There are 12 questions in this paper.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12

Sixth Term Examination Paper

21-S3

Compiled by: Dr Yu 郁博士

www.CasperYC.club

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Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1 **(i)** A curve has parametric equations

$$x = -4\cos^3(t) \quad \text{and} \quad y = 12\sin(t) - 4\sin^3(t).$$

Find the equation of the normal to this curve at the point

$$\left(-4\cos^3(\phi), 12\sin(\phi) - 4\sin^3(\phi)\right),$$

where $0 < \phi < \frac{1}{2}\pi$.

Verify that this normal is a tangent to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

at the point $(8\cos^3(\phi), 8\sin^3(\phi))$.

(ii) A curve has parametric equations

$$x = \cos(t) + t\sin(t) \quad \text{and} \quad y = \sin(t) - t\cos(t).$$

Find the equation of the normal to this curve at the point

$$(\cos(\phi) + \phi\sin(\phi), \sin(\phi) - \phi\cos(\phi)),$$

where $0 < \phi < \frac{1}{2}\pi$.

Determine the perpendicular distance from the origin to this normal, and hence find the equation of a curve, independent of ϕ, to which this normal is a tangent.

2 **(i)** Let

$$x = \frac{a}{b-c} \quad \text{and} \quad y = \frac{b}{c-a} \quad \text{and} \quad z = \frac{c}{a-b}$$

where a, b and c are distinct real numbers.

Show that

$$\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and use this result to deduce that $yz + zx + xy = -1$.

Hence show that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geqslant 2$$

(ii) Let

$$x = \frac{2a}{b+c} \quad \text{and} \quad y = \frac{2b}{c+a} \quad \text{and} \quad z = \frac{2c}{a+b}$$

where a, b and c are positive real numbers.

Using a suitable matrix, show that $xyz + yz + zx + xy = 4$.

Hence show that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b).$$

Show further that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 7(b+c)(c+a)(a+b).$$

3 **(i)** Let

$$I_n = \int_0^{\beta} (\sec(x) + \tan(x))^n \, dx,$$

where n is a non-negative integer and $0 < \beta < \frac{1}{2}\pi$.

For $n \geqslant 1$, show that

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\Big[(\sec(\beta) + \tan(\beta))^n - 1 \Big].$$

Show also that

$$I_n < \frac{1}{n}\Big[(\sec(\beta) + \tan(\beta))^n - 1 \Big].$$

(ii) Let

$$J_n = \int_0^{\beta} \Big[\sec(x)\cos(\beta) + \tan(x) \Big]^n \, dx,$$

where n is a non-negative integer and $0 < \beta < \frac{1}{2}\pi$.

For $n \geqslant 1$, show that

$$J_n < \frac{1}{n}\Big[(1 + \tan(\beta))^n - \cos^n(\beta) \Big].$$

4 Let $\mathbf{n}$ be a vector of unit length and Π be the plane through the origin perpendicular to $\mathbf{n}$. For any vector $\mathbf{x}$, the *projection* of $\mathbf{x}$ onto the plane Π is defined to be the vector $\mathbf{x} - (\mathbf{x}.\mathbf{n})\mathbf{n}$.

The vectors $\mathbf{a}$ and $\mathbf{b}$ each have unit length and the angle between them is θ, which satisfies $0 < \theta < \pi$. The vector $\mathbf{m}$ is given by $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

(i) Show that $\mathbf{m}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

(ii) The vector $\mathbf{c}$ also has unit length. The angle between $\mathbf{a}$ and $\mathbf{c}$ is α, and the angle between $\mathbf{b}$ and $\mathbf{c}$ is β. Both angles are acute and non-zero.

Let $\mathbf{a_1}$ and $\mathbf{b_1}$ be the projections of $\mathbf{a}$ and $\mathbf{b}$, respectively, onto the plane through the origin perpendicular to $\mathbf{c}$.

Show that $\mathbf{a_1}.\mathbf{c} = 0$ and, by considering $|\mathbf{a_1}|^2 = \mathbf{a_1}.\mathbf{a_1}$, show that $|\mathbf{a_1}| = \sin(\alpha)$.

Show also that the angle ϕ between $\mathbf{a_1}$ and $\mathbf{b_1}$ satisfies

$$\cos(\phi) = \frac{\cos(\theta) - \cos(\alpha)\cos(\beta)}{\sin(\alpha)\sin(\beta)}.$$

(iii) Let $\mathbf{m_1}$ be the projection of $\mathbf{m}$ onto the plane through the origin perpendicular to $\mathbf{c}$. Show that $\mathbf{m_1}$ bisects the angle between $\mathbf{a_1}$ and $\mathbf{b_1}$ if and only if

$$\alpha = \beta \quad \text{or} \quad \cos(\theta) = \cos(\alpha - \beta).$$

5 Two curves have polar equations

$$r = a + 2\cos(\theta) \quad \text{and} \quad r = 2 + \cos(2\theta),$$

where $r \geqslant 0$ and a is a constant.

