that e is irrational.

8 Let R_α be the 2×2 matrix that represents a rotation through the angle α and let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(i) Find in terms of a, b and c an angle α such that $R_{-\alpha}AR_\alpha$ is a diagonal matrix (i.e. has the value zero in top-right and bottom-left positions).

(ii) Find values of a, b and c such that the equation of the ellipse

$$x^2 + (y + 2x\cot 2\theta)^2 = 1 \qquad (0 < \theta < \tfrac{1}{4}\pi)$$

can be expressed in the form

$$\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

(iii) Show that, for this A, $R_{-\alpha}AR_\alpha$ is diagonal if $\alpha = \theta$.

(iv) Express the non–zero elements of this matrix in terms of θ.

(v) Deduce, or show otherwise, that the minimum and maximum distances from the centre to the circumference of this ellipse are $\tan\theta$ and $\cot\theta$.

Section B: Mechanics

9 A uniform rigid rod BC is suspended from a fixed point A by light stretched springs AB, AC. The springs are of different natural lengths but the ratio of tension to extension is the same constant κ for each. The rod is *not* hanging vertically. Show that the ratio of the lengths of the stretched springs is equal to the ratio of the natural lengths of the unstretched springs.

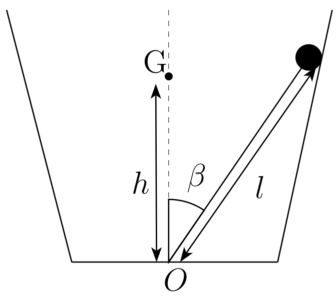
10 By pressing a finger down on it, a uniform spherical marble of radius a is made to slide along a horizontal table top with an initial linear velocity v_0 and an initial *backward* angular velocity ω_0 about the horizontal axis perpendicular to v_0. The frictional force between the marble and the table is constant (independent of speed).

For what value of $v_0/(a\omega_0)$ does the marble

(i) slide to a complete stop,

(ii) come to a stop and then roll back towards its initial position with linear speed $v_0/7$.

11



A heavy symmetrical bell and clapper can both swung freely in a vertical plane about a point O on a horizontal beam at the apex of the bell. The mass of the bell is M and its moment of inertia about the beam is Mk^2. Its centre of mass, G, is a distance h from O. The clapper may be regarded as a small heavy ball on a light rod of length l. Initially the bell is held with its axis vertical and its mouth above the beam. The clapper ball rests against the side of the bell, with the rod making an angle β with the axis. The bell is then released. Show that, at the moment when the clapper and bell separate, the clapper rod makes an angle α with the upwards vertical, where

$$\cot\alpha = \cot\beta - \frac{k^2}{hl}\operatorname{cosec}\beta.$$

Section C: Probability and Statistics

12 **(i)** I toss a biased coin which has a probability p of landing heads and a probability $q = 1 - p$ of landing tails. Let K be the number of tosses required to obtain the first head and let

$$\mathrm{G}(s) = \sum_{k=1}^{\infty} \mathrm{P}(K = k)s^k.$$

Show that

$$\mathrm{G}(s) = \frac{ps}{1 - qs}$$

and hence find the expectation and variance of K.

(ii) I sample cards at random with replacement from a normal pack of 52. Let N be the total number of draws I make in order to sample every card at least once. By expressing N as a sum $N = N_1 + N_2 + \cdots + N_{52}$ of random variables, or otherwise, find the expectation of N. Estimate the numerical value of this expectation, using the approximations $\mathrm{e} \approx 2.7$ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx 0.5 + \ln n$ if n is large.

13 Let X and Y be independent standard normal random variables: the probability density function, f, of each is therefore given by

$$\mathrm{f}(x) = (2\pi)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}.$$

(i) Find the moment generating function $\mathrm{E}(\mathrm{e}^{\theta X})$ of X.

(ii) Find the moment generating function of $aX + bY$ and hence obtain the condition on a and b which ensures that $aX + bY$ has the same distribution as X and Y.

(iii) Let $Z = \mathrm{e}^{\mu + \sigma X}$. Show that

$$\mathrm{E}(Z^\theta) = \mathrm{e}^{\mu\theta + \frac{1}{2}\sigma^2\theta^2},$$

and hence find the expectation and variance of Z.

14 An industrial process produces rectangular plates of mean length μ_1 and mean breadth μ_2. The length and breadth vary independently with non-zero standard deviations σ_1 and σ_2 respectively. Find the means and standard deviations of the perimeter and of the area of the plates. Show that the perimeter and area are not independent.

Section A: Pure Mathematics

1 Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \tfrac{1}{4}\cosh 4x + \tfrac{3}{4}.$$

(i) Find $a_0, a_1, \ldots, a_n$ in terms of n such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

(ii) Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \ldots, 2m$.

2 For all values of a and b, either solve the simultaneous equations

$$\begin{aligned} x + y + az &= 2 \\ x + ay + z &= 2 \\ 2x + y + z &= 2b \end{aligned}$$

or prove that they have no solution.

3 **(i)** Find

$$\int_0^\theta \frac{1}{1 - a\cos x}\,\mathrm{d}x\,,$$

where $0 < \theta < \pi$ and $-1 < a < 1$.

(ii) Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2 - a\cos x}\,\mathrm{d}x = \frac{2}{\sqrt{4 - a^2}}\tan^{-1}\sqrt{\frac{2+a}{2-a}}\,,$$

and also that

$$\int_0^{\frac{3}{4}\pi} \frac{1}{\sqrt{2} + \cos x}\,\mathrm{d}x = \frac{\pi}{2}\,.$$

4 **(i)** Find the integers k satisfying the inequality $k \leqslant 2(k-2)$.

(ii) Given that N is a strictly positive integer consider the problem of finding strictly positive integers whose sum is N and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2 \times 3, P(6) = 3^2, P(7) = 2^2 \times 3, P(8) = 2 \times 3^2$ and $P(9) = 3^3$.

(iii) Find $P(1000)$ explaining your reasoning carefully.

5 Show, using de Moivre's theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where $t = \tan\theta$.

(i) By considering the equation $\tan 7\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right),\ \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

6 **(i)** Let S be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where a is any real non-zero number. Show that S is closed under matrix multiplication and, further, that S is a group under matrix multiplication.

(ii) Let G be a set of $n \times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of G, show that if a given element $\mathbf{A}$ of G is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of G are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

7 **(i)** If $x + y + z = \alpha$, $xy + yz + zx = \beta$ and $xyz = \gamma$, find numbers A, B and C such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C.$$

Solve the equations

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 4. \end{aligned}$$

(ii) The area of a triangle whose sides are a, b and c is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter $\frac{1}{2}(a + b + c)$. If a, b and c are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

8 A transformation T of the real numbers is defined by

$$y = T(x) = \frac{ax - b}{cx - d},$$

where a, b, c, d are real numbers such that $ad \neq bc$. Find all numbers x such that $T(x) = x$. Show that the inverse operation, $x = T^{-1}(y)$ expressing x in terms of y is of the same form as T and find corresponding numbers a', b', c', d'.

(i) Let S_r denote the set of all real numbers excluding r. Show that, if $c \neq 0$, there is a value of r such that T is defined for all $x \in S_r$ and find the image $T(S_r)$. What is the corresponding result if $c = 0$?

(ii) If T_1, given by numbers a_1, b_1, c_1, d_1, and T_2, given by numbers a_2, b_2, c_2, d_2 are two such transformations, show that their composition T_3, defined by $T_3(x) = T_2(T_1(x))$, is of the same form.

(iii) Find necessary and sufficient conditions on the numbers a, b, c, d for T^2, the composition of T with itself, to be the identity. Hence, or otherwise, find transformations T_1, T_2 and their composition T_3 such that T_1^2 and T_2^2 are each the identity but T_3^2 is not.

Section B: Mechanics

9 A particle of mass m is at rest on top of a smooth fixed sphere of radius a. Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$a(5\sqrt{5}+4\sqrt{23})/27$$

from the centre of the sphere.

[Air resistance should be neglected.]

10 Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle α to the horizontal, where $0<\alpha<\pi/2$. Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is W_1 and the coefficient of friction between it and the plane is μ_1. The corresponding qunatities for the lower cylinder are W_2 and μ_2 respectively and the coefficient of friction between the two cylinders is μ. Show that for equilibrium to be possible:

(i) $W_1 \geqslant W_2$;

(ii) $\mu \geqslant \dfrac{W_1+W_2}{W_1-W_2}$;

(iii) $\mu_1 \geqslant \left(\dfrac{2W_1\cot\alpha}{W_1+W_2}-1\right)^{-1}$.

Find the similar inequality to **(iii)** for μ_2.

11 A smooth circular wire of radius a is held fixed in a vertical plane with light elastic strings of natural length a and modulus λ attached to the upper and lower extremities, A and C respectively, of the vertical diameter. The other ends of the two strings are attached to a small ring B which is free to slide on the wire.

(i) Show that, while both strings remain taut, the equation for the motion of the ring is

$$2ma\ddot{\theta} = \lambda(\cos\theta - \sin\theta) - mg\sin\theta,$$

where θ is the angle $\angle CAB$.

(ii) Initially the system is at rest in equilibrium with $\sin\theta = \frac{3}{5}$. Deduce that $5\lambda = 24mg$.

(iii) The ring is now displaced slightly. Show that, in the ensuing motion, it will oscillate with period approximately

$$10\pi\sqrt{\frac{a}{91g}}\,.$$

Section C: Probability and Statistics

12 It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type i, then her next theorem is of type j with probability p_{ij}, where p_{ij} is the entry in the ith row and jth column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Let a_i, $i = 1, 2, 3$, be the probability that a given theorem is of type i, and let b_j be the consequent probability that the next theorem is of type j.

(i) Explain why $b_j = a_1p_{1j} + a_2p_{2j} + a_3p_{3j}$.

(ii) Find values of a_1, a_2 and a_3 such that $b_i = a_i$ for $i = 1, 2, 3$.

(iii) For these values of the a_i find the probabilities q_{ij} that, if a particular theorem is of type j, then the *preceding* theorem was of type i.

13 **(i)** Let X be a random variable which takes only the finite number of different possible real values $x_1, x_2, \ldots, x_n$. Define the expectation $\mathrm{E}(X)$ and the variance $\mathrm{var}(X)$ of X. Show that , if a and b are real numbers, then $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and express $\mathrm{var}(aX + b)$ similarly in terms of $\mathrm{var}(X)$.

(ii) Let λ be a positive real number. By considering the contribution to $\mathrm{var}(X)$ of those x_i for which $|x_i - \mathrm{E}(X)| \geqslant \lambda$, or otherwise, show that

$$\mathrm{P}\left(|X - \mathrm{E}(X)| \geqslant \lambda\right) \leqslant \frac{\mathrm{var}(X)}{\lambda^2}.$$

(iii) Let k be a real number satisfying $k \geqslant \lambda$. If $|x_i - \mathrm{E}(X)| \leqslant k$ for all i, show that

$$\mathrm{P}\left(|X - \mathrm{E}(X)| \geqslant \lambda\right) \geqslant \frac{\mathrm{var}(X) - \lambda^2}{k^2 - \lambda^2}.$$

14 **(i)** Whenever I go cycling I start with my bike in good working order. However if all is well at time t, the probability that I get a puncture in the small interval $(t, t+\delta t)$ is $\alpha\,\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is T?

(ii) When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time t, the repair will be completed in time $(t, t+\delta t)$ is $\beta\,\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time t, write down an equation relating $p(t)$ to $p(t+\delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha+\beta}(1 - \mathrm{e}^{-(\alpha+\beta)t})$$

satisfies this differential equation with the appropriate initial condition.

(iii) Find an expression, involving α, β and T, for the time expected to be spent mending punctures during a journey of total time T. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \qquad \text{if } (\alpha+\beta)T \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha+\beta} \qquad \text{if } (\alpha+\beta)T \text{ is large.}$$

Section A: Pure Mathematics

1 Find the simultaneous solutions of the three linear equations

$$a^2x + ay + z = a^2$$
$$ax + y + bz = 1$$
$$a^2bx + y + bz = b$$

for all possible real values of a and b.

2 **(i)** If

$$\mathrm{I}_n = \int_0^a x^{n+\frac{1}{2}}(a-x)^{\frac{1}{2}}\,\mathrm{d}x,$$

show that $\mathrm{I}_0 = \pi a^2/8$.

(ii) Show that $(2n+4)\mathrm{I}_n = (2n+1)a\mathrm{I}_{n-1}$ and hence evaluate I_n.

3 **(i)** What is the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

for each of the cases: (i) $k > 1$; (ii) $k = 1$; (iii) $0 < k < 1$?

(ii) In case (iii) the equation represents damped simple harmonic motion with damping factor k. Let $x(0) = 0$ and let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of successive maxima and minima, so that if x_n is a maximum then x_{n+1} is the next minimum. Show that $|x_{n+1}/x_n|$ takes a value α which is independent of n, and that

$$k^2 = \frac{(\ln\alpha)^2}{\pi^2 + (\ln\alpha)^2}.$$

4 Let

$$\mathrm{C}_n(\theta) = \sum_{k=0}^{n} \cos k\theta$$

and let

$$\mathrm{S}_n(\theta) = \sum_{k=0}^{n} \sin k\theta,$$

where n is a positive integer and $0 < \theta < 2\pi$.

(i) Show that

$$\mathrm{C}_n(\theta) = \frac{\cos(\frac{1}{2}n\theta)\sin\left(\frac{1}{2}(n+1)\theta\right)}{\sin(\frac{1}{2}\theta)},$$

and obtain the corresponding expression for $\mathrm{S}_n(\theta)$.

(ii) Hence, or otherwise, show that for $0 < \theta < 2\pi$,

$$\left|\mathrm{C}_n(\theta) - \frac{1}{2}\right| \leqslant \frac{1}{2\sin(\frac{1}{2}\theta)}.$$

5 **(i)** Show that $y = \sin^2(m\sin^{-1}x)$ satisfies the differential equation

$$(1-x^2)y^{(2)} = xy^{(1)} + 2m^2(1-2y),$$

(ii) and deduce that, for all $n \geqslant 1$,

$$(1-x^2)y^{(n+2)} = (2n+1)xy^{(n+1)} + (n^2-4m^2)y^{(n)},$$

where $y^{(n)}$ denotes the nth derivative of y.

(iii) Derive the Maclaurin series for y, making it clear what the general term is.

6 The variable non-zero complex number z is such that

$$|z - \mathrm{i}| = 1.$$

(i) Find the modulus of z when its argument is θ. Find also the modulus and argument of $1/z$ in terms of θ and show in an Argand diagram the loci of points which represent z and $1/z$.

(ii) Find the locus C in the Argand diagram such that $w \in C$ if, and only if, the real part of $(1/w)$ is -1.

7 Consider the following sets with the usual definition of multiplication appropriate to each. In each case you may assume that the multiplication is associative. In each case state, giving adequate reasons, whether or not the set is a group.

