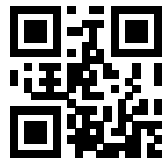


THERE ARE 16 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13 Q14 Q15
Q16

Sixth Term Examination Paper

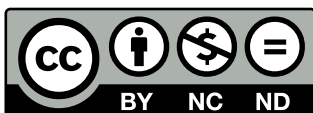
92-S2



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SUGGESTIONS TO DRYUFROMSHANGHAI@QQ.COM

Section A: Pure Mathematics

1 Find the limit, as $n \rightarrow \infty$, of each of the following. You should explain your reasoning briefly.

(i) $\frac{n}{n+1}$, (ii) $\frac{5n+1}{n^2-3n+4}$, (iii) $\frac{\sin n}{n}$,

(iv) $\frac{\sin(1/n)}{(1/n)}$, (v) $(\arctan n)^{-1}$, (vi) $\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+2} - \sqrt{n}}$.

2 Suppose that y satisfies the differential equation

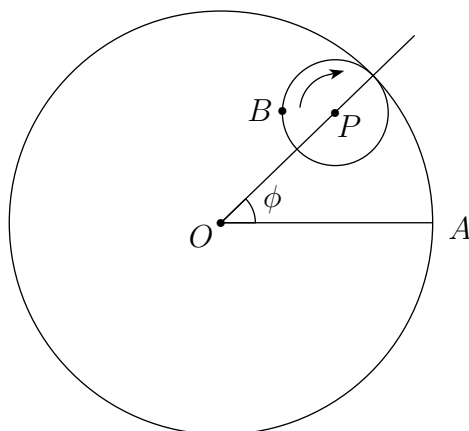
$$y = x \frac{dy}{dx} - \cosh \left(\frac{dy}{dx} \right). \quad (*)$$

(i) By differentiating both sides of $(*)$ with respect to x , show that either

$$\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad x - \sinh \left(\frac{dy}{dx} \right) = 0.$$

(ii) Find the general solutions of each of these two equations. Determine the solutions of $(*)$.

- 3** In the figure, the large circle with centre O has radius 4 and the small circle with centre P has radius 1. The small circle rolls around the inside of the larger one. When P was on the line OA (before the small circle began to roll), the point B was in contact with the point A on the large circle.



- (i) Sketch the curve C traced by B as the circle rolls. Show that if we take O to be the origin of cartesian coordinates and the line OA to be the x -axis (so that A is the point $(4, 0)$) then B is the point

$$(3 \cos \phi + \cos 3\phi, 3 \sin \phi - \sin 3\phi).$$

- (ii) It is given that the area of the region enclosed by the curve C is

$$\int_0^{2\pi} x \frac{dy}{d\phi} d\phi,$$

where B is the point (x, y) . Calculate this area.

- 4** $\diamond$ is an operation which take polynomials in x to polynomials in x ; that is, given a polynomial $h(x)$ there is another polynomial called $\diamond h(x)$. It is given that, if $f(x)$ and $g(x)$ are any two polynomials in x , the following are always true:

(i) $\diamond(f(x)g(x)) = g(x)\diamond f(x) + f(x)\diamond g(x),$

(ii) $\diamond(f(x) + g(x)) = \diamond f(x) + \diamond g(x),$

(iii) $\diamond x = 1$

(iv) λ is a constant then $\diamond(\lambda f(x)) = \lambda \diamond f(x).$

- (i) Show that, if $f(x)$ is a constant (i.e., a polynomial of degree zero), then $\diamond f(x) = 0.$

- (ii) Calculate $\diamond x^2$ and $\diamond x^3.$

- (iii) Prove that $\diamond h(x) = \frac{d}{dx}(h(x))$ for any polynomial $h(x).$

5 (i) Explain what is meant by the order of an element g of a group G .

(ii) The set S consists of all 2×2 matrices whose determinant is 1.

(iii) Find the inverse of the element $\mathbf{A}$ of S , where

$$\mathbf{A} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

(iv) Show that S is a group under matrix multiplication (you may assume that matrix multiplication is associative).

(v) For which elements $\mathbf{A}$ is $\mathbf{A}^{-1} = \mathbf{A}$?

(vi) Which element or elements have order 2?

(vii) Show that the element $\mathbf{A}$ of S has order 3 if, and only if, $w + z + 1 = 0$.

(viii) Write down one such element.

6 (i) Sketch the graphs of $y = \sec x$ and $y = \ln(2 \sec x)$ for $0 \leq x \leq \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2 \sec x)$$

has no solution with $0 \leq x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large.

(ii) Explain why there is a number k_0 such that

$$k_0 x = \ln(2 \sec x)$$

has exactly one solution with $0 \leq x < \frac{1}{2}\pi$.

(iii) Let x_0 be this solution, so that $0 \leq x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2 \sec x)$. Show that

$$x_0 = \cot x_0 \ln(2 \sec x_0).$$

(iv) Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0 .

7 The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c . Express p, q and r in terms of a, b and c .

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

8 Calculate the following integrals

(i) $\int \frac{x}{(x-1)(x^2-1)} dx;$

(ii) $\int \frac{1}{3 \cos x + 4 \sin x} dx;$

(iii) $\int \frac{1}{\sinh x} dx.$

9 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane.

(i) Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters.

(ii) By writing this equation in the form $\mathbf{r} \cdot \mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p , show that

$$-(\lambda + \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

for all scalars λ, μ .

(iii) Deduce that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(iv) Say briefly what happens if A, B, C and O are all in the same plane.