(i) Show that these curves meet when

$$2\cos^2(\theta) - 2\cos(\theta) + 1 - a = 0.$$

Hence show that these curves touch if $a = \frac{1}{2}$ and find the other two values of a for which the curves touch.

(ii) Sketch the curves $r = a + 2\cos(\theta)$ and $r = 2 + \cos(2\theta)$ on the same diagram in the case $a = \frac{1}{2}$. Give the values of r and θ at the points at which the curves touch and justify the other features you show on your sketch.

(iii) On two further diagrams, one for each of the other two values of a, sketch both the curves $r = a + 2\cos(\theta)$ and $r = 2 + \cos(2\theta)$. Give the values of r and θ at the points at which the curves touch and justify the other features you show on your sketch.

6 **(i)** For $x \neq \tan(\alpha)$, the function f_α is defined by

$$\mathrm{f}_\alpha(x) = \tan^{-1}\left(\frac{x\tan(\alpha) + 1}{\tan(\alpha) - x}\right)$$

where $0 < \alpha < \frac{1}{2}\pi$.

Show that

$$\mathrm{f}'_\alpha(x) = \frac{1}{1 + x^2}.$$

Hence sketch $y = \mathrm{f}_\alpha(x)$.

On a separate diagram, sketch $y = \mathrm{f}_\alpha(x) - \mathrm{f}_\beta(x)$ where $0 < \alpha < \beta < \frac{1}{2}\pi$.

(ii) For $0 \leqslant x \leqslant 2\pi$ and $x \neq \frac{1}{2}\pi, \frac{3}{2}\pi$, the function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = \tanh^{-1}(\sin(x)) - \sinh^{-1}(\tan(x)).$$

For $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, show that

$$\mathrm{g}'(x) = 2\sec(x).$$

Use this result to sketch $y = \mathrm{g}(x)$ for $0 \leqslant x \leqslant 2\pi$.

7 **(i)** Let

$$z = \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}}$$

where θ and ϕ are real, and $\theta - \phi = 2n\pi$ for any integer n. Show that

$$z = i\cot\left(\frac{1}{2}(\phi - \theta)\right)$$

and give expressions for the modulus and argument of z.

(ii) The distinct points A and B lie on a circle with radius 1 and centre O. In the complex plane, A and B are represented by the complex numbers a and b,and O is at the origin. The point X is represented by the complex number x, where $x = a + b$ and $a + b \neq 0$. Show that OX is perpendicular to AB.

If the distinct points A, B and C in the complex plane, which are represented by the complex numbers a, b and c, lie on a circle with radius 1 and centre O, and $h = a + b + c$ represents the point H, then H is said to be the *orthocentre* of the triangle ABC.

(iii) The distinct points A, B and C lie on a circle with radius 1 and centre O. In the complex plane, A, B and C are represented by the complex numbers a, b and c, and O is at the origin.

Show that, if the point H, represented by the complex number h, is the orthocentre of the triangle ABC, then either $h = a$ or AH is perpendicular to BC.

(iv) The distinct points A, B, C and D (in that order, anticlockwise) all lie on a circle with radius 1 and centre O. The points P, Q, R and S are the orthocentres of the triangles ABC, BCD, CDA and DAB, respectively. By considering the midpoint of AQ, show that there is a single transformation which maps the quadrilateral $ABCD$ on to the quadrilateral $QRSP$ and describe this transformation fully.

8 A sequence $x_1, x_2, \ldots$ of real numbers is defined by $x_{n+1} = x_n^2 - 2$ for $n \geqslant 1$ and $x_1 = a$.

(i) Show that if $a > 2$ then $x_n \geqslant 2 + 4^{n-1}(a-2)$.

(ii) Show also that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$.

(iii) When $a > 2$, a second sequence $y_1, y_2, \ldots$ is defined by

$$y_n = \frac{Ax_1 x_2 \cdots x_n}{x_{n+1}},$$

where A is a positive constant and $n \geqslant 1$.

Prove that, for a certain value of a, with $a > 2$, which you should find in terms of A,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for all $n \geqslant 1$.

Determine whether, for this value of a, the second sequence converges.

