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Sixth Term Examination Paper

23-S2

Compiled by: Dr Yu 郁博士

www.CasperYC.club

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Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1 **(i)** Show that making the substitution $x = \dfrac{1}{t}$ in the integral

$$\int_b^a \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x,$$

where $b > a > 0$, gives the integral

$$\int_{a^{-1}}^{b^{-1}} \frac{-t}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t.$$

(ii) Evaluate:

(a)

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x;$$

(b)

$$\int_{-2}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

(iii) **(a)** Show that

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int_{\frac{1}{2}}^{2} \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,\mathrm{d}x,$$

and hence evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x.$$

(b) Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1-x}{x\,(1+x^2)^{\frac{1}{2}}}\,\mathrm{d}x.$$

2 **(i)** The real numbers x, y and zsatisfy the equations

$$y = \frac{2x}{1-x^2},$$
$$z = \frac{2y}{1-y^2},$$
$$x = \frac{2z}{1-z^2}.$$

Let $x = \tan\alpha$. Deduce that $y = \tan(2\alpha)$ and show that $\tan\alpha = \tan 8\alpha$.

Find all solutions of the equations, giving each value of x, y and z in the form $\tan\theta$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(ii) Determine the number of real solutions of the simultaneous equations

$$y = \frac{3x - x^3}{1-3x^2},$$
$$z = \frac{3y - y^3}{1-3y^2},$$
$$x = \frac{3z - z^3}{1-3z^2}.$$

(iii) Consider the simultaneous equations

$$y = 2x^2 - 1,$$
$$z = 2y^2 - 1,$$
$$x = 2z^2 - 1.$$

(a) Determine the number of real solutions of these simultaneous equations with $|x| \leqslant 1, |y| \leqslant 1, |z| \leqslant 1$.

(b) By finding the degree of a single polynomial equation which is satisfied by x, show that all solutions of these simultaneous equations have $|x| \leqslant 1, |y| \leqslant 1, |z| \leqslant 1$.

3 Let $\mathrm{p}(x)$ be a polynomial of degree n with $\mathrm{p}(x) > 0$ for all x and let

$$\mathrm{q}(x) = \sum_{k=0}^{n} \mathrm{p}^{(k)}(x),$$

where $\mathrm{p}^{(k)}(x) \equiv \dfrac{\mathrm{d}^k \mathrm{p}(x)}{\mathrm{d}x^k}$ for $k \geqslant 1$ and $\mathrm{p}^{(0)}(x) \equiv \mathrm{p}(x)$.

(i) **(a)** Explain why n must be even and show that $\mathrm{q}(x)$ takes positive values for some values of x.

(b) Show that $\mathrm{q}'(x) = \mathrm{q}(x) - \mathrm{p}(x)$.

(ii) In this part you will be asked to show the same result in three different ways.

(a) Show that the curves $y = \mathrm{p}(x)$ and $y = \mathrm{q}(x)$ meet at every stationary pointof $y = \mathrm{q}(x)$.

Hence show that $\mathrm{q}(x) > 0$ for all x.

(b) Show that $\mathrm{e}^{-x}\mathrm{q}(x)$ is a decreasing function.

Hence show that $\mathrm{q}(x) > 0$ for all x.

(c) Show that

$$\int_0^\infty \mathrm{p}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{p}(x) + \int_0^\infty \mathrm{p}^{(1)}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t.$$

Show further that

$$\int_0^\infty \mathrm{p}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{q}(x).$$

Hence show that $\mathrm{q}(x) > 0$ for all x.

4 **(i)** Show that, if $(x - \sqrt{2})^2 = 3$, then $x^4 - 10x^2 + 1 = 0$.

Deduce that, if $\mathrm{f}(x) = x^4 - 10x^2 + 1$, then $\mathrm{f}(\sqrt{2} + \sqrt{3}) = 0$.

(ii) Find a polynomial g of degree 8 with integer coefficients such that $\mathrm{g}\left(\sqrt{2} + \sqrt{3} + \sqrt{5}\right) = 0$. Write your answer in a form without brackets.

(iii) Let a, b and c be the three roots of $t^3 - 3t + 1 = 0$.

Find a polynomial h of degree 6 with integer coefficients such that $h(a + \sqrt{2}) = 0$, $h(b + \sqrt{2}) = 0$ and $h(c + \sqrt{2}) = 0$. Write your answer in a form without brackets.