(i) the complex numbers of unit modulus;

(ii) the integers modulo 4;

(iii) the matrices

$$\mathrm{M}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 \leqslant \theta < 2\pi$;

(iv) the integers $1, 3, 5, 7$ modulo 8;

(v) the 2×2 matrices all of whose entries are integers;

(vi) the integers $1, 2, 3, 4$ modulo 5.

In the case of each pair of groups above state, with reasons, whether or not they are isomorphic.

8 **(i)** A plane π in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

(ii) The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r} \cdot \mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

(iii) Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

Section B: Mechanics

9 A thin circular disc of mass m, radius r and with its centre of mass at its centre C can rotate freely in a vertical plane about a fixed horizontal axis through a point O of its circumference. A particle P, also of mass m, is attached to the circumference of the disc so that the angle OCP is 2α, where $\alpha \leqslant \pi/2$.

(i) In the position of stable equilibrium OC makes an angle β with the vertical. Prove that

$$\tan\beta = \frac{\sin 2\alpha}{2 - \cos 2\alpha}.$$

(ii) The density of the disc at a point distant x from C is $\rho x/r$. Show that its moment of inertia about the horizontal axis through O is $8mr^2/5$.

(iii) The mid-point of CP is Q. The disc is held at rest with OQ horizontal and C lower than P and it is then released. Show that the speed v with which C is moving when P passes vertically below O is given by

$$v^2 = \frac{15gr\sin\alpha}{2(2+5\sin^2\alpha)}.$$

Find the maximum value of v^2 as α is varied.

10 A cannon is situated at the bottom of a plane inclined at angle β to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed u. Ignoring air resistance, find the angle of firing which will maximise the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1+\sin\beta)}$$

from the cannon.

11 A ship is sailing due west at V knots while a plane, with an airspeed of kV knots, where $k > \sqrt{2}$, patrols so that it is always to the north west of the ship. If the wind in the area is blowing from north to south at V knots and the pilot is instructed to return to the ship every thirty minutes, how long will her outward flight last?

Assume that the maximum distance of the plane from the ship during the above patrol was d_w miles. If the air now becomes dead calm, and the pilot's orders are maintained, show that the ratio d_w/d_c of d_w to the new maximum distance, d_c miles, of the plane from the ship is

$$\frac{k^2-2}{2k(k^2-1)}\sqrt{4k^2-2}.$$

Section C: Probability and Statistics

12 **(i)** The random variables X and Y are independently normally distributed with means 0 and variances 1. Show that the joint probability density function for (X, Y) is

$$\mathrm{f}(x, y) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(x^2+y^2)} \qquad -\infty < x < \infty, -\infty < y < \infty.$$

(ii) If (x, y) are the coordinates, referred to rectangular axes, of a point in the plane, explain what is meant by saying that this density is radially symmetrical.

(iii) The random variables U and V have a joint probability density function which is radially symmetrical (in the above sense). By considering the straight line with equation $U = kV$, or otherwise, show that

$$\mathrm{P}\left(\frac{U}{V} < k\right) = 2\mathrm{P}(U < kV, V > 0).$$

(iv) Hence, or otherwise, show that the probability density function of U/V is

$$\mathrm{g}(k) = \frac{1}{\pi(1+k^2)} \qquad -\infty < k < \infty.$$

13 A message of 10^k binary digits is sent along a fibre optic cable with high probabilities p_0 and p_1 that the digits 0 and 1, respectively, are received correctly. If the probability of a digit in the original message being a 1 is α, find the probability that the entire message is received correctly.

(i) Find the probability β that a randomly chosen digit in the message is received as a 1 and show that $\beta = \alpha$ if, and only if

$$\alpha = \frac{q_0}{q_1 + q_0},$$

where $q_0 = 1 - p_0$ and $q_1 = 1 - p_1$. If this condition is satisfied and the received message consists entirely of zeros, what is the probability that it is correct?

(ii) If now $q_0 = q_1 = q$ and $\alpha = \frac{1}{2}$, find the approximate value of q which will ensure that a message of one million binary digits has a fifty-fifty chance of being received entirely correctly.

(iii) The probability of error q is proportional to the square of the length of the cable. Initially the length is such that the probability of a message of one million binary bits, among which 0 and 1 are equally likely, being received correctly is $\frac{1}{2}$. What wold this probability become if a booster station were installed at its mid-point, assuming that the booster station re-transmits the received version of the message, and assuming that terms of order q^2 may be ignored?

14 A candidate finishes examination questions in time T, where T has probability density function

$$\mathrm{f}(t) = t\mathrm{e}^{-t} \qquad t \geqslant 0,$$

the probabilities for the various questions being independent.

(i) Find the moment generating function of T and hence find the moment generating function for the total time U taken to finish two such questions.

(ii) Show that the probability density function for U is

$$\mathrm{g}(u) = \frac{1}{6}u^3\mathrm{e}^{-u} \qquad u \geqslant 0.$$

(iii) Find the probability density function for the total time taken to answer n such questions.

Section A: Pure Mathematics

1 **(i)** Calculate

$$\int_0^x \operatorname{sech} t \, \mathrm{d}t.$$

(ii) Find the reduction formula involving I_n and I_{n-2}, where

$$I_n = \int_0^x \operatorname{sech}^n t \, \mathrm{d}t$$

(iii) and, hence or otherwise, find I_5 and I_6.

2 **(i)** By setting $y = x + x^{-1}$, find the solutions of

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0.$$

(ii) Solve

$$x^4 + x^3 - 10x^2 - 4x + 16 = 0.$$

3 **(i)** Describe geometrically the possible intersections of a plane with a sphere.

(ii) Let P_1 and P_2 be the planes with equations

$$3x - y - 1 = 0,$$
$$x - y + 1 = 0,$$

respectively, and let S_1 and S_2 be the spheres with equations

$$x^2 + y^2 + z^2 = 7,$$
$$x^2 + y^2 + z^2 - 6y - 4z + 10 = 0,$$

respectively. Let C_1 be the intersection of P_1 and S_1, let C_2 be the intersection of P_2 and S_2 and let L be the intersection of P_1 and P_2. Find the points where L meets each of S_1 and S_2. Determine, giving your reasons, whether the circles C_1 and C_2 are linked.

4 **(i)** Find the two solutions of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y$$

which pass through the point (a, b^2), where $b \neq 0$.

(ii) Find two distinct points $(a_1, 1)$ and $(a_2, 1)$ such that one of the solutions through each of them also passes through the origin.

(iii) Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1, 1)$ and $(a_2, 1)$.

(iv) Now sketch sufficient members of the family of solutions (for varying a and b) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

5 **(i)** The function f is given by $\mathrm{f}(x) = \sin^{-1} x$ for $-1 < x < 1$. Prove that

$$(1 - x^2)\mathrm{f}''(x) - x\mathrm{f}'(x) = 0.$$

(ii) Prove also that

$$(1 - x^2)\mathrm{f}^{(n+2)}(x) - (2n + 1)x\mathrm{f}^{(n+1)}(x) - n^2\mathrm{f}^{(n)}(x) = 0,$$

for all $n > 0$, where $\mathrm{f}^{(n)}$ denotes the nth derivative of f. Hence express $\mathrm{f}(x)$ as a Maclaurin series.

(iii) The function g is given by

$$\mathrm{g}(x) = \ln\sqrt{\frac{1 + x}{1 - x}},$$

for $-1 < x < 1$. Write down a power series expression for $\mathrm{g}(x)$, and show that the coefficient of x^{2n+1} is greater than that in the expansion of f, for each $n > 0$.

6 The four points A, B, C, D in the Argand diagram (complex plane) correspond to the complex numbers a, b, c, d respectively. The point P_1 is mapped to P_2 by rotating about A through $\pi/2$ radians. Then P_2 is mapped to P_3 by rotating about B through $\pi/2$ radians, P_3 is mapped to P_4 by rotating about C through $\pi/2$ radians and P_4 is mapped to P_5 by rotating about D through $\pi/2$ radians, each rotation being in the positive sense. If z_i is the complex number corresponding to P_i, find z_5 in terms of a, b, c, d and z_1.

(i) Show that P_5 will coincide with P_1, irrespective of the choice of the latter if, and only if

$$a - c = \mathrm{i}(b - d)$$

and interpret this condition geometrically.

(ii) The points A, B and C are now chosen to be distinct points on the unit circle and the angle of rotation is changed to θ, where $\theta \neq 0$, on each occasion. Find the necessary and sufficient condition on θ and the points A, B and C for P_4 always to coincide with P_1.

7 **(i)** Let S_3 be the group of permutations of three objects and Z_6 be the group of integers under addition modulo 6. List all the elements of each group, stating the order of each element. State, with reasons, whether S_3 is isomorphic with Z_6.

(ii) Let C_6 be the group of 6th roots of unity. That is, $C_6 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$ where $\alpha = \mathrm{e}^{\mathrm{i}\pi/3}$ and the group operation is complex multiplication. Prove that C_6 is isomorphic with Z_6. Is there any (multiplicative or additive) subgroup of the complex numbers which is isomorphic with S_3? Give a reason for your answer.

8 Let a, b, c, d, p, q, r and s be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where z_1, z_2, z_3 and z_4 are suitably chosen complex numbers, find expressions L_1, L_2, L_3 and L_4, each of which is linear in a, b, c and d and also linear in p, q, r and s, such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

Section B: Mechanics

9 A smooth, axially symmetric bowl has its vertical cross-sections determined by $s = 2\sqrt{ky}$, where s is the arc-length measured from its lowest point V, and y is the height above V. A particle is released from rest at a point on the surface at a height h above V.

(i) Explain why

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + 2gy$$

is constant.

(ii) Show that the time for the particle to reach V is

$$\pi\sqrt{\frac{k}{2g}}.$$

(iii) Two elastic particles of mass m and αm, where $\alpha < 1$, are released simultaneously from opposite sides of the bowl at heights $\alpha^2 h$ and h respectively. If the coefficient of restitution between the particles is α, describe the subsequent motion.

10 The island of Gammaland is totally flat and subject to a constant wind of w kh^{-1}, blowing from the West. Its southernmost shore stretches almost indefinitely, due east and west, from the coastal city of Alphabet. A novice pilot is making her first solo flight from Alphaport to the town of Betaville which lies north-east of Alphaport. Her instructor has given her the correct heading to reach Betaville, flying at the plane's recommended airspeed of v kh^{-1}, where $v > w$.

On reaching Betaport the pilot returns with the opposite heading to that of the outward flight and, so featureless is Gammaland, that she only realises her error as she crosses the coast with Alphaport nowhere in sight. Assuming that she then turns West along the coast, and that her outward flight took t hours, show that her return flight takes

$$\left(\frac{v+w}{v-w}\right) t \text{ hours.}$$

If Betaville is d kilometres from Alphaport, show that, with the correct heading, the return flight should have taken

$$t + \frac{\sqrt{2}wd}{v^2 - w^2} \text{ hours.}$$

11 A step-ladder has two sections AB and AC, each of length $4a$, smoothly hinged at A and connected by a light elastic rope DE, of natural length $a/4$ and modulus W, where D is on AB, E is on AC and $AD = AE = a$. The section AB, which contains the steps, is uniform and of weight W and the weight of AC is negligible.

The step-ladder rests on a smooth horizontal floor and a man of weight $4W$ carefully ascends it to stand on a rung distant βa from the end of the ladder resting on the floor. Find the height above the floor of the rung on which the man is standing when β is the maximum value at which equilibrium is possible.

Section C: Probability and Statistics

12 In certain forms of Tennis two players A and B serve alternate games. Player A has probability p_A of winning a game in which she serves and p_B of winning a game in which player B serves. Player B has probability $q_B = 1 - p_B$ of winning a game in which she serves and probability $q_A = 1 - p_A$ of winning a game in which player A serves.

(i) In Shortened Tennis the first player to lead by 2 games wins the match. Find the probability P_{short} that A wins a Shortened Tennis match in which she serves first and show that it is the same as if B serves first.

(ii) In Standard Tennis the first player to lead by 2 or more games after 4 or more games have been played wins the match. Show that the probability that the match is decided in 4 games is

$$p_A^2 p_B^2 + q_A^2 q_B^2 + 2(p_A p_B + q_A q_B)(p_A q_B + q_A p_B).$$

(iii) If $p_A = p_B = p$ and $q_A = q_B = q$, find the probability P_{stan} that A wins a Standard Tennis match in which she serves first. Show that

$$P_{\text{stan}} - P_{\text{short}} = \frac{p^2 q^2 (p - q)}{p^2 + q^2}.$$

13 During his performance a trapeze artist is supported by two identical ropes, either of which can bear his weight. Each rope is such that the time, in hours of performance, before it fails is exponentially distributed, independently of the other, with probability density function $\lambda \exp(-\lambda t)$ for $t \geqslant 0$ (and 0 for $t < 0$), for some $\lambda > 0$. A particular rope has already been in use for t_0 hours of performance.

(i) Find the distribution for the length of time the artist can continue to use it before it fails. Interpret and comment upon your result.

(ii) Before going on tour the artist insists that the management purchase two new ropes of the above type. Show that the probability density function of the time until both ropes fail is

$$\mathrm{f}(t) = \begin{cases} 2\lambda \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t}) & \text{if } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If each performance lasts for h hours, find the probability that both ropes fail during the nth performance. Show that the probability that both ropes fail during the same performance is $\tanh(\lambda h/2)$.

14 Three points, P, Q and R, are independently randomly chosen on the perimeter of a circle.

(i) Prove that the probability that at least one of the angles of the triangle PQR will exceed $k\pi$ is $3(1-k)^2$ if $\frac{1}{2} \leqslant k \leqslant 1$.

(ii) Find the probability if $\frac{1}{3} \leqslant k \leqslant \frac{1}{2}$.

Section A: Pure Mathematics

1 The curve P has the parametric equations

$$x = \sin\theta, \quad y = \cos 2\theta \qquad \text{for } -\pi/2 \leqslant \theta \leqslant \pi/2.$$

(i) Show that P is part of the parabola $y = 1 - 2x^2$ and sketch P.

(ii) Show that the length of P is $\sqrt{17} + \frac{1}{4}\sinh^{-1} 4$.

(iii) Obtain the volume of the solid enclosed when P is rotated through 2π radians about the line $y = -1$.

2 The curve C has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on C. You may assume that the origin is not a point of inflection.

(ii) The part of C which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

3 The matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{M}$ are given by

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 13 & 5 \\ 3 & 8 & 7 \end{pmatrix},$$

where $a, b, \ldots, r$ are real numbers.

(i) Given that $\mathbf{M} = \mathbf{AB}$, show that $a = 1, b = 4, c = 1, d = 3, e = 1, f = -2, p = 3, q = 2$ and $r = -3$ gives the *unique* solution for $\mathbf{A}$ and $\mathbf{B}$.

(ii) Evaluate $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$,

(iii) Hence, or otherwise, solve the simultaneous equations

$$\begin{aligned} x + 3y + 2z &= 7 \\ 4x + 13y + 5z &= 18 \\ 3x + 8y + 7z &= 25. \end{aligned}$$

4 Sum the following infinite series.

(i) $1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{5}\left(\frac{1}{2}\right)^4 + \cdots + \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n} + \cdots$.

