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Sixth Term Examination Paper

91-S3

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Section A: Pure Mathematics

1 **(i)** Evaluate

$$\sum_{r=1}^{n} \frac{6}{r(r+1)(r+3)}.$$

(ii) Expand $\ln(1+x+x^2+x^3)$ as a series in powers of x, where $|x|<1$, giving the first five non-zero terms and the general term.

(iii) Expand $e^{x\ln(1+x)}$ as a series in powers of x, where $-1 < x \leqslant 1$, as far as the term in x^4.

2 **(i)** The distinct points P_1, P_2, P_3, Q_1, Q_2 and Q_3 in the Argand diagram are represented by the complex numbers z_1, z_2, z_3, w_1, w_2 and w_3 respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with P_i corresponding to Q_i $(i = 1, 2, 3)$ and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_2 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

(ii) Verify that this condition may be written

$$\det \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(iii) Show that if $w_i = z_i^2$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(iv) Show that if $w_i = z_i^3$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin.

[The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1 + z_2 + z_3)$.]

(v) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

3 The function f is defined for $x < 2$ by

$$\mathrm{f}(x) = 2\left|x^2 - x\right| + \left|x^2 - 1\right| - 2\left|x^2 + x\right|.$$

(i) Find the maximum and minimum points and the points of inflection of the graph of f and sketch this graph.

Is f continuous everywhere?
Is f differentiable everywhere?

(ii) Find the inverse of the function f, i.e. expressions for $\mathrm{f}^{-1}(x)$, defined in the various appropriate intervals.

4 The point P moves on a straight line in three-dimensional space. The position of P is observed from the points $O_1(0,0,0)$ and $O_2(8a,0,0)$. At times $t = t_1$ and $t = t_1'$, the lines of sight from O_1 are along the lines

$$\frac{x}{2} = \frac{z}{3}, y = 0 \quad \text{and} \quad x = 0, \frac{y}{3} = \frac{z}{4}$$

respectively. At times $t = t_2$ and $t = t_2'$, the lines of sight from O_2 are

$$\frac{x-8a}{-3} = \frac{y}{1} = \frac{z}{3} \quad \text{and} \quad \frac{x-8a}{-4} = \frac{y}{2} = \frac{z}{5}$$

respectively. Find an equation or equations for the path of P.

5 The curve C has the differential equation in polar coordinates

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta, \qquad \text{for} \quad \frac{\pi}{5} \leqslant \theta \leqslant \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2$.
Show that C forms a closed loop and that the area of the region enclosed by C is

$$\frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right].$$

6 The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where u is a positive real constant.

(i) Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself.

(ii) Find the equations of two straight lines through the origin which transform into themselves.

(iii) A line, not necessary through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$.

(iv) The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v_1'$ and $\tanh v_2'$ respectively. Find the relation satisfied by v_1 and v_2 that is the necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms.

(v) In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1, ℓ_2 and their transforms.

7 **(i)** Prove that

$$\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\cos x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x - \tfrac{1}{4}\pi \ln 2$$

and

$$\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x.$$

Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x$.

[You may assume that all the integrals converge.]

(ii) Given that $\ln u < u$ for $u \geqslant 1$ deduce that

$$\tfrac{1}{2}\ln x < \sqrt{x} \qquad \text{for} \qquad x \geqslant 1.$$

Deduce that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$ and that $x \ln x \to 0$ as $x \to 0$ through positive values.

(iii) Using the results of parts **(i)** and **(ii)**, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} x \cot x\,\mathrm{d}x$.

8 **(i)** The integral I_k is defined by

$$I_k = \int_0^\theta \cos^k x \, \cos kx \, \mathrm{d}x.$$

Prove that $2kI_k = kI_{k-1} + \cos^k\theta \sin k\theta$.