(iv) Find a polynomial k with integer coefficients such that $k\left(\sqrt[3]{2} + \sqrt[3]{3}\right) = 0$. Write your answer in a form without brackets.

5 **(i)** The sequence x_n for $n = 0, 1, 2, \ldots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geqslant 0$.

(a) Explain briefly why $x_n \geqslant 1$ for all n.

(b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$\left|x_{n+1}^2 - 2\right| \leqslant \frac{1}{4}\left|x_n^2 - 2\right|$$

(c) Show that

$$2 - 10^{-6} \leqslant x_{10}^2 \leqslant 2.$$

(ii) The sequence y_n for $n = 0, 1, 2, \ldots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geqslant 0$.

(a) Show that, for $n \geqslant 0$,

$$y_{n+1} - \sqrt{2} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$$

and deduce that $y_n \geqslant 1$ for $n \geqslant 0$.

(b) Show that

$$y_n - \sqrt{2} \leqslant 2\left(\frac{\sqrt{2} - 1}{2}\right)^{2^n}$$

for $n \geqslant 1$.

(c) Using the fact that

$$\sqrt{2} - 1 < \frac{1}{2},$$

or otherwise, show that

$$\sqrt{2} \leqslant y_{10} \leqslant \sqrt{2} + 10^{-600}.$$

6 The sequence F_n, for $n = 0, 1, 2, \ldots$, is defined by $F_0 = 0, F_1 = 1$ and by $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$.

Prove by induction that, for all positive integers n,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n,$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) By considering the matrix $\mathbf{Q}^n$, show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all positive integers n.

(ii) By considering the matrix $\mathbf{Q}^{m+n}$, show that $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ for all positive integers m and n.

(iii) Show that $\mathbf{Q}^2 = \mathbf{I} + \mathbf{Q}$.

In the following parts, you may use without proof the Binomial Theorem for matrices:

$$(\mathbf{I} + \mathbf{A})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k.$$

(a) Show that, for all positive integers n,

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k.$$

(b) Show that, for all positive integers n,

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k$$

and also that

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}.$$

(c) Show that, for all positive integers n,

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k} = 0.$$

7 **(i)** The complex numbers z and w have real and imaginary parts given by $z = a + \mathbf{i}b$ and $w = c + \mathbf{i}d$. Prove that $|zw| = |z||w|$.

(ii) By considering the complex numbers $2 + \mathbf{i}$ and $10 + 11\mathbf{i}$, find positive integers h and k such that $h^2 + k^2 = 5 \times 221$.

(iii) Find positive integers m and n such that $m^2 + n^2 = 8045$.

(iv) You are given that $102^2 + 201^2 = 50805$.

Find positive integers p and q such that $p^2 + q^2 = 36 \times 50805$.

(v) Find three distinct pairs of positive integers r and s such that $r^2 + s^2 = 25 \times 1002082$ and $r < s$.

(vi) You are given that $109 \times 9193 = 1002037$.

Find positive integers t and u such that $t^2 + u^2 = 9193$.

8 A tetrahedron is called isosceles if each pair of edges which do not share a vertex have equal length.

(i) Prove that a tetrahedron is isosceles if and only if all four faces have the same perimeter.

Let $OABC$ be an isosceles tetrahedron and let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(ii) By considering the lengths of OA and BC, show that

$$2\mathbf{b}.\mathbf{c} = |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

Show that

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2.$$

(iii) Let G be the *centroid* of the tetrahedron, defined by $\overrightarrow{OG} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Show that G is equidistant from all four vertices of the tetrahedron.

(iv) By considering the length of the vector $\mathbf{a} - \mathbf{b} - \mathbf{c}$, or otherwise, show that, in an isosceles tetrahedron, none of the angles between pairs of edges which share a vertex can be obtuse. Can any of them be right angles?

Section B: Mechanics

9 A truck of mass M is connected by a light, rigid tow–bar, which is parallel to the ground, to a trailer of mass kM. A constant driving force D which is parallel to the ground acts on the truck, and the only resistance to motion is a frictional force acting on the trailer, with coefficient of friction μ.

- When the truck pulls the trailer up a slope which makes an angle α to the horizontal, the acceleration is a_1 and there is a tension T_1 in the tow-bar.
- When the truck pulls the trailer on horizontal ground, the acceleration is a_2 and there is a tension T_2 in the tow-bar.
- When the truck pulls the trailer down a slope which makes an angle α to the horizontal, the acceleration is a_3 and there is a tension T_3 in the tow-bar.