(ii) $2 - x - x^3 + 2x^4 - \cdots + 2x^{4k} - x^{4k+1} - x^{4k+3} + \cdots$ where $|x| < 1$.

(iii) $\displaystyle\sum_{r=2}^{\infty} \frac{r\,2^{r-2}}{3^{r-1}}$.

(iv) $\displaystyle\sum_{r=2}^{\infty} \frac{2}{r(r^2-1)}$.

5 The set S consists of ordered pairs of complex numbers (z_1, z_2) and a binary operation $\circ$ on S is defined by

$$(z_1, z_2) \circ (w_1, w_2) = (z_1 w_1 - z_2 w_2^*,\ z_1 w_2 + z_2 w_1^*).$$

(i) Show that the operation $\circ$ is associative and determine whether it is commutative.

(ii) Evaluate $(z, 0) \circ (w, 0)$, $(z, 0) \circ (0, w)$, $(0, z) \circ (w, 0)$ and $(0, z) \circ (0, w)$.

(iii) The set S_1 is the subset of S consisting of A, B, $\ldots$, H, where $A = (1, 0)$, $B = (0, 1)$, $C = (i, 0)$, $D = (0, i)$, $E = (-1, 0)$, $F = (0, -1)$, $G = (-i, 0)$ and $H = (0, -i)$. Show that S_1 is closed under $\circ$ and that it has an identity element.

(iv) Determine the inverse and order of each element of S_1.

(v) Show that S_1 is a group under $\circ$.
[You are not required to compute the multiplication table in full.]

(vi) Show that $\{A, B, E, F\}$ is a subgroup of S_1 and determine whether it is isomorphic to the group generated by the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under matrix multiplication.

6 The point in the Argand diagram representing the complex number z lies on the circle with centre K and radius r, where K represents the complex number k.

(i) Show that

$$zz^* - kz^* - k^*z + kk^* - r^2 = 0.$$

(ii) The points P, Q_1 and Q_2 represent the complex numbers z, w_1 and w_2 respectively. The point P lies on the circle with OA as diameter, where O and A represent 0 and $2i$ respectively.

Given that $w_1 = z/(z-1)$, find the equation of the locus L of Q_1 in terms of w_1 and describe the geometrical form of L.

(iii) Given that $w_2 = z^*$, show that the locus of Q_2 is also L. Determine the positions of P for which Q_1 coincides with Q_2.

7 The real numbers x and y satisfy the simultaneous equations

$$\sinh(2x) = \cosh y \qquad \text{and} \qquad \sinh(2y) = 2\cosh x.$$

(i) Show that $\sinh^2 y$ is a root of the equation

$$4t^3 + 4t^2 - 4t - 1 = 0$$

and demonstrate that this gives at most one valid solution for y.

(ii) Show that the relevant value of t lies between 0.7 and 0.8, and use an iterative process to find t to 6 decimal places.

(iii) Find y and hence find x, checking your answers and stating the final answers to four decimal places.

8 A square pyramid has its base vertices at the points A $(a, 0, 0)$, B $(0, a, 0)$, C $(-a, 0, 0)$ and D $(0, -a, 0)$, and its vertex at E $(0, 0, a)$. The point P lies on AE with x-coordinate λa, where $0 < \lambda < 1$, and the point Q lies on CE with x-coordinate $-\mu a$, where $0 < \mu < 1$. The plane BPQ cuts DE at R and the y-coordinate of R is $-\gamma a$.

(i) Prove that

$$\gamma = \frac{\lambda\mu}{\lambda + \mu - \lambda\mu}.$$

(ii) Show that the quadrilateral $BPRQ$ cannot be a parallelogram.

9 For the real numbers a_1, a_2, a_3, $\ldots$,

(i) prove that $a_1^2 + a_2^2 \geqslant 2a_1a_2$,

(ii) prove that $a_1^2 + a_2^2 + a_3^2 \geqslant a_2a_3 + a_3a_1 + a_1a_2$,

(iii) prove that $3(a_1^2 + a_2^2 + a_3^2 + a_4^2) \geqslant 2(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)$,

(iv) state and prove a generalisation of (iii) to the case of n real numbers,

(v) prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant \frac{2n}{n-1}\sum_{i,j} a_ia_j,$$

where the latter sum is taken over all pairs (i, j) with $1 \leqslant i < j \leqslant n$.

10 The transformation T of the point P in the x,y plane to the point P' is constructed as follows: Lines are drawn through P parallel to the lines $y = mx$ and $y = -mx$ to cut the line $y = kx$ at Q and R respectively, m and k being given constants. P' is the fourth vertex of the parallelogram $PQP'R$.

(i) Show that if P is (x_1, y_1) then Q is

$$\left(\frac{mx_1 - y_1}{m-k}, \frac{k(mx_1 - y_1)}{m-k}\right).$$

(ii) Obtain the coordinates of P' in terms of x_1, y_1, m and k, and express T as a matrix transformation.

(iii) Show that areas are transformed under T into areas of the same magnitude.

Section B: Mechanics

11 *In this question, all gravitational forces are to be neglected.*

A rigid frame is constructed from 12 equal uniform rods, each of length a and mass m, forming the edges of a cube. Three of the edges are OA, OB and OC, and the vertices opposite O, A, B and C are O', A', B' and C' respectively. Forces act along the lines as follows, in the directions indicated by the order of the letters:

$$\begin{array}{rrr} 2mg \text{ along } OA, & mg \text{ along } AC', & \sqrt{2}mg \text{ along } O'A, \\ \sqrt{2}mg \text{ along } OA', & 2mg \text{ along } C'B, & mg \text{ along } A'C. \end{array}$$

(i) The frame is freely pivoted at O. Show that the direction of the line about which it will start to rotate is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ with respect to axes along OA, OB and OC respectively.

(ii) Show that the moment of inertia of the rod OA about the axis OO' is $2ma^2/9$ and about a parallel axis through its mid-point is $ma^2/18$. Hence find the moment of inertia of $B'C$ about OO' and show that the moment of inertia of the frame about OO' is $14ma^2/3$. If the frame is freely pivoted about the line OO' and the forces continue to act along the specified lines, find the initial angular acceleration of the frame.

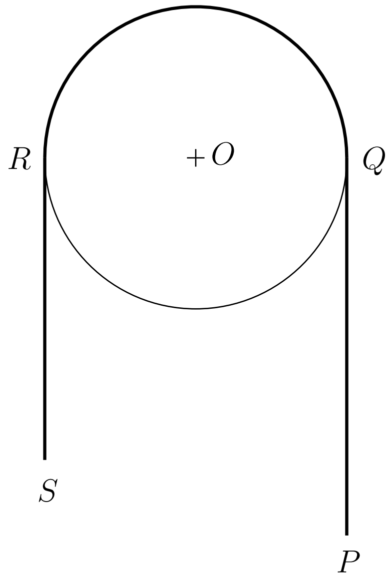
12 $ABCD$ is a horizontal line with $AB = CD = a$ and $BC = 6a$. There are fixed smooth pegs at B and C. A uniform string of natural length $2a$ and modulus of elasticity kmg is stretched from A to D, passing over the pegs at B and C. A particle of mass m is attached to the midpoint P of the string.

(i) When the system is in equilibrium, P is a distance $a/4$ below BC. Evaluate k.

(ii) The particle is pulled down to a point Q, which is at a distance pa below the mid-point of BC, and is released from rest. P rises to a point R, which is at a distance $3a$ above BC. Show that $2p^2 - p - 17 = 0$.

(iii) Show also that the tension in the strings is less when the particle is at R than when the particle is at Q.

13



A uniform circular disc with radius a, mass $4m$ and centre O is freely mounted on a fixed horizontal axis which is perpendicular to its plane and passes through O. A uniform heavy chain PS of length $(4+\pi)a$, mass $(4+\pi)m$ and negligible thickness is hung over the rim of the disc as shown in the diagram: Q and R are the points of the chain at the same level as O. The contact between the chain and the rim of the disc is sufficiently rough to prevent slipping. Initially, the system is at rest with $PQ = RS = 2a$. A particle of mass m is attached to the chain at P and the system is released.

(i) By considering the energy of the system, show that when P has descended a distance x, its speed v is given by

$$(\pi+7)av^2 = 2g(x^2+ax).$$

(ii) By considering the part PQ of the chain as a body of variable mass, show that when S reaches R the tension in the chain at Q is

$$\frac{5\pi-2}{\pi+7}mg.$$

14 A particle rests at a point A on a horizontal table and is joined to a point O on the table by a taut inextensible string of length c. The particle is projected vertically upwards at a speed $64\sqrt{6gc}$. It next strikes the table at a point B and rebounds. The coefficient of restitution for any impact between the particle and the table is $\frac{1}{2}$. After rebounding at B, the particle will rebound alternately at A and B until the string becomes slack.

(i) Show that when the string becomes slack the particle is at height $c/2$ above the table.

(ii) Determine whether the first rebound *between* A and B is nearer to A or to B.

Section C: Probability and Statistics

15 The probability of throwing a head with a certain coin is p and the probability of throwing a tail is $q = 1 - p$. The coin is thrown until at least two heads and at least two tails have been thrown; this happens when the coin has been thrown N times.

(i) Write down an expression for the probability that $N = n$.

(ii) Show that the expectation of N is

$$2\left(\frac{1}{pq} - 1 - pq\right).$$

16 The time taken for me to set an acceptable examination question it T hours. The distribution of T is a truncated normal distribution with probability density f where

$$\mathrm{f}(t) = \begin{cases} \dfrac{1}{k\sigma\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\left(\dfrac{t-\sigma}{\sigma}\right)^2\right) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Sketch the graph of $\mathrm{f}(t)$.

Show that k is approximately 0.841 and obtain the mean of T as a multiple of σ.

Over a period of years, I find that the mean setting time is 3 hours.

(i) Find the approximate probability that none of the 16 questions on next year's paper will take more than 4 hours to set.

(ii) Given that a particular question is unsatisfactory after 2 hours work, find the probability that it will still be unacceptable after a further 2 hours work.

Section A: Pure Mathematics

1 **(i)** Given that

$$f(x) = \ln(1 + e^x),$$

prove that $\ln[f'(x)] = x - f(x)$ and that $f''(x) = f'(x) - [f'(x)]^2$. Hence, or otherwise, expand $f(x)$ as a series in powers of x up to the term in x^4.

(ii) Given that

$$g(x) = \frac{1}{\sinh x \cosh 2x},$$

explain why $g(x)$ can not be expanded as a series of non-negative powers of x but that $xg(x)$ can be so expanded. Explain also why this latter expansion will consist of even powers of x only. Expand $xg(x)$ as a series as far as the term in x^4.

2 The matrices $\mathbf{I}$ and $\mathbf{J}$ are

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

respectively and $\mathbf{A} = \mathbf{I} + a\mathbf{J}$, where a is a non-zero real constant.

(i) Prove that

$$\mathbf{A}^2 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^2 - 1]\mathbf{J} \quad \text{and} \quad \mathbf{A}^3 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^3 - 1]\mathbf{J}$$

and obtain a similar form for $\mathbf{A}^4$.

(ii) If $\mathbf{A}^k = \mathbf{I} + p_k\mathbf{J}$, suggest a suitable form for p_k and prove that it is correct by induction, or otherwise.

3 **(i)** Sketch the curve C_1 whose parametric equations are $x = t^2$, $y = t^3$.

(ii) The circle C_2 passes through the origin O. The points R and S with real non-zero parameters r and s respectively are other intersections of C_1 and C_2. Show that r and s are roots of an equation of the form

$$t^4 + t^2 + at + b = 0,$$

where a and b are real constants.

(iii) By obtaining a quadratic equation, with coefficients expressed in terms of r and s, whose roots would be the parameters of any further intersections of C_1 and C_2, or otherwise, show that O, R and S are the only real intersections of C_1 and C_2.

4 **(i)** A set of curves S_1 is defined by the equation

$$y = \frac{x}{x-a},$$

where a is a constant which is different for different members of S_1. Sketch on the same axes the curves for which $a = -2, -1, 1$ and 2.

(ii) A second of curves S_2 is such that at each intersection between a member of S_2 and a member of S_1 the tangents of the intersecting curves are perpendicular. On the same axes as the already sketched members of S_1, sketch the member of S_2 that passes through the point $(1, -1)$.

(iii) Obtain the first order differential equation for y satisfied at all points on all members of S_1 (i.e. an equation connecting x, y and $\mathrm{d}y/\mathrm{d}x$ which does not involve a).

(iv) State the relationship between the values of $\mathrm{d}y/\mathrm{d}x$ on two intersecting curves, one from S_1 and one from S_2, at their intersection. Hence show that the differential equation for the curves of S_2 is

$$x = y(y-1)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(v) Find an equation for the member of S_2 that you have sketched.

5 The tetrahedron $ABCD$ has A at the point $(0, 4, -2)$. It is symmetrical about the plane $y+z=2$, which passes through A and D. The mid-point of BC is N. The centre, Y, of the sphere $ABCD$ is at the point $(3, -2, 4)$ and lies on AN such that $\overrightarrow{AY} = 3\overrightarrow{YN}$.

(i) Show that $BN = 6\sqrt{2}$ and find the coordinates of B and C.

(ii) The angle AYD is $\cos^{-1}\frac{1}{3}$. Find the coordinates of D. [There are two alternative answers for each point.]

6 **(i)** Given that $I_n = \displaystyle\int_0^\pi \frac{x\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, where n is a positive integer, show that $I_n - I_{n-1} = J_n$, where

$$J_n = \int_0^\pi \frac{x\sin(2n-1)x}{\sin x}\,\mathrm{d}x.$$

(ii) Obtain also a reduction formula for J_n.

(iii) The curve C is given by the cartesian equation

$$y = \frac{x\sin^2(nx)}{\sin^2 x},$$

where n is a positive integer and $0 \leqslant x \leqslant \pi$. Show that the area under the curve C is $\frac{1}{2}n\pi^2$.

7 The points P and R lie on the sides AB and AD, respectively, of the parallelogram $ABCD$. The point Q is the fourth vertex of the parallelogram $APQR$. Prove that BR, CQ and DP meet in a point.

8 Show that

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{2r} \cot^{2r}\theta,$$

where n is a positive integer. Deduce that the equation

$$\sum_{r=0}^{n} (-1)^{r} \binom{2n+1}{2r} x^{r} = 0$$

has roots $\cot^2(k\pi/(2n+1))$ for $k = 1, 2, \ldots, n$.

Show that

(i) $\displaystyle\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3}$,

(ii) $\displaystyle\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) = n(2n+1)$,

(iii) $\displaystyle\sum_{k=1}^{n} \operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n(n+1)}{3}$.

9 The straight line OSA, where O is the origin, bisects the angle between the positive x and y axes. The ellipse E has S as focus. In polar coordinates with S as pole and SA as the initial line, E has equation $\ell = r(1 + e\cos\theta)$.