(ii) Prove that

$$1 + m\cos 2\theta + \binom{m}{2}\cos 4\theta + \cdots + \binom{m}{r}\cos 2r\theta + \cdots + \cos 2m\theta = 2^m \cos^m\theta \cos m\theta.$$

(iii) Using the results of **(i)** and **(ii)**, show that

$$m\frac{\sin 2\theta}{2} + \binom{m}{2}\frac{\sin 4\theta}{4} + \cdots + \binom{m}{r}\frac{\sin 2r\theta}{2r} + \cdots + \frac{\sin 2m\theta}{2m}$$

is equal to

$$\cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cdots + \frac{1}{r}2^{r-1}\cos^r\theta\sin r\theta + \cdots + \frac{1}{m}2^{m-1}\cos^m\theta\sin m\theta.$$

9 The parametric equations E_1 and E_2 define the same ellipse, in terms of the parameters θ_1 and θ_2, (though not referred to the same coordinate axes).

$$E_1: \quad x = a\cos\theta_1, \qquad y = b\sin\theta_1,$$
$$E_2: \quad x = \frac{k\cos\theta_2}{1+e\cos\theta_2}, \quad y = \frac{k\sin\theta_2}{1+e\cos\theta_2},$$

where $0 < b < a$, $0 < e < 1$ and $0 < k$.

(i) Find the position of the axes for E_2 relative to the axes for E_1 and show that $k = a(1-e^2)$ and $b^2 = a^2(1-e^2)$.

[The standard polar equation of an ellipse is $r = \dfrac{\ell}{1+e\cos\theta}$.]

(ii) By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that

$$\int_0^\pi \sqrt{1-e^2\cos^2\theta}\,\mathrm{d}\theta = \int_0^\pi \frac{1-e^2}{(1+e\cos\theta)^2}\sqrt{1+e^2+2e\cos\theta}\,\mathrm{d}\theta.$$

(iii) Given that e is so small that e^6 may be neglected, show that the value of either integral is

$$\tfrac{1}{64}\pi(64 - 16e^2 - 3e^4).$$

10 The equation

$$x^n - qx^{n-1} + r = 0,$$

where $n \geqslant 5$ and q and r are real constants, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. The sum of the products of m distinct roots is denoted by Σ_m (so that, for example, $\Sigma_3 = \sum \alpha_i \alpha_j \alpha_k$ where the sum runs over the values of i, j and k with $n \geqslant i > j > k \geqslant 1$). The sum of mth powers of the roots is denoted by S_m (so that, for example, $S_3 = \sum_{i=1}^{n} \alpha_i^3$).

(i) Prove that $S_p = p^q$ for $1 \leqslant p \leqslant n-1$.

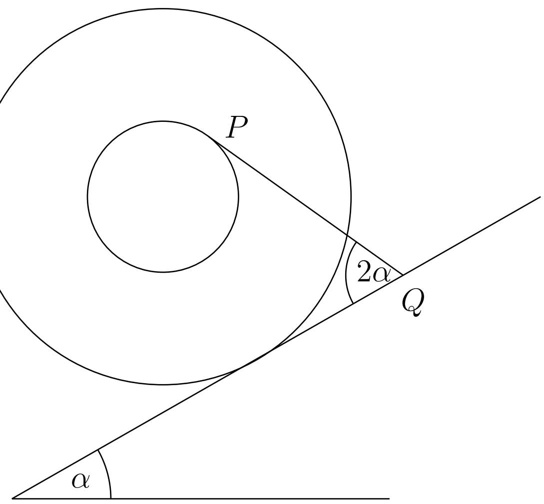
[You may assume that for any nth degree equation and $1 \leqslant p \leqslant n$

$$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 - \cdots + (-1)^{p-1}S_1\Sigma_{p-1} + (-1)^p p\Sigma_p = 0.]$$

(ii) Find expressions for S_n, S_{n+1} and S_{n+2} in terms of q, r and n. Suggest an expression for S_{n+m}, where $m < n$, and prove its validity by induction.

Section B: Mechanics

11



A uniform circular cylinder of radius $2a$ with a groove of radius a cut in its central cross-section has mass M. It rests, as shown in the diagram, on a rough plane inclined at an acute angle α to the horizontal. It is supported by a light inextensible string would round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at Q, the free part of the string, PQ, making an angle 2α with the inclined plane.

(i) The coefficient of friction at the contact between the cylinder and the plane is μ. Show that $\mu \geqslant \frac{1}{3}\tan\alpha$.

(ii) The string PQ is now detached from the plane and the end Q is fastened to a particle of mass $3M$ which is placed on the plane, the position of the string remain unchanged. Given that $\tan\alpha = \frac{1}{2}$ and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.

12 A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a\cos\theta \quad \text{and} \quad y = b\sin\theta$$

where the x-axis is horizontal and the y-axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section.

