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Sixth Term Examination Paper

90-S1

Compiled by: Dr Yu 郁博士

www.CasperYC.club

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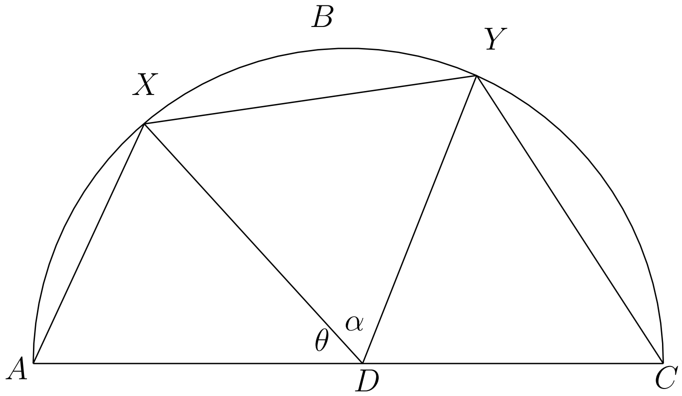
Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1



In the above diagram, $ABCD$ represents a semicircular window of fixed radius r and centre D, and $AXYC$ is a quadrilateral blind.

(i) If $\angle XDY = \alpha$ is fixed and $\angle ADX = \theta$ is variable, determine the value of θ which gives the blind **maximum** area.

(ii) If now α is allowed to vary but r remains fixed, find the maximum possible area of the blind.

2 **(i)** Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$. Show that $1 + \omega + \omega^2 = 0$ and calculate the modulus and argument of $1 + \omega^2$.

(ii) Let n be a positive integer. By evaluating $(1 + \omega^r)^n$ in two ways, taking $r = 1, 2$ and 3, or otherwise, prove that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{k} = \frac{1}{3}\left(2^n + 2\cos\left(\frac{n\pi}{3}\right)\right),$$

where k is the largest multiple of 3 less than or equal to n.

(iii) Without using a calculator, evaluate

$$\binom{25}{0} + \binom{25}{3} + \cdots + \binom{25}{24}$$

and

$$\binom{24}{2} + \binom{24}{5} + \cdots + \binom{24}{23}.$$

$[2^{25} = 33554432.]$

3 Given a curve described by $y = \mathrm{f}(x)$, and such that $y \geqslant 0$, a *push-off* of the curve is a new curve obtained as follows: for each point $(x, \mathrm{f}(x))$ with position vector $\mathbf{r}$ on the original curve, there is a point with position vector $\mathbf{s}$ on the new curve such that $\mathbf{s} - \mathbf{r} = \mathrm{p}(x)\mathbf{n}$, where p is a given function and $\mathbf{n}$ is the downward-pointing unit normal to the original curve at $\mathbf{r}$.

(i) For the curve $y = x^k$, where $x > 0$ and k is a positive integer, obtain the function p for which the push-off is the positive x-axis, and find the value of k such that, for all points on the original curve, $|\mathbf{r}| = |\mathbf{r} - \mathbf{s}|$.

(ii) Suppose that the original curve is $y = x^2$ and p is such that the gradient of the curves at the points with position vectors $\mathbf{r}$ and $\mathbf{s}$ are equal (for every point on the original curve). By writing $\mathrm{p}(x) = \mathrm{q}(x)\sqrt{1 + 4x^2}$, where q is to be determined, or otherwise, find the form of p.

4 **(i)** The sequence $a_1, a_2, \ldots, a_n, \ldots$ forms an arithmetic progression. Establish a formula, involving n, a_1 and a_2 for the sum $a_1 + a_2 + \cdots + a_n$ of the first n terms.

(ii) A sequence $b_1, b_2, \ldots, b_n, \ldots$ is called a *double arithmetic progression* if the sequence of differences

$$b_2 - b_1, b_3 - b_2, \ldots, b_{n+1} - b_n, \ldots$$

is an arithmetic progression. Establish a formula, involving n, b_1, b_2 and b_3, for the sum $b_1 + b_2 + b_3 + \cdots + b_n$ of the first n terms of such a progression.

(iii) A sequence $c_1, c_2, \ldots, c_n, \ldots$ is called a *factorial progression* if $c_{n+1} - c_n = n!d$ for some non-zero d and every $n \geqslant 1$. Suppose $1, b_2, b_3, \ldots$ is a double arithmetic progression, and also that b_2, b_4, b_6 and 220 are the first four terms in a factorial progression. Find the sum $1 + b_2 + b_3 + \cdots + b_n$.