(i) Show that, at the point on E given by $\theta = \alpha$, the gradient of the tangent to the ellipse is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\alpha - \cos\alpha - e}{\sin\alpha + \cos\alpha + e}.$$

(ii) The points on E given by $\theta = \alpha$ and $\theta = \beta$ are the ends of a diameter of E. Show that

$$\tan(\alpha/2)\tan(\beta/2) = -\frac{1+e}{1-e}.$$

[**Hint.** A diameter of an ellipse is a chord through its centre.]

10 **(i)** Sketch the curve C whose polar equation is

$$r = 4a\cos 2\theta \qquad \text{for } -\frac{1}{4}\pi < \theta < \frac{1}{4}\pi.$$

(ii) The ellipse E has parametric equations

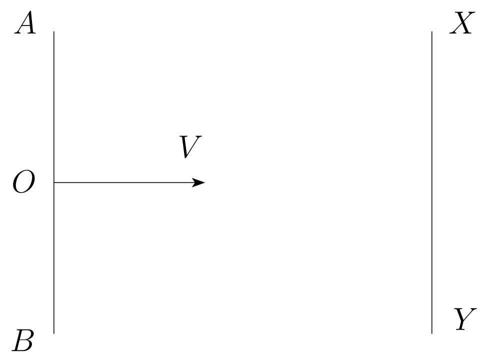
$$x = 2a\cos\phi, \qquad y = a\sin\phi.$$

Show, without evaluating the integrals, that the perimeters of C and E are equal.

(iii) Show also that the areas of the regions enclosed by C and E are equal.

Section B: Mechanics

11



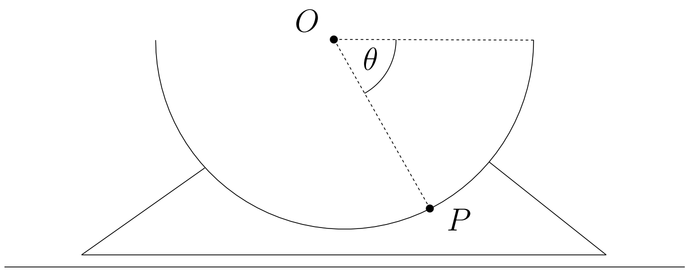
AOB represents a smooth vertical wall and XY represents a parallel smooth vertical barrier, both standing on a smooth horizontal table. A particle P is projected along the table from O with speed V in a direction perpendicular to the wall. At the time of projection, the distance between the wall and the barrier is $(75/32)VT$, where T is a constant. The barrier moves directly towards the wall, remaining parallel to the wall, with initial speed $4V$ and with constant acceleration $4V/T$ directly away from the wall.

(i) The particle strikes the barrier XY and rebounds. Show that this impact takes place at time $5T/8$.

(ii) The barrier is sufficiently massive for its motion to be unaffected by the impact. Given that the coefficient of restitution is $1/2$, find the speed of P immediately after impact.

(iii) P strikes AB and rebounds. Given that the coefficient of restitution for this collision is also $1/2$, show that the next collision of P with the barrier is at time $9T/8$ from the start of the motion.

12



A smooth hemispherical bowl of mass $2m$ is rigidly mounted on a light carriage which slides freely on a horizontal table as shown in the diagram. The rim of the bowl is horizontal and has centre O. A particle P of mass m is free to slide on the inner surface of the bowl. Initially, P is in contact with the rim of the bowl and the system is at rest.

(i) The system is released and when OP makes an angle θ with the horizontal the velocity of the bowl is v? Show that

$$3v = a\dot{\theta}\sin\theta$$

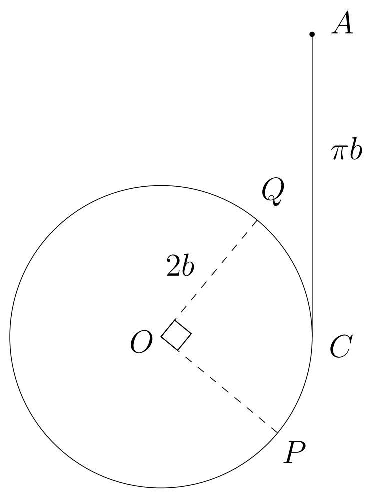
(ii) and that

$$v^2 = \frac{2ga\sin^3\theta}{3(3-\sin^2\theta)},$$

where a is the interior radius of the bowl.

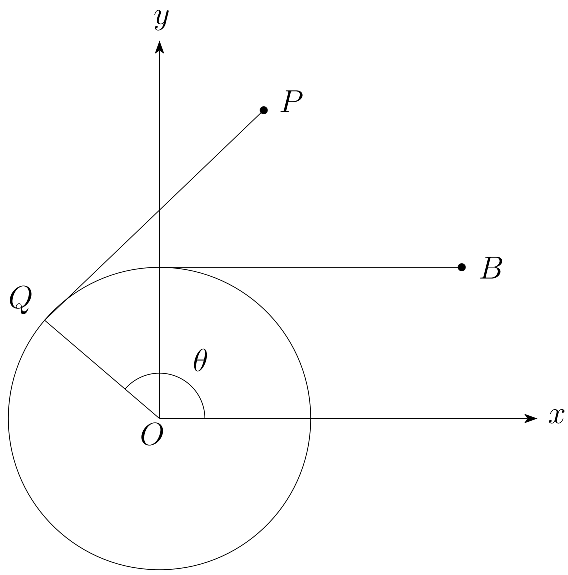
(iii) Find, in terms of m, g and θ, the reaction between the bowl and the particle.

13



A uniform circular disc of radius $2b$, mass m and centre O is free to turn about a fixed horizontal axis through O perpendicular to the plane of the disc. A light elastic string of modulus kmg, where $k > 4/\pi$, has one end attached to a fixed point A and the other end to the rim of the disc at P. The string is in contact with the rim of the disc along the arc PC, and OC is horizontal. The natural length of the string and the length of the line AC are each πb and AC is vertical. A particle Q of mass m is attached to the rim of the disc and $\angle POQ = 90°$ as shown in the diagram. The system is released from rest with OP vertical and P below O. Show that P reaches C and that then the upward vertical component of the reaction on the axis is $mg(10 - \pi k)/3$.

14



A horizontal circular disc of radius a and centre O lies on a horizontal table and is fixed to it so that it cannot rotate. A light inextensible string of negligible thickness is wrapped round the disc and attached at its free end to a particle P of mass m. When the string is all in contact with the disc, P is at A. The string is unwound so that the part not in contact with the disc is taut and parallel to OA. P is then at B. The particle is projected along the table from B with speed V perpendicular to and away from OA. In the general position, the string is tangential to the disc at Q and $\angle AOQ = \theta$.

(i) Show that, in the general position, the x-coordinate of P with respect to the axes shown in the figure is $a\cos\theta + a\theta\sin\theta$, and find y-coordinate of P.

(ii) Hence, or otherwise, show that the acceleration of P has components $a\theta\dot{\theta}^2$ and $a\dot{\theta}^2 + a\theta\ddot{\theta}$ along and perpendicular to PQ, respectively.

(iii) The friction force between P and the table is $2\lambda mv^2/a$, where v is the speed of P and λ is a constant. Show that

$$\frac{\ddot{\theta}}{\dot{\theta}} = -\left(\frac{1}{\theta} + 2\lambda\theta\right)\dot{\theta}$$

and find $\dot{\theta}$ in terms of θ, λ and a.

(iv) Find also the tension in the string when $\theta = \pi$.

Section C: Probability and Statistics

15 A goat G lies in a square field $OABC$ of side a. It wanders randomly round its field, so that at any time the probability of its being in any given region is proportional to the area of this region.

(i) Write down the probability that its distance, R, from O is less than r if $0 < r \leqslant a$,

(ii) and show that if $r \geqslant a$ the probability is

$$\left(\frac{r^2}{a^2} - 1\right)^{\frac{1}{2}} + \frac{\pi r^2}{4a^2} - \frac{r^2}{a^2}\cos^{-1}\left(\frac{a}{r}\right).$$

(iii) Find the median of R and probability density function of R.

(iv) The goat is then tethered to the corner O by a chain of length a. Find the conditional probability that its distance from the fence OC is more than $a/2$.

16 The probability that there are exactly n misprints in an issue of a newspaper is $\mathrm{e}^{-\lambda}\lambda^n/n!$ where λ is a positive constant. The probability that I spot a particular misprint is p, independent of what happens for other misprints, and $0 < p < 1$.

(i) If there are exactly $m + n$ misprints, what is the probability that I spot exactly m of them?

(ii) Show that, if I spot exactly m misprints, the probability that I have failed to spot exactly n misprints is

$$\frac{(1-p)^n\lambda^n}{n!}\mathrm{e}^{-(1-p)\lambda}.$$

Section A: Pure Mathematics

1 **(i)** Evaluate

$$\sum_{r=1}^{n} \frac{6}{r(r+1)(r+3)}.$$

(ii) Expand $\ln(1+x+x^2+x^3)$ as a series in powers of x, where $|x|<1$, giving the first five non-zero terms and the general term.

(iii) Expand $\mathrm{e}^{x\ln(1+x)}$ as a series in powers of x, where $-1<x\leqslant 1$, as far as the term in x^4.

2 **(i)** The distinct points P_1, P_2, P_3, Q_1, Q_2 and Q_3 in the Argand diagram are represented by the complex numbers z_1, z_2, z_3, w_1, w_2 and w_3 respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with P_i corresponding to Q_i $(i=1,2,3)$ and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1-z_2}{z_2-z_3} = \frac{w_1-w_2}{w_1-w_3}.$$

(ii) Verify that this condition may be written

$$\det\begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(iii) Show that if $w_i = z_i^2$ $(i=1,2,3)$ then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(iv) Show that if $w_i = z_i^3$ $(i=1,2,3)$ then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin. [The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1+z_2+z_3)$.]

(v) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

3 The function f is defined for $x < 2$ by

$$\mathrm{f}(x) = 2\left|x^2 - x\right| + \left|x^2 - 1\right| - 2\left|x^2 + x\right|.$$

(i) Find the maximum and minimum points and the points of inflection of the graph of f and sketch this graph. Is f continuous everywhere? Is f differentiable everywhere?

(ii) Find the inverse of the function f, i.e. expressions for $\mathrm{f}^{-1}(x)$, defined in the various appropriate intervals.

4 The point P moves on a straight line in three-dimensional space. The position of P is observed from the points $O_1(0,0,0)$ and $O_2(8a,0,0)$. At times $t = t_1$ and $t = t_1'$, the lines of sight from O_1 are along the lines

$$\frac{x}{2} = \frac{z}{3}, y = 0 \quad \text{and} \quad x = 0, \frac{y}{3} = \frac{z}{4}$$

respectively. At times $t = t_2$ and $t = t_2'$, the lines of sight from O_2 are

$$\frac{x-8a}{-3} = \frac{y}{1} = \frac{z}{3} \quad \text{and} \quad \frac{x-8a}{-4} = \frac{y}{2} = \frac{z}{5}$$

respectively. Find an equation or equations for the path of P.

5 The curve C has the differential equation in polar coordinates

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta, \qquad \text{for} \quad \frac{\pi}{5} \leqslant \theta \leqslant \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2$.

Show that C forms a closed loop and that the area of the region enclosed by C is

$$\frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right].$$

6 The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where u is a positive real constant.

(i) Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself.

(ii) Find the equations of two straight lines through the origin which transform into themselves.

(iii) A line, not necessary through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$.

(iv) The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v_1'$ and $\tanh v_2'$ respectively. Find the relation satisfied by v_1 and v_2 that is the necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms.

(v) In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1, ℓ_2 and their transforms.

7 **(i)** Prove that

$$\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\cos x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x - \tfrac{1}{4}\pi\ln 2$$

and

$$\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x.$$

Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x$.

[You may assume that all the integrals converge.]

(ii) Given that $\ln u < u$ for $u \geqslant 1$ deduce that

$$\tfrac{1}{2}\ln x < \sqrt{x} \qquad \text{for} \qquad x \geqslant 1.$$

Deduce that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$ and that $x\ln x \to 0$ as $x \to 0$ through positive values.

(iii) Using the results of parts **(i)** and **(ii)**, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} x\cot x\,\mathrm{d}x$.

8 **(i)** The integral I_k is defined by

$$I_k = \int_0^{\theta} \cos^k x\, \cos kx\,\mathrm{d}x.$$

Prove that $2kI_k = kI_{k-1} + \cos^k\theta\sin k\theta$.

(ii) Prove that

$$1 + m\cos 2\theta + \binom{m}{2}\cos 4\theta + \cdots + \binom{m}{r}\cos 2r\theta + \cdots + \cos 2m\theta = 2^m\cos^m\theta\cos m\theta.$$

(iii) Using the results of **(i)** and **(ii)**, show that

$$m\frac{\sin 2\theta}{2} + \binom{m}{2}\frac{\sin 4\theta}{4} + \cdots + \binom{m}{r}\frac{\sin 2r\theta}{2r} + \cdots + \frac{\sin 2m\theta}{2m}$$

is equal to

$$\cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cdots + \frac{1}{r}2^{r-1}\cos^r\theta\sin r\theta + \cdots + \frac{1}{m}2^{m-1}\cos^m\theta\sin m\theta.$$

9 The parametric equations E_1 and E_2 define the same ellipse, in terms of the parameters θ_1 and θ_2, (though not referred to the same coordinate axes).

$$E_1 : \quad x = a\cos\theta_1, \quad y = b\sin\theta_1,$$
$$E_2 : \quad x = \frac{k\cos\theta_2}{1+e\cos\theta_2}, \quad y = \frac{k\sin\theta_2}{1+e\cos\theta_2},$$

where $0 < b < a$, $0 < e < 1$ and $0 < k$.

(i) Find the position of the axes for E_2 relative to the axes for E_1 and show that $k = a(1-e^2)$ and $b^2 = a^2(1-e^2)$.

[The standard polar equation of an ellipse is $r = \dfrac{\ell}{1+e\cos\theta}$.]

(ii) By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that
$$\int_0^\pi \sqrt{1-e^2\cos^2\theta}\,\mathrm{d}\theta = \int_0^\pi \frac{1-e^2}{(1+e\cos\theta)^2}\sqrt{1+e^2+2e\cos\theta}\,\mathrm{d}\theta.$$

(iii) Given that e is so small that e^6 may be neglected, show that the value of either integral is
$$\tfrac{1}{64}\pi(64-16e^2-3e^4).$$

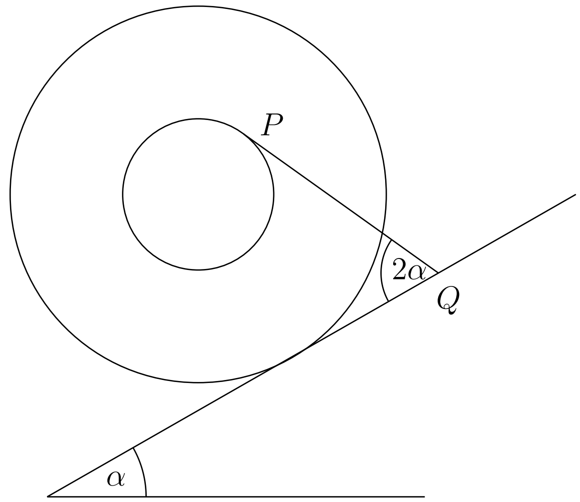
10 The equation
$$x^n - qx^{n-1} + r = 0,$$
where $n \geqslant 5$ and q and r are real constants, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. The sum of the products of m distinct roots is denoted by Σ_m (so that, for example, $\Sigma_3 = \sum \alpha_i\alpha_j\alpha_k$ where the sum runs over the values of i, j and k with $n \geqslant i > j > k \geqslant 1$). The sum of mth powers of the roots is denoted by S_m (so that, for example, $S_3 = \sum_{i=1}^n \alpha_i^3$).

