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# Sixth Term Examination Paper

96-S1

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# Section A: Pure Mathematics

**1** A cylindrical biscuit tin has volume $V$ and surface area $S$ (including the ends).

**(i)** Show that the minimum possible surface area for a given value of $V$ is $S = 3(2\pi V^2)^{1/3}$.

**(ii)** For this value of $S$ show that the volume of the largest sphere which can fit inside the tin is $\frac{2}{3}V$,

**(iii)** and find the volume of the smallest sphere into which the tin fits.

**2** **(i)** Show that

$$\int_0^1 (1 + (\alpha - 1)x)^n \, \mathrm{d}x = \frac{\alpha^{n+1} - 1}{(n+1)(\alpha - 1)}$$

when $\alpha \neq 1$ and $n$ is a positive integer.

**(ii)** Show that if $0 \leqslant k \leqslant n$ then the coefficient of $\alpha^k$ in the polynomial

$$\int_0^1 (\alpha x + (1 - x))^n \, \mathrm{d}x$$

is

$$\binom{n}{k} \int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x \,.$$

**(iii)** Hence, or otherwise, show that

$$\int_0^1 x^k (1 - x)^{n-k} \, \mathrm{d}x = \frac{k!(n-k)!}{(n+1)!} \,.$$

**3** Let $n$ be a positive integer.

**(i)** Factorise $n^5 - n^3$, and show that it is divisible by 24.

**(ii)** Prove that $2^{2n} - 1$ is divisible by 3.

**(iii)** If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

**4** Show that

$$\int_0^1 \frac{1}{x^2+2ax+1}\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1-a^2}}\tan^{-1}\sqrt{\dfrac{1-a}{1+a}} & \text{if } |a|<1, \\ \dfrac{1}{2\sqrt{a^2-1}}\ln\left|a+\sqrt{a^2-1}\right| & \text{if } |a|>1. \end{cases}$$

**5** **(i)** Find all rational numbers $r$ and $s$ which satisfy

$$(r+s\sqrt{3})^2 = 4-2\sqrt{3}.$$

**(ii)** Find all real numbers $p$ and $q$ which satisfy

$$(p+q\mathrm{i})^2 = (3-2\sqrt{3})+2(1-\sqrt{3})\mathrm{i}.$$

**(iii)** Solve the equation

$$(1+\mathrm{i})z^2-2z+2\sqrt{3}-2=0,$$

writing your solutions in as simple a form as possible.

[ No credit will be given to answers involving use of calculators. ]

**6** Let $\mathrm{f}(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$ for $0<x\leqslant\pi$.

**(i)** Using the formula

$$2\sin\tfrac{1}{2}x\cos kx = \sin(k+\tfrac{1}{2})x-\sin(k-\tfrac{1}{2})x$$

(which you may assume), or otherwise, show that

$$\mathrm{f}(x) = 1+2\sum_{k=1}^{n}\cos kx\,.$$

**(ii)** Find $\displaystyle\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^\pi \mathrm{f}(x)\cos x\,\mathrm{d}x$.

**7** **(i)** At time $t = 0$ a tank contains one unit of water. Water flows out of the tank at a rate proportional to the amount of water in the tank. The amount of water in the tank at time $t$ is $y$. Show that there is a constant $b < 1$ such that $y = b^t$.

**(ii)** Suppose instead that the tank contains one unit of water at time $t = 0$, but that in addition to water flowing out as described, water is added at a steady rate $a > 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y \ln b = a,$$

and hence find $y$ in terms of $a, b$ and $t$.

**8** **(i)** By using the formula for the sum of a geometric series, or otherwise, express the number $0.38383838\ldots$ as a fraction in its lowest terms.

**(ii)** Let $x$ be a real number which has a recurring decimal expansion

$$x = 0 \cdot a_1 a_2 a_2 \cdots ,$$

so that there exists positive integers $N$ and $k$ such that $a_{n+k} = a_n$ for all $n > N$. Show that

$$x = \frac{b}{10^N} + \frac{c}{10^N(10^k - 1)},$$

where $b$ and $c$ are integers to be found. Deduce that $x$ is rational.

# Section B: Mechanics

**9** A bungee–jumper of mass $m$ is attached by means of a light rope of natural length $l$ and modulus of elasticity $mg/k$, where $k$ is a constant, to a bridge over a ravine. She jumps from the bridge and falls vertically towards the ground.

**(i)** If she only just avoids hitting the ground, show that the height $h$ of the bridge above the floor of the ravine satisfies

$$h^2 - 2hl(k+1) + l^2 = 0,$$

**(ii)** Hence find $h$. Show that the maximum speed $v$ which she attains during her fall satisfies

$$v^2 = (k+2)gl.$$

**10** A spaceship of mass $M$ is at rest. It separates into two parts in an explosion in which the total kinetic energy released is $E$. Immediately after the explosion the two parts have masses $m_1$ and $m_2$ and speeds $v_1$ and $v_2$ respectively. Show that the minimum possible relative speed $v_1 + v_2$ of the two parts of the spaceship after the explosion is $(8E/M)^{1/2}$.

**11** A particle is projected under the influence of gravity from a point $O$ on a level plane in such a way that, when its horizontal distance from $O$ is $c$, its height is $h$. It then lands on the plane at a distance $c+d$ from $O$.

**(i)** Show that the angle of projection $\alpha$ satisfies

$$\tan\alpha = \frac{h(c+d)}{cd}$$

**(ii)** and that the speed of projection $v$ satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c+d)^2 h}{cd}\right).$$

# Section C: Probability and Statistics

**12** An examiner has to assign a mark between 1 and $m$ inclusive to each of $n$ examination scripts ($n \leqslant m$). He does this randomly, but never assigns the same mark twice.

**(i)** If $K$ is the highest mark that he assigns, explain why

$$\mathrm{P}(K = k) = \binom{k-1}{n-1} \Big/ \binom{m}{n}$$

for $n \leqslant k \leqslant m$,

**(ii)** and deduce that

$$\sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}.$$

Find the expected value of $K$.

**13** I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess correctly, adding another cup each time. The price he pays for the stamp is $\pounds N$, where $N$ is the number of cups present when Jim fails to guess correctly.

**(i)** Find $\mathrm{P}(N = k)$.

**(ii)** Show that $\mathrm{E}(N) = \mathrm{e}$ and calculate $\mathrm{Var}(N)$.

**14** A biased coin, with a probability $p$ of coming up heads and a probability $q = 1 - p$ of coming up tails, is tossed repeatedly. Let $A$ be the event that the first run of $r$ successive heads occurs before the first run of $s$ successive tails. If $H$ is the even that on the first toss the coin comes up heads and $T$ is the event that it comes up tails.

**(i)** Show that

$$\mathrm{P}(A|H) = p^{\alpha} + (1 - p^{\alpha})\mathrm{P}(A|T),$$
$$\mathrm{P}(A|T) = (1 - q^{\beta})\mathrm{P}(A|H),$$

where $\alpha$ and $\beta$ are to be determined.

**(ii)** Use these two equations to find $\mathrm{P}(A|H)$, $\mathrm{P}(A|T)$, and hence $\mathrm{P}(A)$.