10 Let α be a fixed angle, $0 < \alpha \leq \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $\arg\left(\frac{z-1}{z}\right) = \alpha,$

(ii) $\arg\left(\frac{z-1}{z}\right) = \alpha - \pi,$

(iii) $\left|\frac{z-1}{z}\right| = 1.$

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

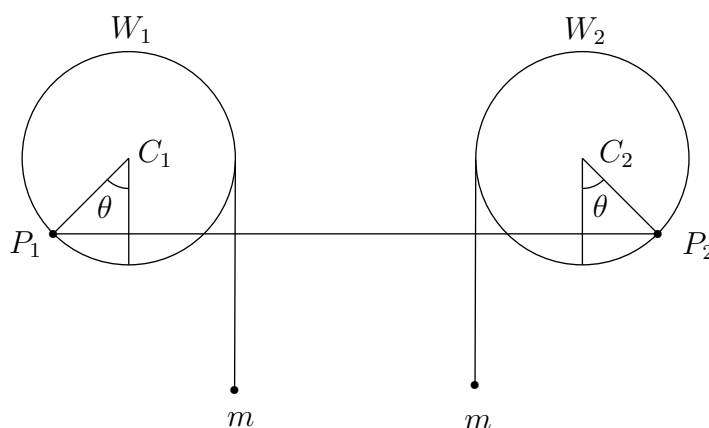
show, by considering $\arg w$, that w is real.

Section B: Mechanics

- 11** I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h . It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V , at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geq g(2h + d).$$

- 12** In the figure, W_1 and W_2 are wheels, both of radius r . Their centres C_1 and C_2 are fixed at the same height, a distance d apart, and each wheel is free to rotate, without friction, about its centre. Both wheels are in the same vertical plane. Particles of mass m are suspended from W_1 and W_2 as shown, by light inextensible strings wound round the wheels. A light elastic string of natural length d and modulus elasticity λ is fixed to the rims of the wheels at the points P_1 and P_2 . The lines joining C_1 to P_1 and C_2 to P_2 both make an angle θ with the vertical. The system is in equilibrium.



Show that

$$\sin 2\theta = \frac{mgd}{\lambda r}.$$

For what value or values of λ (in terms of m, d, r and g) are there

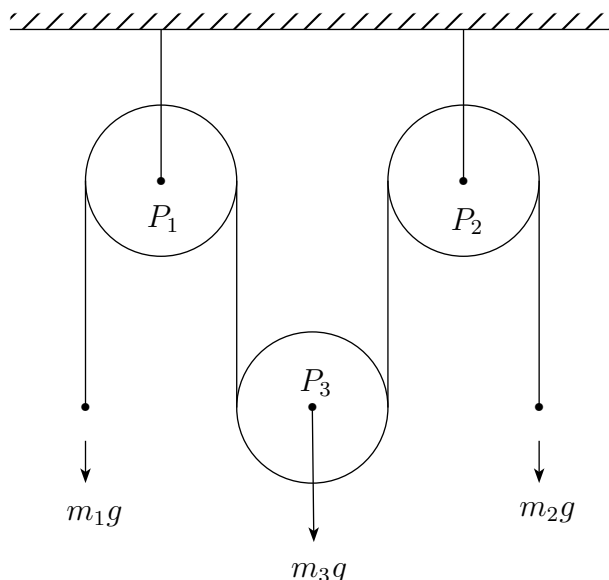
- (i) no equilibrium positions,
- (ii) just one equilibrium position,
- (iii) exactly two equilibrium positions,
- (iv) more than two equilibrium positions?

- 13** Two particles P_1 and P_2 , each of mass m , are joined by a light smooth inextensible string of length ℓ . P_1 lies on a table top a distance d from the edge, and P_2 hangs over the edge of the table and is suspended a distance b above the ground. The coefficient of friction between P_1 and the table top is μ , and $\mu < 1$. The system is released from rest. Show that P_1 will fall off the edge of the table if and only if

$$\mu < \frac{b}{2d - b}.$$

Suppose that $\mu > b/(2d - b)$, so that P_1 comes to rest on the table, and that the coefficient of restitution between P_2 and the floor is e . Show that, if $e > 1/(2\mu)$, then P_1 comes to rest before P_2 bounces a second time.

14



In the diagram P_1 and P_2 are smooth light pulleys fixed at the same height, and P_3 is a third smooth light pulley, freely suspended. A smooth light inextensible string runs over P_1 , under P_3 and over P_2 , as shown: the parts of the string not in contact with any pulley are vertical. A particle of mass m_3 is attached to P_3 . There is a particle of mass m_1 attached to the end of the string below P_1 and a particle of mass m_2 attached to the other end, below P_2 . The system is released from rest. Find the tension in the string, and show that the pulley P_3 will remain at rest if

$$4m_1m_2 = m_3(m_1 + m_2).$$

Section C: Probability and Statistics

- 15** A point moves in unit steps on the x -axis starting from the origin. At each step the point is equally likely to move in the positive or negative direction. The probability that after s steps it is at one of the points $x = 2, x = 3, x = 4$ or $x = 5$ is $P(s)$.

(i) Show that $P(5) = \frac{3}{16}$, $P(6) = \frac{21}{64}$ and

$$P(2k) = \binom{2k+1}{k-1} \left(\frac{1}{2}\right)^{2k}$$

where k is a positive integer. Find a similar expression for $P(2k+1)$.

(ii) Determine the values of s for which $P(s)$ has its greatest value.

- 16** A taxi driver keeps a packet of toffees and a packet of mints in her taxi. From time to time she takes either a toffee (with probability p) or mint (with probability $q = 1 - p$). At the beginning of the week she has n toffees and m mints in the packets. On the N th occasion that she reaches for a sweet, she discovers (for the first time) that she has run out of that kind of sweet. What is the probability that she was reaching for a toffee?