(i) Prove that $S_p = p^q$ for $1 \leqslant p \leqslant n-1$. [You may assume that for any nth degree equation and $1 \leqslant p \leqslant n$
$$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 - \cdots + (-1)^{p-1}S_1\Sigma_{p-1} + (-1)^p p\Sigma_p = 0.]$$

(ii) Find expressions for S_n, S_{n+1} and S_{n+2} in terms of q, r and n. Suggest an expression for S_{n+m}, where $m < n$, and prove its validity by induction.

Section B: Mechanics

11



A uniform circular cylinder of radius $2a$ with a groove of radius a cut in its central cross-section has mass M. It rests, as shown in the diagram, on a rough plane inclined at an acute angle α to the horizontal. It is supported by a light inextensible string would round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at Q, the free part of the string, PQ, making an angle 2α with the inclined plane.

(i) The coefficient of friction at the contact between the cylinder and the plane is μ. Show that $\mu \geqslant \frac{1}{3}\tan\alpha$.

(ii) The string PQ is now detached from the plane and the end Q is fastened to a particle of mass $3M$ which is placed on the plane, the position of the string remain unchanged. Given that $\tan\alpha = \frac{1}{2}$ and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.

12 A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a\cos\theta \qquad y = b\sin\theta$$

where the x-axis is horizontal and the y-axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section.

(i) By first finding $\ddot{x}$ and $\ddot{y}$, or otherwise, show that the acceleration along the inward normal at the point with parameter θ is

$$\frac{ab\dot{\theta}^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

(ii) The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter θ where

$$5\sin^3\theta + 12\sin\theta - 8 = 0.$$

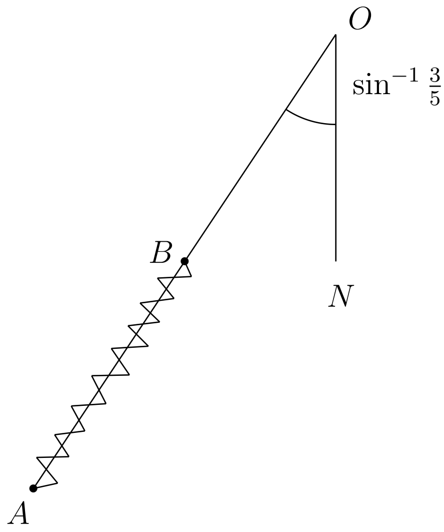
13 A smooth particle P_1 is projected from a point O on the horizontal floor of a room with has a horizontal ceiling at a height h above the floor. The speed of projection is $\sqrt{8gh}$ and the direction of projection makes an acute angle α with the horizontal. The particle strikes the ceiling and rebounds, the impact being perfectly elastic.

(i) Show that for this to happen α must be at least $\frac{1}{6}\pi$ and that the range on the floor is then

$$8h\cos\alpha\left(2\sin\alpha - \sqrt{4\sin^2\alpha - 1}\right).$$

(ii) Another particle P_2 is projected from O with the same velocity as P_1 but its impact with the ceiling is perfectly inelastic. Find the difference D between the ranges of P_1 and P_2 on the floor and show that, as α varies, D has a maximum value when $\alpha = \frac{1}{4}\pi$.

14



The end O of a smooth light rod OA of length $2a$ is a fixed point. The rod OA makes a fixed angle $\sin^{-1}\frac{3}{5}$ with the downward vertical ON, but is free to rotate about ON. A particle of mass m is attached to the rod at A and a small ring B of mass m is free to slide on the rod but is joined to a spring of natural length a and modulus of elasticity kmg. The vertical plane containing the rod OA rotates about ON with constant angular velocity $\sqrt{5g/2a}$ and B is at rest relative to the rod.

(i) Show that the length of OB is
$$\frac{(10k+8)a}{10k-9}.$$

(ii) Given that the reaction of the rod on the particle at A makes an angle $\tan^{-1}\frac{13}{21}$ with the horizontal, find the value of k. Find also the magnitude of the reaction between the rod and the ring B.

Section C: Probability and Statistics

15 A pack of $2n$ (where $n \geqslant 4$) cards consists of two each of n different sorts.

(i) If four cards are drawn from the pack without replacement show that the probability that no pairs of identical cards have been drawn is

$$\frac{4(n-2)(n-3)}{(2n-1)(2n-3)}.$$

(ii) Find the probability that exactly one pair of identical cards is included in the four.

(iii) If k cards are drawn without replacement and $2 < k < 2n$, find an expression for the probability that there are exactly r pairs of identical cards included when $r < \frac{1}{2}k$.

(iv) For even values of k show that the probability that the drawn cards consist of $\frac{1}{2}k$ pairs is

$$\frac{1 \times 3 \times 5 \times \cdots \times (k-1)}{(2n-1)(2n-3)\cdots(2n-k+1)}.$$

16 The random variables X and Y take integer values x and y respectively which are restricted by $x \geqslant 1$, $y \geqslant 1$ and $2x + y \leqslant 2a$ where a is an integer greater than 1. The joint probability is given by

$$\mathrm{P}(X = x, Y = y) = c(2x + y),$$

where c is a positive constant, within this region and zero elsewhere.

(i) Obtain, in terms of x, c and a, the marginal probability $\mathrm{P}(X = x)$ and show that

$$c = \frac{6}{a(a-1)(8a+5)}.$$

(ii) Show that when y is an even number the marginal probability $\mathrm{P}(Y = y)$ is

$$\frac{3(2a-y)(2a+2+y)}{2a(a-1)(8a+5)}$$

and find the corresponding expression when y is off.

(iii) Evaluate $\mathrm{E}(Y)$ in terms of a.

Section A: Pure Mathematics

1 **(i)** Show, using de Moivre's theorem, or otherwise, that

$$\tan 9\theta = \frac{t(t^2-3)(t^6-33t^4+27t^2-3)}{(3t^2-1)(3t^6-27t^4+33t^2-1)}, \qquad \text{where } t=\tan\theta.$$

(ii) By considering the equation $\tan 9\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{9}\right), \qquad \tan^2\left(\frac{2\pi}{9}\right) \qquad \text{and} \qquad \tan^2\left(\frac{4\pi}{9}\right).$$

(iii) Deduce the value of

$$\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right).$$

(iv) Show that

$$\tan^6\left(\frac{\pi}{9}\right)+\tan^6\left(\frac{2\pi}{9}\right)+\tan^6\left(\frac{4\pi}{9}\right)=33273.$$

2 The distinct points $O\,(0,0,0)$, $A\,(a^3,a^2,a)$, $B\,(b^3,b^2,b)$ and $C\,(c^3,c^2,c)$ lie in 3-dimensional space.

(i) Prove that the lines OA and BC do not intersect.

(ii) Given that a and b can vary with $ab=1$, show that $\angle AOB$ can take any value in the interval $0<\angle AOB<\frac{1}{2}\pi$, but no others.

3 The elements a,b,c,d belong to the group G with binary operation $*$. Show that

(i) if a,b and $a*b$ are of order 2, then a and b commute;

(ii) $c*d$ and $d*c$ have the same order;

(iii) if $c^{-1}*b*c=b^r$, then $c^{-1}*b^s*c=b^{sr}$ and $c^{-n}*b^s*c^n=b^{sr^n}$.

4 **(i)** Given that $\sin\beta \neq 0$, sum the series

$$\cos\alpha + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+2r\beta) + \cdots + \cos(\alpha+2n\beta)$$

and

$$\cos\alpha + \binom{n}{1}\cos(\alpha+2\beta) + \cdots + \binom{n}{r}\cos(\alpha+2r\beta) + \cdots + \cos(\alpha+2n\beta).$$

(ii) Given that $\sin\theta \neq 0$, prove that

$$1 + \cos\theta\sec\theta + \cos 2\theta\sec^2\theta + \cdots + \cos r\theta\sec^r\theta + \cdots + \cos n\theta\sec^n\theta = \frac{\sin(n+1)\theta\sec^n\theta}{\sin\theta}.$$

5 **(i)** Prove that, for any integers n and r, with $1 \leqslant r \leqslant n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

(ii) Hence or otherwise, prove that

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots + uv^{(n)},$$

where u and v are functions of x and $z^{(r)}$ means $\dfrac{\mathrm{d}^r z}{\mathrm{d}x^r}$.

(iii) Prove that, if $y = \sin^{-1} y$, then $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$.

6 The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$ is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{2}{5}\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(i) Show that T leaves the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ unchanged in direction but multiplied by a scalar, and that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is similarly transformed.

(ii) The circle C whose equation is $x^2 + y^2 = 1$ transforms under T to a curve E. Show that E has equation

$$8X^2 + 12XY + 17Y^2 = 80,$$

and state the area of the region bounded by E. Show also that the greatest value of X on E is $2\sqrt{17/5}$.

(iii) Find the equation of the tangent to E at the point which corresponds to the point $\frac{1}{5}(3, 4)$ on C.

7 The points $P\,(0, a)$, $Q\,(a, 0)$ and $R\,(a, -a)$ lie on the curve C with cartesian equation

$$xy^2 + x^3 + a^2y - a^3 = 0, \qquad \text{where } a > 0.$$

(i) At each of P, Q and R, express y as a Taylor series in h, where h is a small increment in x, as far as the term in h^2. Hence, or otherwise, sketch the shape of C near each of these points.

(ii) Show that, if (x, y) lies on C, then

$$4x^4 - 4a^3x - a^4 \leqslant 0.$$

(iii) Sketch the graph of $y = 4x^4 - 4a^3 - a^4$.

(iv) Given that the y-axis is an asymptote to C, sketch the curve C.

8 Let P, Q and R be functions of x.

(i) Prove that, for any function y of x, the function

$$Py'' + Qy' + Ry$$

can be written in the form $\dfrac{\mathrm{d}}{\mathrm{d}x}(py' + qy)$, where p and q are functions of x, if and only if $P'' - Q' + R = 0$.

(ii) Solve the differential equation

$$(x - x^4)y'' + (1 - 7x^3)y' - 9x^2y = (x^3 + 3x)\mathrm{e}^x,$$

given that when $x = 2, y = 2\mathrm{e}^2$ and $y' = 0$.

9 The real variables θ and u are related by the equation $\tan\theta = \sinh u$ and $0 \leqslant \theta < \frac{1}{2}\pi$. Let $v = \mathrm{sech} u$. Prove that

(i) $v = \cos\theta$;

(ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}u} = v$;

(iii) $\sin 2\theta = -2\dfrac{\mathrm{d}v}{\mathrm{d}u}$ and $\cos 2\theta = -\cosh u \dfrac{\mathrm{d}^2v}{\mathrm{d}u^2}$;

(iv) $\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + \dfrac{\mathrm{d}v}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left(\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = 0.$

10 **(i)** By considering the graphs of $y = kx$ and $y = \sin x$, show that the equation $kx = \sin x$, where $k > 0$, may have $0, 1, 2$ or 3 roots in the interval $(4n + 1)\frac{\pi}{2} < x < (4n + 5)\frac{\pi}{2}$, where n is a positive integer.

(ii) For a certain given value of n, the equation has exactly one root in this interval. Show that k lies in an interval which may be written $\sin\delta < k < \dfrac{2}{(4n + 1)\pi}$, where $0 < \delta < \frac{1}{2}\pi$ and

$$\cos\delta = \left((4n + 5)\frac{\pi}{2} - \delta\right)\sin\delta.$$

(iii) Show that, if n is large, then $\delta \approx \dfrac{2}{(4n + 5)\pi}$ and obtain a second, improved, approximation.

Section B: Mechanics

11 The points O, A, B and C are the vertices of a uniform square lamina of mass M. The lamina can turn freely under gravity about a horizontal axis perpendicular to the plane of the lamina through O. The sides of the lamina are of length $2a$. When the lamina is haning at rest with the diagonal OB vertically downwards it is struck at the midpoint of OC by a particle of mass $6M$ moving horizontally in the plane of the lamina with speed V. The particle adheres to the lamina. Find, in terms of a, M and g, the value which V^2 must exceed for the lamina and particle to make complete revolutions about the axis.

12 A uniform smooth wedge of mass m has congruent triangular end faces $A_1B_1C_1$ and $A_2B_2C_2$, and A_1A_2, B_1B_2 and C_1C_2 are perpendicular to these faces. The points A, B and C are the midpoints of A_1A_2, B_1B_2 and C_1C_2 respectively. The sides of the triangle ABC have lengths $AB = AC = 5a$ and $BC = 6a$. The wedge is placed with BC on a smooth horizontal table, a particle of mass $2m$ is placed at A on AC, and the system is released from rest. The particle slides down AC, strikes the table, bounces perfectly elastically and lands again on the table at D. At this time the point C of the wedge has reached the point E.

Show that $DE = \frac{192}{19}a$.

13 A particle P is projected, from the lowest point, along the smooth inside surface of a fixed sphere with centre O. It leaves the surface when OP makes an angle θ with the upward vertical. Find the smallest angle that must be exceeded by θ to ensure that P will strike the surface below the level of O.

[You may find it helpful to find the time at which the particle strikes the sphere.]

14 The edges OA, OB, OC of a rigid cube are taken as coordinate axes and O', A', B', C' are the vertices diagonally opposite O, A, B, C, respectively. The four forces acting on the cube are

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \text{ at } O\ (0,0,0), \quad \begin{pmatrix}\lambda\\0\\1\end{pmatrix} \text{ at } O'\ (a,a,a), \quad \begin{pmatrix}-1\\0\\2\end{pmatrix} \text{ at } B\ (0,a,0), \quad \text{and} \quad \begin{pmatrix}1\\\mu\\\nu\end{pmatrix} \text{ at } B'\ (a,0,a).$$

The moment of the system about O is zero: find λ, μ and ν.

(i) Given that $\alpha = \beta = \gamma = 0$, find the system consisting of a single force at B together with a couple which is equivalent to the given system.

(ii) Given that $\alpha = 2, \beta = 3$ and $\gamma = 2$, find the equation of the locus about each point of which the moment of the system is zero. Find the number of units of work done on the cube when it moves (without rotation) a distance in the direction of this line under the action of the given forces only.