(i) By first finding $\ddot{x}$ and $\ddot{y}$, or otherwise, show that the acceleration along the inward normal at the point with parameter θ is

$$\frac{ab\dot{\theta}^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

(ii) The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter θ where

$$5\sin^3\theta + 12\sin\theta - 8 = 0.$$

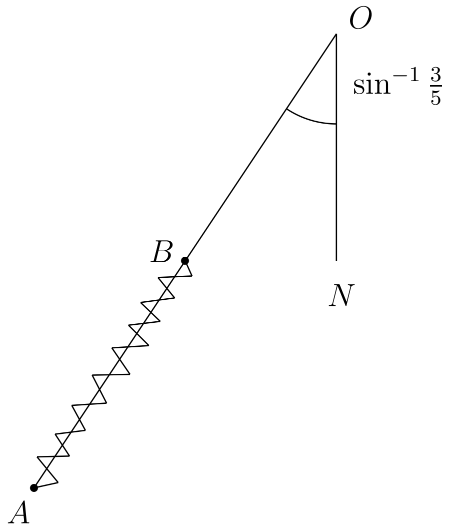
13 A smooth particle P_1 is projected from a point O on the horizontal floor of a room with has a horizontal ceiling at a height h above the floor. The speed of projection is $\sqrt{8gh}$ and the direction of projection makes an acute angle α with the horizontal. The particle strikes the ceiling and rebounds, the impact being perfectly elastic.

(i) Show that for this to happen α must be at least $\frac{1}{6}\pi$ and that the range on the floor is then

$$8h\cos\alpha\left(2\sin\alpha - \sqrt{4\sin^2\alpha - 1}\right).$$

(ii) Another particle P_2 is projected from O with the same velocity as P_1 but its impact with the ceiling is perfectly inelastic. Find the difference D between the ranges of P_1 and P_2 on the floor and show that, as α varies, D has a maximum value when $\alpha = \frac{1}{4}\pi$.

14



The end O of a smooth light rod OA of length $2a$ is a fixed point. The rod OA makes a fixed angle $\sin^{-1}\frac{3}{5}$ with the downward vertical ON, but is free to rotate about ON. A particle of mass m is attached to the rod at A and a small ring B of mass m is free to slide on the rod but is joined to a spring of natural length a and modulus of elasticity kmg. The vertical plane containing the rod OA rotates about ON with constant angular velocity $\sqrt{5g/2a}$ and B is at rest relative to the rod.

(i) Show that the length of OB is

$$\frac{(10k+8)a}{10k-9}.$$

(ii) Given that the reaction of the rod on the particle at A makes an angle $\tan^{-1}\frac{13}{21}$ with the horizontal, find the value of k. Find also the magnitude of the reaction between the rod and the ring B.

Section C: Probability and Statistics

15 A pack of $2n$ (where $n \geqslant 4$) cards consists of two each of n different sorts.

(i) If four cards are drawn from the pack without replacement show that the probability that no pairs of identical cards have been drawn is

$$\frac{4(n-2)(n-3)}{(2n-1)(2n-3)}.$$

(ii) Find the probability that exactly one pair of identical cards is included in the four.

(iii) If k cards are drawn without replacement and $2 < k < 2n$, find an expression for the probability that there are exactly r pairs of identical cards included when $r < \frac{1}{2}k$.

(iv) For even values of k show that the probability that the drawn cards consist of $\frac{1}{2}k$ pairs is

$$\frac{1 \times 3 \times 5 \times \cdots \times (k-1)}{(2n-1)(2n-3)\cdots(2n-k+1)}.$$

16 The random variables X and Y take integer values x and y respectively which are restricted by $x \geqslant 1$, $y \geqslant 1$ and $2x + y \leqslant 2a$ where a is an integer greater than 1. The joint probability is given by

$$\mathrm{P}(X = x, Y = y) = c(2x + y),$$

where c is a positive constant, within this region and zero elsewhere.

(i) Obtain, in terms of x, c and a, the marginal probability $\mathrm{P}(X = x)$ and show that

$$c = \frac{6}{a(a-1)(8a+5)}.$$

(ii) Show that when y is an even number the marginal probability $\mathrm{P}(Y = y)$ is

$$\frac{3(2a-y)(2a+2+y)}{2a(a-1)(8a+5)}$$

and find the corresponding expression when y is off.

(iii) Evaluate $\mathrm{E}(Y)$ in terms of a.