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Sixth Term Examination Paper

99-S1

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Section A: Pure Mathematics

1 **(i)** How many integers greater than or equal to zero and less than a million are not divisible by 2 or 5? What is the average value of these integers?

(ii) How many integers greater than or equal to zero and less than 4179 are not divisible by 3 or 7? What is the average value of these integers?

2 A point moves in the x–y plane so that the sum of the squares of its distances from the three fixed points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is always a^2.

(i) Find the equation of the locus of the point and interpret it geometrically.

(ii) Explain why a^2 cannot be less than the sum of the squares of the distances of the three points from their centroid.

[The *centroid* has coordinates $(\bar{x}, \bar{y})$ where $3\bar{x} = x_1 + x_2 + x_3$, $3\bar{y} = y_1 + y_2 + y_3$.]

3 The n positive numbers $x_1, x_2, \dots, x_n$, where $n \geqslant 3$, satisfy

$$x_1 = 1 + \frac{1}{x_2}\,, x_2 = 1 + \frac{1}{x_3}\,, \dots, x_{n-1} = 1 + \frac{1}{x_n}\,,$$

and also

$$x_n = 1 + \frac{1}{x_1}\,.$$

Show that

(i) $x_1, x_2, \dots, x_n > 1$,

(ii) $x_1 - x_2 = -\dfrac{x_2 - x_3}{x_2 x_3}$,

(iii) $x_1 = x_2 = \cdots = x_n$.

(iv) Hence find the value of x_1.

4 Sketch the following subsets of the x–y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x-1| + |y-1| \leqslant 1$;

(iii) $|x-1| - |y+1| \leqslant 1$;

(iv) $|x|\,|y-2| \leqslant 1$.

5 For this question, you may use the following approximations, valid if θ is small: $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.

A satellite X is directly above the point Y on the Earth's surface and can just be seen (on the horizon) from another point Z on the Earth's surface. The radius of the Earth is R and the height of the satellite above the Earth is h.

(i) Find the distance d of Z from Y along the Earth's surface.

(ii) If the satellite is in low orbit (so that h is small compared with R), show that

$$d \approx k(Rh)^{1/2},$$

where k is to be found.

(iii) If the satellite is very distant from the Earth (so that R is small compared with h), show that

$$d \approx aR + b(R^2/h),$$

where a and b are to be found.

6 **(i)** Find the greatest and least values of $bx + a$ for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.

(ii) Find the greatest and least values of $cx^2 + bx + a$, where $c \geqslant 0$, for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases that can arise for different values of b and c.

7 **(i)** Show that $\sin(k\sin^{-1}x)$, where k is a constant, satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + k^2y = 0. \qquad (*)$$

(ii) In the particular case when $k=3$, find the solution of equation $(*)$ of the form

$$y = Ax^3 + Bx^2 + Cx + D,$$

that satisfies $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ at $x=0$.

(iii) Use this result to express $\sin 3\theta$ in terms of powers of $\sin\theta$.

8 The function f satisfies $0 \leqslant \mathrm{f}(t) \leqslant K$ when $0 \leqslant t \leqslant x$.

(i) Explain by means of a sketch, or otherwise, why

$$0 \leqslant \int_0^x \mathrm{f}(t)\,\mathrm{d}t \leqslant Kx.$$

(ii) By considering $\displaystyle\int_0^1 \frac{t}{n(n-t)}\,\mathrm{d}t$, or otherwise, show that, if $n>1$,

$$0 \leqslant \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leqslant \frac{1}{n-1} - \frac{1}{n}$$

and deduce that

$$0 \leqslant \ln N - \sum_{n=2}^{N} \frac{1}{n} \leqslant 1.$$

(iii) Deduce that as $N \to \infty$

$$\sum_{n=1}^{N} \frac{1}{n} \to \infty.$$

(iv) Noting that $2^{10} = 1024$, show also that if $N < 10^{30}$ then

$$\sum_{n=1}^{N} \frac{1}{n} < 101.$$

Section B: Mechanics

9 A tortoise and a hare have a race to the vegetable patch, a distance X kilometres from the starting post, and back. The tortoise sets off immediately, at a steady v kilometers per hour. The hare goes to sleep for half an hour and then sets off at a steady speed V kilometres per hour. The hare overtakes the tortoise half a kilometre from the starting post, and continues on to the vegetable patch, where she has another half an hour's sleep before setting off for the return journey at her previous pace. One and quarter kilometres from the vegetable patch, she passes the tortoise, still plodding gallantly and steadily towards the vegetable patch.

(i) Show that

$$V = \frac{10}{4X - 9}$$

and find v in terms of X.

(ii) Find X if the hare arrives back at the starting post one and a half hours after the start of the race.

10 A particle is attached to a point P of an unstretched light uniform spring AB of modulus of elasticity λ in such a way that AP has length a and PB has length b. The ends A and B of the spring are now fixed to points in a vertical line a distance l apart. The particle oscillates along this line.

(i) Show that the motion is simple harmonic.

(ii) Show also that the period is the same whatever the value of l and whichever end of the string is uppermost.

11 The force of attraction between two stars of masses m_1 and m_2 a distance r apart is $\gamma m_1 m_2 / r^2$. The Starmakers of Kryton place three stars of equal mass m at the corners of an equilateral triangle of side a.

(i) Show that it is possible for each star to revolve round the centre of mass of the system with angular velocity $(3\gamma m/a^3)^{1/2}$.

(ii) Find a corresponding result if the Starmakers place a fourth star, of mass λm, at the centre of mass of the system.

Section C: Probability and Statistics

12 **(i)** Prove that if $x > 0$ then $x + x^{-1} \geqslant 2$.

I have a pair of six–faced dice, each with faces numbered from 1 to 6. The probability of throwing i with the first die is q_i and the probability of throwing j with the second die is r_j $(1 \leqslant i, j \leqslant 6)$.

The two dice are thrown independently and the sum noted. By considering the probabilities of throwing 2, 12 and 7, show the sums 2, 3, . . . , 12 are not equally likely.

(ii) The first die described above is thrown twice and the two numbers on the die noted. Is it possible to find values of q_j so that the probability that the numbers are the same is less than $1/36$?

13 Bar magnets are placed randomly end–to–end in a straight line. If adjacent magnets have ends of opposite polarities facing each other, they join together to form a single unit. If they have ends of the same polarity facing each other, they stand apart. Find the expectation and variance of the number of separate units in terms of the total number N of magnets.

14 When I throw a dart at a target, the probability that it lands a distance X from the centre is a random variable with density function

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

I score points according to the position of the dart as follows: if $0 \leqslant X < \frac{1}{4}$, my score is 4; if $\frac{1}{4} \leqslant X < \frac{1}{2}$, my score is 3; if $\frac{1}{2} \leqslant X < \frac{3}{4}$, my score is 2; if $\frac{3}{4} \leqslant X \leqslant 1$, my score is 1.

(i) Show that my expected score from one dart is 15/8.

(ii) I play a game with the following rules. I start off with a total score 0, and each time I throw a dart my score on that throw is added to my total. Then:

if my new total is greater than 3, I have lost and the game ends;
if my new total is 3, I have won and the game ends;
if my new total is less than 3, I throw again.

Show that, if I have won such a game, the probability that I threw the dart three times is 343/2231.