Section C: Probability and Statistics

15 An unbiased twelve-sided die has its faces marked $A, A, A, B, B, B, B, B, B, B, B, B$. In a series of throws of the die the first M throws show A, the next N throws show B and the $(M+N+1)$th throw shows A. Write down the probability that $M = m$ and $N = n$, where $m \geqslant 0$ and $n \geqslant 1$. Find

(i) the marginal distributions of M and N,

(ii) the mean values of M and N.

(iii) Investigate whether M and N are independent.

(iv) Find the probability that N is greater than a given integer k, where $k \geqslant 1$, and find $\mathrm{P}(N > M)$.

(v) Find also $\mathrm{P}(N = M)$ and show that $\mathrm{P}(N < M) = \frac{1}{52}$.

16 **(i)** A rod of unit length is cut into pieces of length X and $1 - X$; the latter is then cut in half. The random variable X is uniformly distributed over $[0, 1]$. For some values of X a triangle can be formed from the three pieces of the rod. Show that the conditional probability that, if a triangle can be formed, it will be obtuse-angled is $3 - 2\sqrt{2}$.

(ii) The bivariate distribution of the random variables X and Y is uniform over the triangle with vertices $(1, 0), (1, 1)$ and $(0, 1)$. A pair of values x, y is chosen at random from this distribution and a (perhaps degenerate) triangle ABC is constructed with $BC = x$ and $CA = y$ and $AB = 2 - x - y$. Show that the construction is always possible and that $\angle ABC$ is obtuse if and only if

$$y > \frac{x^2 - 2x + 2}{2 - x}.$$

Deduce that the probability that $\angle ABC$ is obtuse is $3 - 4\ln 2$.

Section A: Pure Mathematics

1 **(i)** Prove that the area of the zone of the surface of a sphere between two parallel planes cutting the sphere is given by

$$2\pi \times \text{(radius of sphere)} \times \text{(perpendicular distance between the planes)}.$$

(ii) A tangent from the origin O to the curve with cartesian equation

$$(x-c)^2 + y^2 = c^2,$$

where a and c are positive constants with $c > a$, touches the curve at P. The x-axis cuts the curve at Q and R, the points lying in the order OQR on the axis. The line OP and the arc PR are rotated through 2π radians about the line OQR to form a surface. Find the area of this surface.

2 The points A, B and C lie on the surface of the ground, which is an inclined plane. The point B is 100m due north of A, and C is 60m due east of B. The vertical displacements from A to B, and from B to C, are each 5m downwards. A plane coal seam lies below the surface and is to be located by making vertical bore-holes at A, B and C. The bore-holes strike the coal seam at 95m, 45m and 76m below A, B and C respectively.

(i) Show that the coal seam is inclined at $\cos^{-1}(\frac{4}{5})$ to the horizontal.

(ii) The coal seam comes to the surface along a line. Find the bearing of this line.

3 **(i)** The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{pmatrix},$$

where m is an integer greater than 1. Prove that

$$\mathbf{M}^{m-1} + \mathbf{M}^{m-2} + \cdots + \mathbf{M}^2 + \mathbf{M} + \mathbf{I} = \mathbf{O},$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(ii) The sequence $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ is defined by

$$\mathbf{X}_{k+1} = \mathbf{P}\mathbf{X}_k + \mathbf{Q},$$

where $\mathbf{P}, \mathbf{Q}$ and $\mathbf{X}_0$ are given 2×2 matrices. Suggest a suitable expression for $\mathbf{X}_k$ in terms of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{X}_0$, and justify it by induction.

(iii) The binary operation $*$ is defined as follows:

$\mathbf{X}_i * \mathbf{X}_j$ is the result of substituting $\mathbf{X}_j$for $\mathbf{X}_0$in the expression for $\mathbf{X}_i$.

Show that if $\mathbf{P} = \mathbf{M}$, the set $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots\}$ forms a finite group under the operation $*$.

4 Sketch the curve whose cartesian equation is

$$y = \frac{2x(x^2-5)}{x^2-4},$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence, or otherwise, determine (giving reasons) the number of real roots of the following equations:

(i) $4x^2(x^2-5) = (5x-2)(x^2-4)$;

(ii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)$;

(iii) $4z^2(z-5)^2 = (z-4)^2(z+1)$.

5 Given that $y = \cosh(n \cosh^{-1} x)$, for $x \geqslant 1$, prove that

$$y = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Explain why, when $n = 2k + 1$ and $k \in \mathbb{Z}^+$, y can also be expressed as the polynomial

$$a_0 x + a_1 x^3 + a_2 x^5 + \cdots + a_k x^{2k+1}.$$

Find a_0, and show that

(i) $a_1 = (-1)^{k-1} 2k(k+1)(2k+1)/3$;

(ii) $a_2 = (-1)^k 2(k-1)k(k+2)(2k+1)/15$.

Find also the value of $\displaystyle\sum_{r=0}^{k} a_r$.

6 **(i)** Show that, for a given constant γ $(\sin\gamma \neq 0)$ and with suitable choice of the constants A and B, the line with cartesian equation $lx + my = 1$ has polar equations

$$\frac{1}{r} = A\cos\theta + B\cos(\theta - \gamma).$$

(ii) The distinct points P and Q on the conic with polar equations

$$\frac{a}{r} = 1 + e\cos\theta$$

correspond to $\theta = \gamma - \delta$ and $\theta = \gamma + \delta$ respectively, and $\cos\delta \neq 0$. Obtain the polar equation of the chord PQ. Hence, or otherwise, obtain the equation of the tangent at the point where $\theta = \gamma$.

(iii) The tangents at L and M to a conic with focus S meet at T. Show that ST bisects the angle LSM and find the position of the intersection of ST and LM in terms of your chosen parameters for L and M.

7 The linear transformation T is a shear which transforms a point P to the point P' defined by

(i) $\overrightarrow{PP'}$ makes an acute angle α (anticlockwise) with the x-axis,

(ii) $\angle POP'$ is clockwise (i.e. the rotation from OP to OP' clockwise is less than π),

(iii) $PP' = k \times PN$, where PN is the perpendicular onto the line $y = x\tan\alpha$, where k is a given non-zero constant.

If T is represented in matrix form by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$, show that

$$\mathbf{M} = \begin{pmatrix} 1 - k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1 + k\sin\alpha\cos\alpha \end{pmatrix}.$$

Show that the necessary and sufficient condition for $\begin{pmatrix} p & q \\ r & t \end{pmatrix}$ to commute with $\mathbf{M}$ is

$$t - p = 2q\tan\alpha = -2r\cot\alpha.$$

8 **(i)** Given that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4(x - y) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 12(\mathrm{e}^{2t} + \mathrm{e}^{-2t}),$$

obtain a differential equation for x which does not contain y.

(ii) Hence, or otherwise, find x and y in terms of t given that $x = y = 0$ when $t = 0$.

9 Obtain the sum to infinity of each of the following series.

(i) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots + \frac{r}{2^{r-1}} + \cdots;$

(ii) $1 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2^2} + \cdots + \frac{1}{r} \times \frac{1}{2^{r-1}} + \cdots;$

(iii) $\frac{1 \times 3}{2!} \times \frac{1}{3} + \frac{1 \times 3 \times 5}{3!}\frac{1}{3^2} + \cdots + \frac{1 \times 3 \times \cdots \times (2k-1)}{k!} \times \frac{1}{3^{k-1}} + \cdots.$

[Questions of convergence need not be considered.]

10 **(i)** Prove that

$$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)(r+4) = \tfrac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$$

and deduce that

$$\sum_{r=1}^{n} r^5 < \tfrac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

(ii) Prove that, if $n > 1$,

$$\sum_{r=0}^{n-1} r^5 > \tfrac{1}{6}(n-5)(n-4)(n-3)(n-2)(n-1)n.$$

(iii) Let f be an increasing function. If the limits

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{a}{n}\mathrm{f}\left(\frac{ra}{n}\right) \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{r=1}^{n} \frac{a}{n}\mathrm{f}\left(\frac{ra}{n}\right)$$

both exist and are equal, the definite integral $\displaystyle\int_0^a \mathrm{f}(x)\,\mathrm{d}x$ is defined to be their common value. Using this definition, prove that

$$\int_0^a x^5\,\mathrm{d}x = \tfrac{1}{6}a^6.$$

Section B: Mechanics

11 A smooth uniform sphere, with centre A, radius $2a$ and mass $3m$, is suspended from a fixed point O by means of a light inextensible string, of length $3a$, attached to its surface at C. A second smooth unifom sphere, with centre B, radius $3a$ and mass $25m$, is held with its surface touching O and with OB horizontal. The second sphere is released from rest, falls and strikes the first sphere.

(i) The coefficient of restitution between the spheres is $3/4$. Find the speed U of A immediately after the impact in terms of the speed V of B immediately before impact.

(ii) The same system is now set up with a light rigid rod replacing the string and rigidly attached to the sphere so that OCA is a straight line. The rod can turn freely about O. The sphere with centre B is dropped as before. Show that the speed of A immediately after impact is $125U/127$.

12 A smooth horizontal plane rotates with constant angular velocity Ω about a fixed vertical axis through a fixed point O of the plane. The point A is fixed in the plane and $OA = a$. A particle P lies on the plane and is joined to A by a light rod of length $b(> a)$ freely pivoted at A. Initially OAP is a straight line and P is moving with speed $(a + 2\sqrt{ab})\Omega$ perpendicular to OP in the same sense as Ω. At time t, AP makes an angle ϕ with OA produced.

(i) Obtain an expression for the component of the acceleration of P perpendicular to AP in terms of $\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}, \phi, a, b$ and Ω.

(ii) Hence find $\frac{\mathrm{d}\phi}{\mathrm{d}t}$, in terms of ϕ, a, b and Ω, and show that P never crosses OA.

13 The points A, B, C, D and E lie on a thin smooth horizontal table and are equally spaced on a circle with centre O and radius a. At each of these points there is a small smooth hole in the table. Five elastic strings are threaded through the holes, one end of each beging attached at O under the table and the other end of each being attached to a particle P of mass m on top of the table. Each of the string has natural length a and modulus of elasticity λ.

(i) If P is displaced from O to any point F on the table and released from rest, show that P moves with simple harmonic motion of period T, where

$$T = 2\pi\sqrt{\frac{am}{5\lambda}}.$$

(ii) The string PAO is replaced by one of natural length a and modulus $k\lambda$. P is displaced along OA from its equilibrium position and released. Show that P still moves in a straight line with simple harmonic motion, and, given that the period is $T/2$, find k.

14 **(i)** A solid circular disc has radius a and mass m. The density is proportional to the distance from the centre O. Show that the moment of inertia about an axis through C perpendicular to the plane of the disc is $\frac{3}{5}ma^2$.

(ii) A light inelastic string has one end fixed at A. It passes under and supports a smooth pulley B of mass m. It then passes over a rough pulley C which is a disc of the type described in **(i)**, free to turn about its axis which is fixed and horizontal. The string carries a particle D of mass M at its other end. The sections of the string which are not in contact with the pulleys are vertical. The system is released from rest and moves under gravity for t seconds. At the end of this interval the pulley B is suddenly stopped. Given that $m < 2M$, find the resulting impulse on D in terms of m, M, g and t.

[You may assume that the string is long enough for there to be no collisions between the elements of the system, and that the pulley C is rough enough to prevent slipping throughout.]

Section C: Probability and Statistics

15 The continuous random variable X is uniformly distributed over the interval $[-c, c]$. Write down expressions for the probabilities that:

(i) n independently selected values of X are all greater than k,

(ii) n independently selected values of X are all less than k,

where k lies in $[-c, c]$.

A sample of $2n+1$ values of X is selected at random and Z is the median of the sample. Show that Z is distributed over $[-c, c]$ with probability density function

$$\frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2 - z^2)^n.$$

Deduce the value of $\displaystyle\int_{-c}^{c}(c^2 - z^2)^n \, \mathrm{d}z$.

Evaluate $\mathrm{E}(Z)$ and $\mathrm{var}(Z)$.

16 It is believed that the population of Ruritania can be described as follows:

(i) 25% are fair-haired and the rest are dark-haired;

(ii) 20% are green-eyed and the rest hazel-eyed;

(iii) the population can also be divided into narrow-headed and broad-headed;

(iv) no arrow-headed person has green eyes and fair hair;

(v) those who are green-eyed are as likely to be narrow-headed as broad-headed;

(vi) those who are green-eyed and broad-headed are as likely to be fair-headed as dark-haired;

(vii) half of the population is broad-headed and dark-haired;

(viii) a hazel-haired person is as likely to be fair-haired and broad-headed as dark-haired and narrow-headed.

Find the proportion believed to be narrow-headed.

I am acquainted with only six Ruritanians, all of whom are broad-headed. Comment on this observation as evidence for or against the given model.

A random sample of 200 Ruritanians is taken and is found to contain 50 narrow-heads. On the basis of the given model, calculate (to a reasonable approximation) the probability of getting 50 or fewer narrow-heads. Comment on the result.

Section A: Pure Mathematics

1 **(i)** Sketch the graph of

$$y = \frac{x^2 \mathrm{e}^{-x}}{1+x},$$

for $-\infty < x < \infty$.

(ii) Show that the value of

$$\int_0^\infty \frac{x^2 \mathrm{e}^{-x}}{1+x}\,\mathrm{d}x$$

lies between 0 and 1.

2 The real numbers $u_0, u_1, u_2, \ldots$ satisfy the difference equation

$$\alpha u_{n+2} + bu_{n+1} + cu_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

where a, b and c are real numbers such that the quadratic equation

$$ax^2 + bx + c = 0$$

has two distinct real roots α and β.

(i) Show that the above difference equation is satisfied by the numbers u_n defined by

$$u_n = A\alpha^n + B\beta^n,$$

where

$$A = \frac{u_1 - \beta u_0}{\alpha - \beta} \qquad \text{and} \qquad B = \frac{u_1 - \alpha u_0}{\beta - \alpha}.$$

(ii) Show also, by induction, that these numbers provide the only solution.

(iii) Find the numbers v_n $(n = 0, 1, 2, \ldots)$ which satisfy

$$8(n+2)(n+1)v_{n+2} - 2(n+3)(n+1)v_{n+1} - (n+3)(n+2)v_n = 0$$

with $v_0 = 0$ and $v_1 = 1$.

3 Give a parametric form for the curve in the Argand diagram determined by $|z - \mathrm{i}| = 2$.

Let $w = (z + \mathrm{i})/(z - \mathrm{i})$. Find and sketch the locus, in the Argand diagram, of the point which represents the complex number w when

(i) $|z - \mathrm{i}| = 2$;

(ii) z is real;

(iii) z is imaginary.