Section B: Mechanics

9 **(i)** An equilateral triangle ABC has sides of length a. The points P, Q and R lie on the sides BC, CA and AB, respectively, such that the length BP is x and QR is parallel to CB. Show that

$$\left(\sqrt{3}\cot(\phi)+1\right)\left(\sqrt{3}\cot(\theta)+1\right)x = 4(a-x),$$

where $\theta = \angle CPQ$ and $\phi = \angle BRP$.

(ii) A horizontal triangular frame with sides of length a and vertices A, B and C is fixed on a smooth horizontal table. A small ball is placed at a point P inside the frame, in contact with side BC at a distance x from B. It is struck so that it moves round the triangle PQR described above, bouncing off the frame at Q and then R before returning to point P. The frame is smooth and the coefficient of restitution between the ball and the frame is e. Show that

$$x = \frac{a\mathrm{e}}{1+\mathrm{e}}.$$

(iii) Show further that if the ball continues to move round PQR after returning to P, then $\mathrm{e} = 1$.

10 The origin O of coordinates lies on a smooth horizontal table and the $x-$ and $y-$ axes lie in the plane of the table. A cylinder of radius a is fixed to the table with its axis perpendicular to the x–y plane and passing through O, and with its lower circular end lying on the table. One end, P, of a light inextensible string PQ of length b is attached to the bottom edge of the cylinder at $(a, 0)$. The other end, Q, is attached to a particle of mass m, which rests on the table.

Initially PQ is straight and perpendicular to the radius of the cylinder at P, so that Q is at (a, b). The particle is then given a horizontal impulse parallel to the $x-$ axis so that the string immediately begins to wrap around the cylinder. At time t, the part of the string that is still straight has rotated through an angle θ, where $a\theta < b$.

(i) Obtain the Cartesian coordinates of the particle at this time.

Find also an expression for the speed of the particle in terms of $\theta, \dot{\theta}, a$ and b.

(ii) Show that

$$\dot{\theta}(b - a\theta) = u,$$

where u is the initial speed of the particle.

(iii) Show further that the tension in the string at time t is

$$\frac{mu^2}{\sqrt{b^2 - 2aut}}.$$

Section C: Probability and Statistics

11 The continuous random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

where λ is a positive constant.

The random variable Y is the greatest integer less than or equal to X, and $Z = X - Y$.

(i) Show that, for any non-negative integer n,

$$\mathrm{Pr}(Y = n) = \left(1 - \mathrm{e}^{-\lambda}\right)\mathrm{e}^{-n\lambda}.$$

(ii) Show that

$$\mathrm{Pr}(Z < z) = \frac{1 - \mathrm{e}^{-\lambda z}}{1 - \mathrm{e}^{-\lambda}} \quad \text{for } 0 \leqslant z \leqslant 1.$$

(iii) Evaluate $\mathrm{E}(Z)$.

(iv) Obtain an expression for

$$\mathrm{Pr}\left(Y = n \text{ and } z_1 < Z < z_2\right),$$

where $0 \leqslant z_1 < z_2 \leqslant 1$ and n is a non-negative integer.

Determine whether Y and Z are independent.

12 **(i)** In a game, each member of a team of n players rolls a fair six–sided die.

The total score of the team is the number of pairs of players rolling the same number. For example, if 7 players roll $3, 3, 3, 3, 6, 6, 2$ the total score is 7, as six different pairs of players both score 3 and one pair of players both score 6.

Let X_{ij}, for $1 \leqslant i < j \leqslant n$, be the random variable that takes the value 1 if players i and j roll the same number and the value 0 otherwise.

Show that X_{12} is independent of X_{23}.

Hence find the mean and variance of the team's total score.

(ii) Show that, if Y_i, for $1 \leqslant i \leqslant m$, are random variables with mean zero, then

$$\mathrm{Var}\,(Y_1 + Y_2 + \ldots + Y_m) = \sum_{i=1}^{m} \mathrm{E}(Y_i^2) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}(Y_i Y_j).$$

(iii) In a different game, each member of a team of n players rolls a fair six–sided die.

The total score of the team is the number of pairs of players rolling the same even number minus the number of pairs of players rolling the same odd number. For example, if 7 players roll $3, 3, 3, 3, 6, 6, 2$ the total score is -5.

Let Z_{ij}, for $1 \leqslant i < j \leqslant n$, be the random variable that takes the value 1 if players i and j roll the same even number, the value -1 if players i and j roll the same odd number and the value 0 otherwise.

Show that Z_{12} is not independent of Z_{23}.

Find the mean of the team's total score and show that the variance of the team's total score is $\frac{1}{36}n(n^2 - 1)$.