All accelerations are taken to be positive when in the direction of motion of the truck.

(i) Show that $T_1 = T_3$ and that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$.

(ii) It is given that $\mu < 1$.

(a) Show that

$$a_2 < \frac{1}{2}(a_1 + a_3) < a_3.$$

(b) Show further that

$$a_1 < a_2.$$

10 In this question, the x-and y-axes are horizontal and the z-axis is vertically upwards.

(i) A particle P_α is projected from the origin with speed u at an acute angle α above the positive x-axis.

The curve E is given by $z = A - Bx^2$ and $y = 0$.
If E and the trajectory of P_α touch exactly once, show that

$$u^2 - 2gA = u^2(1 - 4AB)\cos^2\alpha.$$

E and the trajectory of P_α touch exactly once for all α with $0 < \alpha < \frac{1}{2}\pi$. Write down the values of A and B in terms of u and g.

An explosion takes place at the origin and results in a large number of particles being simultaneously projected with speed u in different directions. You may assume that all the particles move freely under gravity for $t \geqslant 0$.

(ii) Describe the set of points which can be hit by particles from the explosion, explaining your answer.

(iii) Show that, at a time t after the explosion, the particles lie on a sphere whose centre and radius you should find.

(iv) Another particle Q is projected horizontally from the point $(0, 0, A)$ with speed u in the positive x direction.

Show that, at all times, Q lies on the curve E.

(v) Show that for particles Q and P_α to collide, Q must be projected a time $\dfrac{u(1-\cos\alpha)}{g\sin\alpha}$ after the explosion.

Section C: Probability and Statistics

11 **(i)** X_1 and X_2 are both random variables which take values $x_1, x_2, \ldots, x_n$, with probabilities $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively.

The value of random variable Y is defined to be that of X_1 with probability p and that of X_2 with probability $q = 1 - p$.

If X_1 has mean μ_1 and variance σ_1^2, and X_2 has mean μ_2 and variance σ_2^2, find the mean of Y and show that the variance of Y is $p\sigma_1^2 + q\sigma_2^2 + pq(\mu_1 - \mu_2)^2$.

(ii) To find the value of random variable B, a fair coin is tossed and a fair six-sided die is rolled. If the coin shows heads, then $B = 1$ if the die shows a six and $B = 0$ otherwise; if the coin shows tails, then $B = 1$ if the die does **not** show a six and $B = 0$ if it does. The value of Z_1 is the sum of n independent values of B, where n is large.

Show that Z_1 is a Binomial random variable with probability of success $\frac{1}{2}$.

Using a Normal approximation, show that the probability that Z_1 is within 10% of its mean tends to 1 as $n \to \infty$.

(iii) To find the value of random variable Z_2, a fair coin is tossed and n fair six-sided dice are rolled, where n is large. If the coin shows heads, then the value of Z_2 is the number of dice showing a six; if the coin shows tails, then the value of Z_2 is the number of dice **not** showing a six.

Use part **(i)** to write down the mean and variance of Z_2.

Explain why a Normal distribution with this mean and variance will not be a good approximation to the distribution of Z_2.

Show that the probability that Z_2 is within 10% of its mean tends to 0 as $n \to \infty$.

12 Each of the independent random variables $X_1, X_2, \ldots, X_n$ has the probability density function $\mathrm{f}(x) = \frac{1}{2}\sin x$ for $0 \leqslant x \leqslant \pi$ (and zero otherwise). Let Y be the random variable whose value is the maximum of the values of $X_1, X_2, \ldots, X_n$.

(i) Explain why $\Pr(Y \leqslant t) = [\Pr(X_1 \leqslant t)]^n$ and hence, or otherwise, find the probability density function of Y.

Let $m(n)$ be the median of Y and $\mu(n)$ be the mean of Y.

(ii) Find an expression for $m(n)$ in terms of n. How does $m(n)$ change as n increases?

(iii) Show that

$$\mu(n) = \pi - \frac{1}{2^n}\int_0^{\pi} (1 - \cos x)^n \, \mathrm{d}x\,.$$

(a) Show that $\mu(n)$ increases with n.

(b) Show that $\mu(2) < m(2)$.