4 A kingdom consists of a vast plane with a central parabolic hill. In a vertical cross-section through the centre of the hill, with the x-axis horizontal and the z-axis vertical, the surface of the plane and hill is given by

$$x = \begin{cases} \dfrac{1}{2a}(a^2 - x^2) & \text{for } |x| \leqslant a, \\ 0 & \text{for } |x| > a. \end{cases}$$

The whole surface is formed by rotating this cross-section about the z-axis. In the (x, z) plane through the centre of the hill, the king has a summer residence at $(-R, 0)$ and a winter residence at $(R, 0)$, where $R > a$. He wishes to connect them by a road, consisting of the following segments:

(i) a path in the (x, z) plane joining $(-R, 0)$ to $(-b, (a^2 - b^2)/2a)$, where $0 \leqslant b \leqslant a$.

(ii) a horizontal semicircular path joining the two points $(\pm b, (a^2 - b^2)/2a)$, if $b \neq 0$;

(iii) a path in the (x, z) plane joining $(b, (a^2 - b^2)/2a)$ to $(R, 0)$.

The king wants the road to be as short as possible. Advise him on his choice of b.

5 A firm of engineers obtains the right to dig and exploit an undersea tunnel. Each day the firm borrows enough money to pay for the day's digging, which costs $£c$, and to pay the daily interest of $100k\%$ on the sum already borrowed. The tunnel takes T days to build, and, once finished, earns $£d$ a day, all of which goes to pay the daily interest and repay the debt until it is fully paid. The financial transactions take place at the end of each day's work.

(i) Show that S_n, the total amount borrowed by the end of day n, is given by

$$S_n = \frac{c[(1+k)^n - 1]}{k}$$

for $n \leqslant T$.

(ii) Given that $S_{T+m} > 0$, where $m > 0$, express S_{T+m} in terms of c, d, k, T and m.

(iii) Show that, if $d/c > (1+k)^T - 1$, the firm will eventually pay off the debt.

6 **(i)** Let $\mathrm{f}(x) = \sin 2x \cos x$. Find the 1988th derivative of $\mathrm{f}(x)$.

(ii) Show that the smallest positive value of x for which this derivative is zero is $\frac{1}{3}\pi + \epsilon$, where ϵ is approximately equal to

$$\frac{3^{-1988}\sqrt{3}}{2}.$$

7 For $n = 0, 1, 2, \ldots$, the functions y_n satisfy the differential equation

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \omega^2 x^2 y_n = -(2n+1)\omega y_n,$$

where ω is a positive constant, and $y_n \to 0$ and $\mathrm{d}y_n/\mathrm{d}x \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

(i) Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = \mathrm{e}^{-\lambda x^2} \qquad \text{and} \qquad y_1(x) = x\mathrm{e}^{-\lambda x^2}$$

for some constant λ, to be determined.

(ii) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m \frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n \frac{\mathrm{d}y_m}{\mathrm{d}x}\right) = 2(m-n)\omega y_m y_n,$$

and

(iii) deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x)\, \mathrm{d}x = 0.$$

8 **(i)** Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

(ii) For $i = 1, 2$, and 3, let P_i be the point $(at_i^2, 2at_i)$, where t_1, t_2 and t_3 are all distinct. Let A_1 be the area of the triangle formed by the tangents at P_1, P_2 and P_3, and let A_2 be the area of the triangle formed by the normals at P_1, P_2 and P_3. Using the fact that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of

$$\tfrac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

show that $A_3 = (t_1 + t_2 + t_3)^2 A_1$.

(iii) Deduce a necessary and sufficient condition in terms of t_1, t_2 and t_3 for the normals at P_1, P_2 and P_3 to be concurrent.

9 Let G be a finite group with identity e. For each element $g \in G$, the order of g, $o(g)$, is defined to be the smallest positive integer n for which $g^n = e$.

(i) Show that, if $o(g) = n$ and $g^N = e$, then n divides N.

(ii) Let g and h be elements of G. Prove that, for any integer m,

$$gh^m g^{-1} = (ghg^{-1})^m.$$

(iii) Let g and h be elements of G, such that $g^5 = e, h \neq e$ and $ghg^{-1} = h^2$. Prove that $g^2hg^{-2} = h^4$ and find $o(h)$.

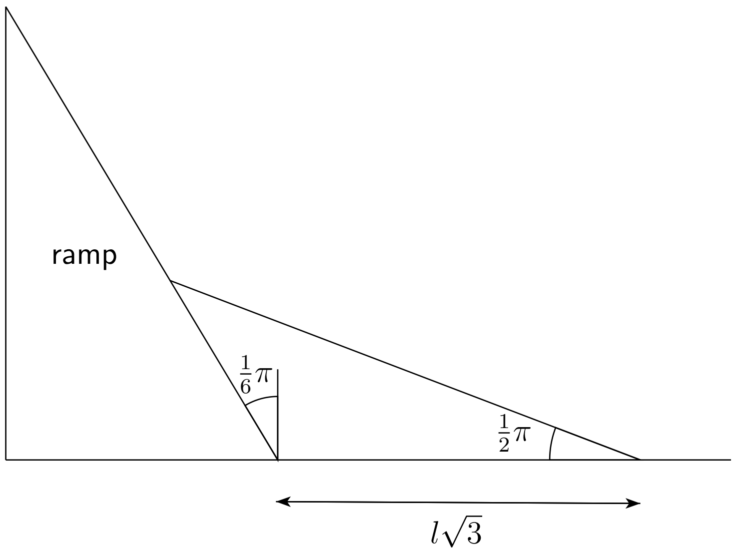
10 Four greyhounds A, B, C and D are held at positions such that $ABCD$ is a large square. At a given instant, the dogs are released and A runs directly towards B at constant speed v, B runs directly towards C at constant speed v, and so on.

(i) Show that A's path is given in polar coordinates (referred to an origin at the centre of the field and a suitable initial line) by $r = \lambda \mathrm{e}^{-\theta}$, where λ is a constant.

(ii) Generalise this result to the case of n dogs held at the vertices of a regular n-gon ($n \geqslant 3$).

Section B: Mechanics

11 A uniform ladder of length l and mass m rests with one end in contact with a smooth ramp inclined at an angle of $\pi/6$ to the vertical. The foot of the ladder rests, on horizontal ground, at a distance $l/\sqrt{3}$ from the foot of the ramp, and the coefficient of friction between the ladder and the ground is μ. The ladder is inclined at an angle $\pi/6$ to the horizontal, in the vertical plane containing a line of greatest slope of the ramp. A labourer of mass m intends to climb slowly to the top of the ladder.



(i) Find the value of μ if the ladder slips as soon as the labourer reaches the midpoint.

(ii) Find the minimum value of μ which will ensure that the labourer can reach the top of the ladder.

12 A smooth billiard ball moving on a smooth horizontal table strikes another identical ball which is at rest. The coefficient of restitution between the balls is $e(< 1)$.

(i) Show that after the collision the angle between the velocities of the balls is less than $\frac{1}{2}\pi$.

(ii) Show also that the maximum angle of deflection of the first ball is

$$\sin^{-1}\left(\frac{1+e}{3-e}\right).$$

13 A goalkeeper stands on the goal-line and kicks the football directly into the wind, at an angle α to the horizontal. The ball has mass m and is kicked with velocity $\mathbf{v}_0$. The wind blows horizontally with constant velocity $\mathbf{w}$ and the air resistance on the ball is mk times its velocity relative to the wind velocity, where k is a positive constant.

(i) Show that the equation of motion of the ball can be written in the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + k\mathbf{v} = \mathbf{g} + k\mathbf{w},$$

where $\mathbf{v}$ is the ball's velocity relative to the ground, and $\mathbf{g}$ is the acceleration due to gravity.

(ii) By writing down horizontal and vertical equations of motion for the ball, or otherwise, find its position at time t after it was kicked.

(iii) On the assumption that the goalkeeper moves out of the way, show that if $\tan\alpha = |\mathbf{g}|/(k|\mathbf{w}|)$, then the goalkeeper scores an own goal.

14 A small heavy bead can slide smoothly in a vertical plane on a fixed wire with equation

$$y = x - \frac{x^2}{4a},$$

where the y-axis points vertically upwards and a is a positive constant. The bead is projected from the origin with initial speed V along the wire.

(i) Show that for a suitable value of V, to be determined, a motion is possible throughout which the bead exerts no pressure on the wire.

(ii) Show that θ, the angle between the particle's velocity at time t and the x-axis, satisfies

$$\frac{4a^2\dot{\theta}^2}{\cos^6\theta} + 2ga(1 - \tan^2\theta) = V^2.$$

Section C: Probability and Statistics

15 Each day, books returned to a library are placed on a shelf in order of arrival, and left there. When a book arrives for which there is no room on the shelf, that book and all books subsequently returned are put on a trolley. At the end of each day, the shelf and trolley are cleared. There are just two-sizes of book: thick, requiring two units of shelf space; and thin, requiring one unit. The probability that a returned book is thick is p, and the probability that it is thin is $q = 1 - p$. Let $M(n)$ be the expected number of books that will be put on the shelf, when the length of the shelf is n units and n is an integer, on the assumption that more books will be returned each day than can be placed on the shelf. Show, giving reasoning, that

(i) $M(0) = 0$;

(ii) $M(1) = q$;

(iii) $M(n) - qM(n-1) - pM(n-2) = 1$, for $n \geqslant 2$.

Verify that a possible solution to these equations is

$$M(n) = A(-p)^n + B + Cn,$$

where A, B and C are numbers independent of n which you should express in terms of p.

16 Balls are chosen at random without replacement from an urn originally containing m red balls and $M - m$ green balls. Find the probability that exactly k red balls will be chosen in n choices $(0 \leqslant k \leqslant m, 0 \leqslant n \leqslant M)$.

The random variables X_i $(i = 1, 2, \ldots, n)$ are defined for $n \leqslant M$ by

$$X_i = \begin{cases} 0 & \text{if the } i\text{th ball chosen is green} \\ 1 & \text{if the } i\text{th ball chosen is red.} \end{cases}$$

Show that

(i) $\mathrm{P}(X_i = 1) = \dfrac{m}{M}$.

(ii) $\mathrm{P}(X_i = 1 \text{ and } X_j = 1) = \dfrac{m(m-1)}{M(M-1)}$, for $i \neq j$.

Find the mean and variance of the random variable X defined by

$$X = \sum_{i=1}^{n} X_i.$$

Section A: Pure Mathematics

1 Find the set of positive integers n for which n does not divide $(n-1)!$. Justify your answer. [Note that small values of n may require special consideration.]

2 Let $I_{m,n} = \displaystyle\int \cos^m x \sin nx \, \mathrm{d}x$, where m and n are non-negative integers.

(i) Prove that for $m, n \geqslant 1$,

$$(m+n)I_{m,n} = -\cos^m \cos nx + mI_{m-1,n-1}.$$

(ii) Show that $\displaystyle\int_0^{\pi} \cos^m x \sin nx \, \mathrm{d}x = 0$ whenever m, n are both even or both odd.

(iii) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \sin 3x \, \mathrm{d}x$.

3 **(i)** If $z = x + \mathrm{i}y$, with x, y real, show that

$$|x|\cos\alpha + |y|\sin\alpha \leqslant |z| \leqslant |x| + |y|$$

for all real α.

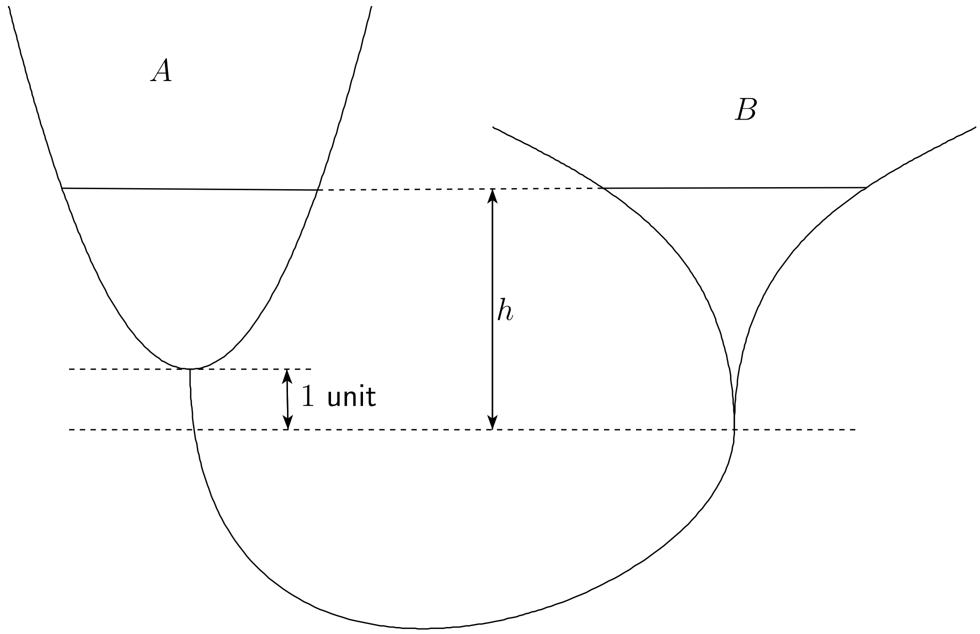
(ii) By considering $(5-\mathrm{i})^4(1+\mathrm{i})$, show that

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Prove similarly that

$$\frac{\pi}{4} = 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right).$$

4



Two funnels A and B have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1} x$ $(x > 0)$ above the y-axis. The bottom of B is one unit lower than the bottom of A and they are connected by a thin rubber tube with a tap in it. The tap is closed and A is filled with water to a depth of 4 units. The tap is then closed. When the water comes to rest, both surfaces are at a height h above the bottom of B, as shown in the diagram. Show that h satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$

5 A secret message consists of the numbers $1, 3, 7, 23, 24, 37, 39, 43, 43, 43, 45, 47$ arranged in some order as $a_1, a_2, \ldots, a_{12}$. The message is encoded as $b_1, b_2, \ldots, b_{12}$ with $0 \leqslant b_j \leqslant 49$ and

$$b_{2j} \equiv a_{2j} + n_0 + j \pmod{50},$$
$$b_{2j+1} \equiv a_{2j+1} + n_1 + j \pmod{50},$$

for some integers n_0 and n_1. If the coded message is $35, 27, 2, 36, 15, 35, 8, 40, 40, 37, 24, 48,$ find the original message, explaining your method carefully.

6 The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 5y = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} + ax - 2y = 2\cos t,$$

subject to $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ at $t = 0$.

(i) Solve these equations for x and y in the case when $a = 1$.

(ii) Without solving the equations explicitly, state briefly how the form of the solutions for x and y if $a > 1$ would differ from the form when $a = 1$.

7 **(i)** Prove that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^t \frac{x^{2n+2}}{1+x^2}\,\mathrm{d}x.$$

(ii) Hence show that, if $0 \leqslant t \leqslant 1$, then

$$\frac{t^{2n+3}}{2(2n+3)} \leqslant \left| \tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1} \right| \leqslant \frac{t^{2n+3}}{2n+3}.$$

(iii) Show that, as $n \to \infty$,

$$4\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)} \to \pi,$$

but that the error in approximating π by $4\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)}$ is at least 10^{-2} if n is less than or equal to 98.

8 **(i)** Show that, if the lengths of the diagonals of a parallelogram are specified, then the parallogram has maximum area when the diagonals are perpendicular.

