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Sixth Term Examination Paper

24-S2

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Section A: Pure Mathematics

1 In the equality

$$4 + 5 + 6 + 7 + 8 = 9 + 10 + 11,$$

the sum of the five consecutive integers from 4 upwards is equal to the sum of the next three consecutive integers.

Throughout this question, the variables n, k and c represent positive integers.

(i) Show that the sum of the $n + k$ consecutive integers from c upwards is equal to the sum of the next n consecutive integers if and only if

$$2n^2 + k = 2ck + k^2.$$

(ii) Find the set of possible values of n, and the corresponding values of c, in each of the cases

(a) $k = 1$

(b) $k = 2$.

(iii) Show that there are no solutions for c and n if $k = 4$.

(iv) Consider now the case where $c = 1$.

(a) Find two possible values of k and the corresponding values of n.

(b) Show, given a possible value N of n, and the corresponding value K of k, that

$$N' = 3N + 2K + 1$$

will also be a possible value of n, with

$$K' = 4N + 3K + 1$$

as the corresponding value of k.

(c) Find two further possible values of k and the corresponding values of n.

2 *In this question, you need not consider issues of convergence.*

(i) Find the binomial series expansion of $(8+x^3)^{-1}$, valid for $|x|<2$.

Hence show that, for each integer $m \geqslant 0$,

$$\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right).$$

(ii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right)=\int_0^1 \frac{1}{x+2}\,\mathrm{d}x\,,$$

and evaluate the integral.

(iii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{72(2k+1)}{(3k+1)(3k+2)}\right)=\pi\sqrt{a}-\ln b,$$

where a and b are integers which you should determine.

3 The unit circle is the circle with radius 1 and centre the origin, O.

N and P are distinct points on the unit circle.
N has coordinates $(-1, 0)$, and P has coordinates $(\cos\theta, \sin\theta\ )$, where $-\pi < \theta < \pi$.
The line NP intersects the y–axis at Q, which has coordinates $(0, q)$.

(i) Show that $q = \tan\frac{1}{2}\theta$.

(ii) In this part, $q \neq 1$.

(a) Let $\mathrm{f}_1(q) = \dfrac{1+q}{1-q}$. Show that $\mathrm{f}_1(q) = \tan\dfrac{1}{2}\left(\theta + \dfrac{1}{2}\pi\right)$.

(b) Let Q_1 be the point with coordinates $(0, \mathrm{f}_1(q))$ and
P_1 be the point of intersection (other than N) of the line NQ_1 and the unit circle.
Describe geometrically the relationship between P and P_1.

(iii) **(a)** P_2 is the image of P under an anti–clockwise rotation about O through angle $\frac{1}{3}\pi$.
The line NP_2 intersects the y–axis at the point Q_2 with co–ordinates $(0, \mathrm{f}_2(q))$.
Find $\mathrm{f}_2(q)$ in terms of q, for $q \neq \sqrt{3}$.

(b) In this part, $q \neq -1$. Let $\mathrm{f}_3(q) = \dfrac{1-q}{1+q}$, let Q_3 be the point with coordinates $(0, \mathrm{f}_3(q))$ and let P_3 be the point of intersection (other than N) of the line NQ_3 and the unit circle.
Describe geometrically the relationship between P and P_3.

(c) In this part, $0 < q < 1$. Let $\mathrm{f}_4(q) = \mathrm{f}_2^{-1}\left(\mathrm{f}_3\left(\mathrm{f}_2(q)\right)\right)$, let Q_4 be the point with coordinates $(0, \mathrm{f}_4(q))$ and let P_4 be the point of intersection (other than N) of the line NQ_4 and the unit circle. Describe geometrically the relationship between P and P_4.

4 In this question, if O, C and D are non–collinear points in three dimensional space, we will call the non–zero vector $\mathbf{v}$ a *bisecting vector* for angle COD if $\mathbf{v}$ lies in the plane COD, the angle between $\mathbf{v}$ and $\overrightarrow{OC}$ is equal to the angle between $\mathbf{v}$ and $\overrightarrow{OD}$, and both angles are less than $90°$.

(i) Let O, X and Y be non–collinear points in three–dimensional space, and define $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$.

Let $\mathbf{b} = |\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}$.

(a) Show that $\mathbf{b}$ is a bisecting vector for angle XOY.
Explain, using a diagram, why any other bisecting vector for angle XOY is a positive multiple of $\mathbf{b}$.

(b) Find the value of λ such that the point B, defined by $\overrightarrow{OB} = \lambda\mathbf{b}$, lies on the line XY.
Find also the ratio in which the point B divides XY.

(c) Show, in the case when OB is perpendicular to XY, that the triangle XOY is isosceles.

(ii) Let O, P, Q and R be points in three–dimensional space, no three of which are collinear.
A bisecting vector is chosen for each of the angles POQ, QOR and ROP.
Show that the three angles between them are either all acute, all obtuse or all right angles.

5 **(i)** The functions f_1 and F_1, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_1(n) = n^2 + 6n + 11,$$
$$\mathrm{F}_1(n) = n^2 + 2.$$

Show that F_1 has the same range as f_1.