5 **(i)** Evaluate

$$\int_1^3 \frac{1}{6x^2 + 19x + 15}\, \mathrm{d}x\,.$$

(ii) Sketch the graph of the function f, where $\mathrm{f}(x) = x^{1760} - x^{220} + q$, and q is a constant. Find the possible numbers of *distinct* roots of the equation $\mathrm{f}(x) = 0$, and state the inequalities satisfied by q.

6 Let $ABCD$ be a parallelogram. By using vectors, or otherwise, prove that:

(i) $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$;

(ii) $|AC^2 - BD^2|$ is 4 times the area of the rectangle whose sides are *any* side of the parallelogram and the projection of an adjacent side on that side.

(iii) State and prove a result like **(ii)** about $|AB^2 - AD^2|$ and the diagonals.

7 **(i)** Let y, u, v, P and Q all be functions of x. Show that the substitution $y = uv$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

leads to an equation for $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ in terms of x, Q and u, provided that u satisfies a suitable first order differential equation.

(ii) Hence or otherwise solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}}\,,$$

given that $y(1) = 0$. For what set of values of x is the solution valid?

8 **(i)** Show that

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right) = \frac{\sin\alpha}{4\sin\left(\frac{\alpha}{4}\right)},$$

where $\alpha \neq k\pi$, k is an integer.

(ii) Prove that, for such α,

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right) = \frac{\sin\alpha}{2^n\sin\left(\frac{\alpha}{2^n}\right)},$$

where n is a positive integer.

(iii) Deduce that

$$\alpha = \frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\cdots},$$

and hence that

$$\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

9 **(i)** Let A and B be the points $(1,1)$ and $(b,1/b)$ respectively, where $b>1$. The tangents at A and B to the curve $y=1/x$ intersect at C. Find the coordinates of C.

(ii) Let A', B' and C' denote the projections of A, B and C, respectively, to the x-axis. Obtain an expression for the sum of the areas of the quadrilaterals $ACC'A'$ and $CBB'C'$.

(iii) Hence or otherwise prove that, for $z>0$,

$$\frac{2z}{2+z} \leqslant \ln(1+z) \leqslant z.$$

Section B: Mechanics

10 In a certain race, runners run 5 km in a straight line to a fixed point and then turn and run back to the starting point. A steady wind of 3 ms^{-1} is blowing from the start to the turning point. At steady racing pace, a certain runner expends energy at a constant rate of 300 W. Two resistive forces act. One is of constant magnitude 50 N. The other, arising from air resistance, is of magnitude $2w\,\mathrm{N}$, where w ms^{-1} is the runner's speed relative to the air.

(i) Give a careful argument to derive formulae from which the runner's steady speed in each half of the race may be found. Calculate, to the nearest second, the time the runner will take for the whole race.

[Effects due to acceleration and deceleration at the start and turn may be ignored.]

(ii) The runner may use alternative tactics, expending the same total energy during the race as a whole, but applying different constant powers, x_1 W in the outward trip, and x_2 W on the return trip. Prove that, with the wind as above, if the outward and return speeds are v_1 ms^{-1} and v_2 ms^{-1} respectively, then $v_1 + v_2$ is independent of the choices for x_1 and x_2.

(iii) Hence show that these alternative tactics allow the runner to run the whole race approximately 15 seconds faster.

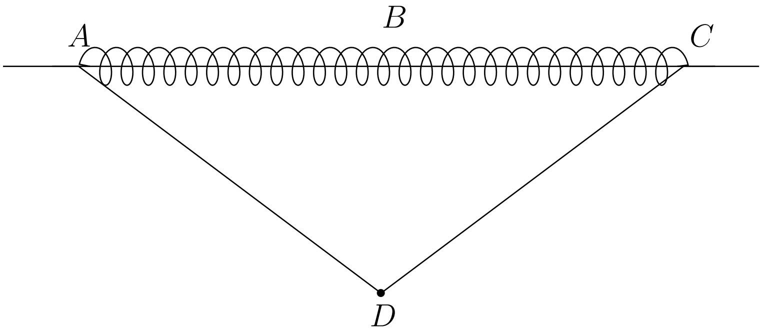
11 A shell of mass m is fired at elevation $\pi/3$ and speed v. Superman, of mass $2m$, catches the shell at the top of its flight, by gliding up behind it in the same horizontal direction with speed $3v$. As soon as Superman catches the shell, he instantaneously clasps it in his cloak, and immediately pushes it vertically downwards, without further changing its horizontal component of velocity, but giving it a downward vertical component of velocity of magnitude $3v/2$.

(i) Calculate the total time of flight of the shell in terms of v and g.