(ii) Show also that the area of a parallelogram is less than or equal to half the square of the length of its longer diagonal.

(iii) The set A of points (x, y) is given by

$$|a_1x + b_1y - c_1| \leqslant \delta,$$
$$|a_2x + b_2y - c_2| \leqslant \delta,$$

with $a_1b_2 \neq a_2b_1$.

Sketch this set and show that it is possible to find $(x_1, y_1), (x_2, y_2) \in A$ with

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \geqslant \frac{8\delta^2}{|a_1b_2 - a_2b_1|}.$$

9 Let $(G, *)$ and $(H, \circ)$ be two groups and $G \times H$ be the set of ordered pairs (g, h) with $g \in G$ and $h \in H$. A multiplication on $G \times H$ is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Show that, with this multiplication, $G \times H$ is a group.

State whether the following are true or false and prove your answers.

(i) $G \times H$ is abelian if and only if both G and H are abelian.

(ii) $G \times H$ contains a subgroup isomorphic to G.

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$.

(iv) $S_2 \times S_3$ is isomorphic to S_6.

[$\mathbb{Z}_n$ is the cyclic group of order n, and S_n is the permutation group on n objects.]

10 The *Bernoulli polynomials* $P_n(x)$, where n is a non-negative integer, are defined by $P_0(x) = 1$ and, for $n \geqslant 1$,

$$\frac{\mathrm{d}P_n}{\mathrm{d}x} = nP_{n-1}(x), \qquad \int_0^1 P_n(x)\,\mathrm{d}x = 0$$

(i) Show by induction or otherwise, that

$$P_n(x+1) - P_n(x) = nx^{n-1}, \quad \text{for } n \geqslant 1.$$

(ii) Deduce that

$$n\sum_{m=0}^{k} m^{n-1} = P_n(k+1) - P_n(0)$$

(iii) Hence show that $\sum_{m=0}^{1000} m^3 = (500500)^2$

Section B: Mechanics

11 A woman stands in a field at a distance of $a\,\mathrm{m}$ from the straight bank of a river which flows with negligible speed. She sees her frightened child clinging to a tree stump standing in the river $b\,\mathrm{m}$ downstream from where she stands and $c\,\mathrm{m}$ from the bank. She runs at a speed of $u\,\mathrm{ms}^{-1}$ and swims at $v\,\mathrm{ms}^{-1}$ in straight lines.

(i) Find an equation to be satisfied by $x,$ where $x\,\mathrm{m}$ is the distance upstream from the stump at which she should enter the river if she is to reach the child in the shortest possible time.

(ii) Suppose now that the river flows with speed v ms^{-1} and the stump remains fixed. Show that, in this case, x must satisfy the equation

$$2vx^2(b-x) = u(x^2-c^2)[a^2+(b-x)^2]^{\frac{1}{2}}.$$

(iii) For this second case, draw sketches of the woman's path for the three possibilities $b > c$, $b = c$ and $b < c$.

12 A firework consists of a uniform rod of mass M and length $2a$, pivoted smoothly at one end so that it can rotate in a fixed horizontal plane, and a rocket attached to the other end. The rocket is a uniform rod of mass $m(t)$ and length $2l(t)$, with $m(t) = 2\alpha l(t)$ and α constant. It is attached to the rod by its front end and it lies at right angles to the rod in the rod's plane of rotation. The rocket burns fuel in such a way that $\mathrm{d}m/\mathrm{d}t = -\alpha\beta$, with β constant. The burnt fuel is ejected from the back of the rocket, with speed u and directly backwards relative to the rocket. Show that, until the fuel is exhausted, the firework's angular velocity ω at time t satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\alpha\beta au}{2[Ma^2+2\alpha l(3a^2+l^2)]}.$$

13 A uniform rod, of mass $3m$ and length $2a$, is freely hinged at one end and held by the other end in a horizontal position. A rough particle, of mass m, is placed on the rod at its mid-point.

(i) If the free end is then released, prove that, until the particle begins to slide on the rod, the inclination θ of the rod to the horizontal satisfies the equation

$$5a\dot{\theta}^2 = 8g\sin\theta.$$

(ii) The coefficient of friction between the particle and the rod is $\frac{1}{2}$. Show that, when the particle begins to slide, $\tan\theta = \frac{1}{26}$.

14 It is given that the gravitational force between a disc, of radius a, thickness δx and uniform density ρ, and a particle of mass m at a distance $b(\geqslant 0)$ from the disc on its axis is

$$2\pi mk\rho\delta x\left(1 - \frac{b}{(a^2+b^2)^{\frac{1}{2}}}\right),$$

where k is a constant.

(i) Show that the gravitational force on a particle of mass m at the surface of a uniform sphere of mass M and radius r is kmM/r^2.

(ii) Deduce that in a spherical cloud of particles of uniform density, which all attract one another gravitationally, the radius r and inward velocity $v = -\dfrac{\mathrm{d}r}{\mathrm{d}t}$ of a particle at the surface satisfy the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{kM}{r^2},$$

where M is the mass of the cloud.

(iii) At time $t = 0$, the cloud is instantaneously at rest and has radius R. Show that $r = R\cos^2\alpha$ after a time

$$\left(\frac{R^3}{2kM}\right)^{\frac{1}{2}}(\alpha + \tfrac{1}{2}\sin 2\alpha).$$

Section C: Probability and Statistics

15 A patient arrives with blue thumbs at the doctor's surgery. With probability p the patient is suffering from Fenland fever and requires treatment costing $\pounds 100$. With probability $1-p$ he is suffering from Steppe syndrome and will get better anyway. A test exists which infallibly gives positive results if the patient is suffering from Fenland fever but also has probability q of giving positive results if the patient is not. The test cost $\pounds 10$. The doctor decides to proceed as follows. She will give the test repeatedly until *either* the last test is negative, in which case she dismisses the patient with kind words, *or* she has given the test n times with positive results each time, in which case she gives the treatment. In the case $n=0$, she treats the patient at once. She wishes to minimise the expected cost $\pounds E_n$ to the National Health Service.

(i) Show that

$$E_{n+1} - E_n = 10p - 10(1-p)q^n(9-10q),$$

and deduce that if $p = 10^{-4}, q = 10^{-2}$, she should choose $n = 3$.

(ii) Show that if q is larger than some fixed value q_0, to be determined explicitly, then whatever the value of p, she should choose $n = 0$.

16 **(i)** $X_1, X_2, \ldots, X_n$ are independent identically distributed random variables drawn from a uniform distribution on $[0,1]$. The random variables A and B are defined by

$$A = \min(X_1, \ldots, X_n), \qquad B = \max(X_1, \ldots, X_n).$$

For any fixed k, such that $0 < k < \frac{1}{2}$, let

$$p_n = \mathrm{P}(A \leqslant k \text{ and } B \geqslant 1-k).$$

What happens to p_n as $n \to \infty$? Comment briefly on this result.

(ii) Lord Copper, the celebrated and imperious newspaper proprietor, has decided to run a lottery in which each of the $4,000,000$ readers of his newspaper will have an equal probability p of winning $\pounds 1,000,000$ and their changes of winning will be independent. He has fixed all the details leaving to you, his subordinate, only the task of choosing p. If nobody wins $\pounds 1,000,000$, you will be sacked, and if more than two readers win $\pounds 1,000,000$, you will also be sacked. Explaining your reasoning, show that however you choose p, you will have less than a 60% change of keeping your job.

Section A: Pure Mathematics

1 **(i)** Guess an expression for

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right),$$

valud for $n \geqslant 2$, and prove by mathematical induction the correctness of your guess.

(ii) Show that, if n is a positive integer,

$$\sum_{r=0}^{k}(-1)^r\binom{n}{r} = (-1)^k\binom{n-1}{k}, \qquad \text{for } k = 0, 1, \ldots, n-1.$$

2 **(i)** Show by using the binomial expansion, or otherwise, that $(1+x)^n > nx$ whenever $x \geqslant 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K, an integer N can be found such that $y^n > K$ for all integers $n \geqslant N$.

(ii) Show further that if $y > 1$ then, given any K, an integer N can be found such that $\dfrac{y^n}{n} > K$ for all integers $n \geqslant N$.

3 For the complex numbers z_1 and z_2 interpret geometrically the inequality

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Prove that, if $|a_i| \leqslant 2$ for $i = 1, 2, \ldots, n$, then the equation

$$a_1 z + a_2 z^2 + \cdots + a_n z^n = 1$$

has no solutions with $|z| \leqslant \frac{1}{3}$.

4 **(i)** Prove that the straight line

$$ty = x + at^2$$

touches the parabola $y^2 = 4ax$ and find the coordinates of the point of contact.

(ii) The tangents from a point to the parabola meet the directrix $(x = -a)$ in points L and M. Show that, if LM is of a fixed length ℓ, the point must lie on

$$(x+a)^2(y^2 - 4ax) = \ell^2 x^2.$$

5 The equation

$$\sin x = \lambda x, \qquad x \geqslant 0,$$

where $\lambda > 0$, has a finite number N of non-zero solutions x_n, $i = 1, \dots, N$, where N depends on λ, provided $\lambda < 1$.

(i) Show by a graphical argument that there are no non-zero solutions for $\lambda > 1$. Show also that for $\lambda = 1 - \epsilon^2$, with $\epsilon > 0$ and very small compared to 1, there is a non-zero solution approximately equal to $\epsilon\sqrt{6}$.

(ii) Suppose that $N = 2R + 1$ where R is an integer, and that $x_1 < x_2 < \cdots < x_{2R+1}$. By drawing an appropriate graph, explain why

$$(2n-2)\pi < x_{2n-1} < (2n-1)\pi \quad \text{for } n = 1, \dots, R+1,$$
$$2n\pi < x_{2n} \quad < (2n + \tfrac{1}{2})\pi \quad \text{for } n = 1, \dots, R.$$

Hence derive an approximate value for N in terms of λ, when λ is very small.

6 **(i)** Let

$$I_n = \int_0^\infty \operatorname{sech}^n u \, \mathrm{d}u.$$

Show that for $n > 0$

$$\int_0^\infty \operatorname{sech}^{n+2} u \, \sinh^2 u \, \mathrm{d}u = \frac{1}{n+1} I_n,$$

(ii) and deduce that

$$(n+1) I_{n+2} = n I_n.$$

(iii) Find the value of I_6.

7 **(i)** Show that the differential equation

$$x^2y'' + (x-2)(xy' - y) = 0$$

has a solution proportional to x^α for some α.

(ii) By making the substitution $y = x^\alpha v$, or otherwise, find the general solution of this equation.

8 **(i)** Using vectors, or otherwise, prove that the sum of the squares of the edges of any tetrahedron equals four times the sum of the squares of the lines joining the midpoints of opposite edges.

(ii) By using the inequality

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}|\,|\mathbf{b}|$$

with a suitable choices of three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, or otherwise, prove that

$$3x + 2y + 6 \leqslant 7\sqrt{x^2 + y^2 + 1}.$$

(iii) For what values of x and y does the equality sign hold?

9 **(i)** Prove that the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix},$$

where x, y and z are real numbers, is a group G under matrix multiplication. (You may assume that matrix multiplication is associative.)

(ii) Does the subset consisting of these matrices where x, y, z are restricted to the integers form a subgroup of G? Is there an element $\mathbf{A}$ in G, with $\mathbf{A}$ not equal to the identity matrix, such that $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{B}$ belonging to G? Justify your answer.

10 **(i)** Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4. Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

(ii) Prove that 17 divides $2^{3n+1} + 3(5^{2n+1})$ for all integers $n \geqslant 0$.

Section B: Mechanics

11 Two identical snowploughs plough the same stretch of road in the same direction. The first starts at time $t = 0$ when the depth of snow is h metres, and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. It may be assumed that each snowplough moves at a speed equal to $k/(\alpha z)$ ms^{-1} with α a constant, where z is the depth of snow it is ploughing, and that it clears all the snow.

(i) Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{\alpha x} - 1)\frac{h}{k} \text{ seconds.}$$

(ii) Show that at time $t > T$, the second snowplough has moved y metres, where t satisfies

$$\frac{1}{\alpha}\frac{\mathrm{d}t}{\mathrm{d}y} = t - (\mathrm{e}^{\alpha y} - 1)\frac{h}{k}.$$

Verify that the required solution of this equation is

$$t = (\mathrm{e}^{\alpha y} - 1)\frac{h}{k} + \left(T - \frac{\alpha h y}{k}\right)\mathrm{e}^{\alpha y}$$

and deduce that the snowploughs collide when they have moved a distance $kT/(\alpha h)$ metres.

12 One end A of a uniform straight rod AB of mass M and length L rests against a smooth vertical wall. The other end B is attached to a light inextensible string BC of length αL which is fixed to the wall at a point C vertically above A. The rod is in equilibrium with the points A, B and C not collinear.

(i) Determine the inclination of the rod to the vertical and the set of possible values of α.

(ii) Show that the tension in the string is

$$\frac{Mg\alpha}{2}\left(\frac{3}{\alpha^2 - 1}\right)^{\frac{1}{2}}.$$

13 A particle of mass m is attached to a light circular hoop of radius a which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at its highest point and is then displaced slightly.

(i) Show that while the hoop is rolling on the table, the speed v of the particle when the radius to the particle makes an angle 2θ with the upward vertical is given by

$$v = 2(ga)^{\frac{1}{2}} \sin\theta.$$

(ii) Write down expressions in terms of θ for x, the horizontal displacement of the particle from its initial position, and y, its height above the table, and use them to show that

$$\theta = \tfrac{1}{2}(g/a)^{\frac{1}{2}} \tan\theta$$

and

$$\ddot{y} = -2g\sin^2\theta.$$

(iii) By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \pi/4$.

14 A uniform straight rod of mass m and length $4a$ can rotate freely about its midpoint on a smooth horizontal table. Initially the rod is at rest. A particle of mass m travelling on the table with speed u at right angles to the rod collides perfectly elastically with the rod at a distance a from the centre of the rod.

(i) Show that the angular speed, ω, of the rod after the collision is given by

$$a\omega = 6u/7.$$

(ii) Show also that the particle and rod undergo a subsequent collision.

Section C: Probability and Statistics

15 The mountain villages A, B, C and D lie at the vertices of a tetrahedron, and each pair of villages is joined by a road. After a snowfall the probability that any road is blocked is p, and is independent of the conditions on any other road.

(i) Find the probability that it is possible to travel from any village to any other village by some route after a snowfall.

(ii) In the case $p = \frac{1}{2}$ show that this probability is $19/32$.

16 The new president of the republic has no children. According to custom, he decides on accession that he will have no more children once he has r sons. It may be assumed that each baby born to him is equally likely to be a boy or a girl, irrespective of the sexes of his previous children. Let C be the final number of children in his family.

(i) Find, in terms of r, the expectation of C.

(ii) By considering the numbers of boys and girls among $2r - 1$ children, or otherwise, show that $\mathrm{P}(C < 2r) = \frac{1}{2}$.

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