(ii) The function g_1, with domain $\mathbb{Z}$, is defined by

$$\mathrm{g}_1(n) = n^2 - 2n + 5.$$

Show that the ranges of f_1 and g_1 have empty intersection.

(iii) The functions f_2 and g_2, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_2(n) = n^2 - 2n - 6,$$
$$\mathrm{g}_2(n) = n^2 - 4n + 2.$$

Find any integers that lie in the intersection of the ranges of the two functions.

(iv) Show that $p^2 + pq + q^2 \geqslant 0$ for all real p and q.

The functions f_3 and g_3, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_3(n) = n^3 - 3n^2 + 7n,$$
$$\mathrm{g}_3(n) = n^3 + 4n - 6.$$

Find any integers that lie in the intersection of the ranges of the two functions.

6 *In this question, you need not consider issues of convergence.*

(i) The sequence T_n, for $n = 0, 1, 2, \ldots$, is defined by $T_0 = 1$ and, for $n \geqslant 1$, by

$$T_n = \frac{2n-1}{2n} T_{n-1}$$

Prove by induction that

$$T_n = \frac{1}{2^{2n}} \binom{2n}{n},$$

for $n = 0, 1, 2, \ldots$.

$\left[\text{Note that } \binom{0}{0} = 1.\right]$

(ii) Show that in the binomial series for $(1-x)^{-\frac{1}{2}}$,

$$(1-x)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} a_r x^r\,,$$

successive coefficients are related by

$$a_r = \frac{2r-1}{2r} a_{r-1}$$

for $r = 1, 2, \ldots$.

Hence prove that $a_r = T_r$ for all $r = 0, 1, 2, \ldots$.

(iii) Let b_r be the coefficient of x^r in the binomial series for $(1-x)^{-\frac{3}{2}}$, so that

$$(1-x)^{-\frac{3}{2}} = \sum_{r=0}^{\infty} b_r x^r$$

By considering $\dfrac{b_r}{a_r}$, find an expression involving a binomial coefficient for b_r, for $r = 0, 1, 2, \ldots$.

(iv) By considering the product of the binomial series for $(1-x)^{-\frac{1}{2}}$ and $(1-x)^{-1}$, prove that

$$\frac{(2n+1)}{2^{2n}} \binom{2n}{n} = \sum_{r=0}^{n} \frac{1}{2^{2r}} \binom{2r}{r}$$

for $n = 1, 2, \ldots$.

7 **(i)** Sketch the curve C_1 with equation

$$\left(y^2+(x-1)^2-1\right)\left(y^2+(x+1)^2-1\right)=0.$$

(ii) Consider the curve C_2 with equation

$$\left(y^2+(x-1)^2-1\right)\left(y^2+(x+1)^2-1\right)=\frac{1}{16}.$$

(a) Show that the line $y=k$ meets the curve C_2 at points for which

$$x^4+2\left(k^2-2\right)x^2+\left(k^4-\frac{1}{16}\right)=0.$$

Hence determine the number of intersections between curve C_2 and the line $y=k$ for each positive value of k.

(b) Determine whether the points on curve C_2 with the greatest possible y–coordinate are further from, or closer to, the y–axis than those on curve C_1.

(c) Show that it is not possible for both $y^2+(x-1)^2-1$ and $y^2+(x+1)^2-1$ to be negative, and deduce that curve C_2 lies entirely outside curve C_1.

(d) Sketch the curves C_1 and C_2 on the same axes.

8 In this question, the following theorem may be used without proof.
Let $u_1, u_2, \ldots$ be a sequence of real numbers. If the sequence is

- bounded above, so $u_n \leqslant b$ for all n, where b is some fixed number
- and increasing, so $u_n \leqslant u_{n+1}$ for all n

then there is a number $L \leqslant b$ such that $u_n \to L$ as $n \to \infty$.

For positive real numbers x and y, define $\mathrm{a}(x, y) = \frac{1}{2}(x + y)$ and $\mathrm{g}(x, y) = \sqrt{xy}$.
Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geqslant 0$

$$x_{n+1} = \mathrm{a}\left(x_n, y_n\right),$$
$$y_{n+1} = \mathrm{g}\left(x_n, y_n\right).$$

(i) By considering $\left(\sqrt{x_n} - \sqrt{y_n}\right)^2$, show that $y_{n+1} < x_{n+1}$, for $n \geqslant 0$.
Show further that, for $n \geqslant 0$

- $x_{n+1} < x_n$
- $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \to M$ as $n \to \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}\left(x_n - y_n\right)$ and hence that $x_n - y_n \to 0$ as $n \to \infty$.

Explain why x_n also tends to M as $n \to \infty$.

(ii) Let

$$\mathrm{I}(p, q) = \int_0^\infty \frac{1}{\sqrt{\left(p^2 + x^2\right)\left(q^2 + x^2\right)}}\,\mathrm{d}x.$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2}\left(x - \frac{pq}{x}\right)$ in the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)\left(pq + t^2\right)}}\,\mathrm{d}t$$

that

$$\mathrm{I}(p, q) = \mathrm{I}(\mathrm{a}(p, q), \mathrm{g}(p, q)).$$

Hence evaluate $\mathrm{I}\left(x_0, y_0\right)$ in terms of M.