(ii) Calculate also, to the nearest degree, the angle Superman's flight trajectory initially makes with the horizontal after releasing the shell, as he soars upwards like a bird.

[Superman and the shell may be regarded as particles.]

12



In the above diagram, ABC represents a light spring of natural length $2l$ and modulus of elasticity λ, which is coiled round a smooth fixed horizontal rod. B is the midpoint of AC. The two ends of a light inelastic string of length $2l$ are attached to the spring at A and C. A particle of mass m is fixed to the string at D, the midpoint of the string. The system can be in equilibrium with the angle CAD equal to $\pi/6$.

(i) Show that

$$mg = \lambda\left(\frac{2}{\sqrt{3}} - 1\right).$$

(ii) Write the length AC as $2xl$, obtain an expression for the potential energy of the system as a function of x.

(iii) The particle is held at B, and the spring is restored to its natural length $2l$. The particle is then released and falls vertically. Obtain an equation satisfied by x when the particle next comes to rest. Verify numerically that a possible solution for x is approximately 0.66.

13 A rough circular cylinder of mass M and radius a rests on a rough horizontal plane. The curved surface of the cylinder is in contact with a smooth rail, parallel to the axis of the cylinder, which touches the cylinder at a height $a/2$ above the plane. Initially the cylinder is held at rest. A particle of mass m rests in equilibrium on the cylinder, and the normal reaction of the cylinder on the particle makes an angle of θ with the upward vertical. The particle is on the same side of the centre of the cylinder as the rail. The coefficient of friction between the cylinder and the particle and between the cylinder and the plane are both μ.

(i) Obtain the condition on θ for the particle to rest in equilibrium.

(ii) Show that, if the cylinder is released, equilibrium of both particle and cylinder is possible provided another inequality involving μ and θ (which should be found explicitly) is satisfied.

(iii) Determine the largest possible value of θ for equilibrium, if $m = 7M$ and $\mu = 0.75$.

Section C: Probability and Statistics

14 A bag contains 5 white balls, 3 red balls and 2 black balls. In the game of Blackball, a player draws a ball at random from the bag, looks at it and replaces it. If he has drawn a white ball, he scores one point, while for a red ball he scores two points, these scores being added to his total score before he drew the ball. If he has drawn a black ball, the game is over and his final score is zero. After drawing a red or white ball, he can either decide to stop, when his final score for the game is the total so far, or he may elect to draw another ball. The starting score is zero.

(i) Juggins' strategy is to continue drawing until either he draws a black ball (when of course he must stop, with final score zero), or until he has drawn three (non-black) balls, when he elects to stop. Find the probability that in any game he achieves a final score of zero by employing this strategy. Find also his expected final score.

(ii) Muggins has so far scored N points, and is deciding whether to draw another ball. Find the expected score if another ball is drawn, and suggest a strategy to achieve the greatest possible average final score in each game.

15 **(i)** A coin has probability p $(0 < p < 1)$ of showing a head when tossed.
Give a careful argument to show that the kth head in a series of consecutive tosses is achieved after **exactly** n tosses with probability

$$\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad (n \geqslant k).$$

(ii) Given that it took an even number of tosses to achieve exactly $k-1$ heads, find the probability that exactly k heads are achieved after an even number of tosses.

(iii) If this coin is tossed until exactly 3 heads are obtained, what is the probability that **exactly** 2 of the heads occur on even–numbered tosses?

16 A bus is supposed to stop outside my house every hour on the hour. From long observation I know that a bus will always arrive some time between 10 minutes before and ten minutes after the hour. The probability it arrives at a given instant increases linearly (from zero at 10 minutes before the hour) up to a maximum value at the hour, and then decreases linearly at the same rate after the hour.

(i) Obtain the probability density function of T, the time in minutes after the scheduled time at which a bus arrives.

(ii) If I get up when my alarm clock goes off, I arrive at the bus stop at 7.55am. However, with probability 0.5, I doze for 3 minutes before it rings again. In that case with probability 0.8 I get up then and reach the bus stop at 7.58am, or, with probability 0.2, I sleep a little longer, not reaching the stop until 8.02am. What is the probability that I catch a bus by 8.10am?

(iii) I buy a louder alarm clock which ensures that I reach the stop at exactly the same time each morning. This clock keeps perfect time, but may be set to an incorrect time. If it is correct, the alarm goes off so that I should reach the stop at 7.55am. After 100 mornings I find that I have had to wait for a bus until **after** 9am (according to the new clock) on 5 occasions. Is this evidence that the new clock is incorrectly set?

[The time of arrival of different buses are independent of each other.]