Section B: Mechanics

9 A long straight trench, with rectangular cross section, has been dug in otherwise horizontal ground. The width of the trench is d and its depth $2d$. A particle is projected at speed v, where $v^2 = \lambda dg$, at an angle α to the horizontal, from a point on the ground a distance d from the nearer edge of the trench. The vertical plane in which it moves is perpendicular to the trench.

(i) The particle lands on the base of the trench without first touching either of its sides.

(a) By considering the vertical displacement of the particle when its horizontal displacement is d, show that $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ and deduce that $\lambda > 1$.

(b) Show also that $(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$ and deduce that $\alpha > 45°$.

(ii) Show that, provided $\lambda > 1, \alpha$ can always be chosen so that the particle lands on the base of the trench without first touching either of its sides.

10 A triangular prism lies on a horizontal plane. One of the rectangular faces of the prism is vertical; the second is horizontal and in contact with the plane; the third, oblique rectangular face makes an angle α with the horizontal. The two triangular faces of the prism are right angled triangles and are vertical. The prism has mass M and it can move without friction across the plane.

A particle of mass m lies on the oblique surface of the prism. The contact between the particle and the plane is rough, with coefficient of friction μ.

(i) Show that if $\mu < \tan\alpha$, then the system cannot be in equilibrium.

Let $\mu = \tan\lambda$, with $0 < \lambda < \alpha < \frac{1}{4}\pi$.
A force P is exerted on the vertical rectangular face of the prism, perpendicular to that face and directed towards the interior of the prism. The particle and prism accelerate, but the particle remains in the same position relative to the prism.

(ii) Show that the magnitude, F, of the frictional force between the particle and the prism is

$$F = \frac{m}{M+m}|(M+m)g\sin\alpha - P\cos\alpha|.$$

Find a similar expression for the magnitude, N, of the normal reaction between the particle and the prism.

(iii) Hence show that the force P must satisfy

$$(M+m)g\tan(\alpha - \lambda) \leqslant P \leqslant (M+m)g\tan(\alpha + \lambda).$$

Section C: Probability and Statistics

11 **(i)** Sketch a graph of $y = x^{\frac{1}{x}}$ for $x > 0$, showing the location of any turning points.

Find the maximum value of $n^{\frac{1}{n}}$, where n is a positive integer.

N people are to have their blood tested for the presence or absence of an enzyme. Each person, independently of the others, has a probability p of having the enzyme present in a sample of their blood, where $0 < p < 1$. The blood test always correctly determines whether the enzyme is present or absent in a sample.

The following method is used.

- The people to be tested are split into r groups of size k, with $k > 1$ and $rk = N$.
- In every group, a sample from each person in that group is mixed into one large sample, which is then tested.
- If the enzyme is not present in the combined sample from a group, no further testing of the people in that group is needed.
- If the enzyme is present in the combined sample from a group, a second sample from each person in that group is tested separately.

(ii) Find, in terms of N, k and p, the expected number of tests.

(iii) Given that N is a multiple of 3, find the largest value of p for which it is possible to find an integer value of k such that $k > 1$ and the expected number of tests is at most N.

Show that this value of p is greater than $\dfrac{1}{4}$.

(iv) Show that, if pk is sufficiently small, the expected number of tests is approximately $N\left(\frac{1}{k} + pk\right)$.
In the case where $p = 0.01$, show that choosing $k = 10$ gives an expected number of tests which is only about 20% of N.

12 *In this question, you may use without proof the results*

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \quad \text{and} \quad \sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

Throughout the question, n and k are integers with $n \geqslant 3$ and $k \geqslant 2$.

(i) In a game, k players, including Ada, are each given a random whole number from 1 to n (that is, for each player, each of these numbers is equally likely and assigned independently of all the others). A player wins the game if they are given a smaller number than all the other players, so there may be no winner in this game.

Find an expression, in terms of n, k and a, for the probability that Ada is given number a, where $1 \leqslant a \leqslant n-1$, and all the other players are given larger numbers. Hence show that, if $k = 4$, the probability that there is a winner in this game is

$$\frac{(n-1)^2}{n^2}.$$

(ii) In a second game, k players, including Ada and Bob, are each given a random whole number from 1 to n. A player wins the game if they are given a smaller number than all the other players or if they are given a larger number than all the other players, so it is possible for there to be zero, one or two winners in this game. Find an expression, in terms of n, k and d, for the probability that Ada is given number a and Bob is given number $a + d + 1$, where $1 \leqslant d \leqslant n-2$ and $1 \leqslant a \leqslant n-d-1$, and all the other players are given numbers greater than a and less than $a + d + 1$. Hence show that, if $k = 4$, the probability that there are two winners in this game is

$$\frac{(n-2)(n-1)^2}{n^3}.$$

If $k = 4$, what is the minimum value of n for which there are more likely to be exactly two winners than exactly one winner in this game?