There are 11 questions in this paper.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11

Sixth Term Examination Paper

19-S1

Compiled by: Dr Yu 郁博士

www.CasperYC.club

Lasted updated: May 8, 2025

Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1 A straight line passes through the fixed point $(1, k)$ and has gradient $-\tan\theta$, where $k > 0$ and $0 < \theta < \frac{1}{2}\pi$. Find, in terms of θ and k, the coordinates of the points X and Y, where the line meets the x-axis and the y-axis respectively

(i) Find an expression for the area A of triangle OXY in terms of k and θ. (The point O is the origin).

You are given that, as θ varies, A has a minimum value. Find an expression in terms of k for this minimum value.

(ii) Show that the length L of the perimeter of triangle OXY is given by

$$L = 1 + \tan\theta + \sec\theta + k(1 + \cot\theta + \csc\theta).$$

You are given that, as θ varies, L has a minimum value. Show that this minimum value occurs when $\theta = \alpha$ where

$$\frac{1 - \cos\alpha}{1 - \sin\alpha} = k.$$

Find and simplify an expression for the minimum value of L in terms of α.

2 **(i)** The curve C is given parametrically by the equations

$$x = 3t^2, y = 2t^3.$$

Show that the equation of the tangent to C at the point $(3p^2, 2p^3)$ is $y = px - p^3$.

(ii) Find the point of intersection of the tangents to C at the distinct points $(3p^2, 2p^3)$ and $(3q^2, 2q^3)$.

(iii) Hence show that, if these two tangents are perpendicular, their point of intersection is $(u^2 + 1, -u)$, where $u = p + q$.

(iv) The curve $\widetilde{C}$ is given parametrically by the equations

$$x = u^2 + 1, y = -u.$$

Find the coordinates of the points that lie on both C and $\widetilde{C}$.

(v) Sketch C and $\widetilde{C}$ on the same axes.

3 **(i)** By first multiplying the numerator and the denominator of the integrand by $(1 - \sin x)$, evaluate

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin(x)}\,\mathrm{d}x.$$

(ii) Evaluate also:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{1+\sec(x)}\,\mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{3}} \frac{1}{(1+\sin(x))^2}\,\mathrm{d}x.$$

4 **(i)** Find integers m and n such that

$$\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}.$$

(ii) Let $\mathrm{f}(x) = x^4 - 10x^2 + 12x - 2$. Given that the equation $\mathrm{f}(x) = 0$ has four real roots, explain why $\mathrm{f}(x)$ can be written in the form

$$\mathrm{f}(x) = (x^2 + sx + p)(x^2 - sx + q)$$

for some real constants s, p and q, and find three equations for s, p and q.

(iii) Show that

$$s^2(s^2-10)^2 + 8s^2 - 144 = 0$$

and find the three possible values of s^2.

(iv) Use the smallest of these values of s^2 to solve completely the equation $\mathrm{f}(x) = 0$, simplifying your answers as far as you can.

5 **(i)** The four points P, Q, R and S are the vertices of a plane quadrilateral.

What is the geometrical shape of $PQRS$ if $\overrightarrow{PQ} = \overrightarrow{SR}$?

What is the geometrical shape of $PQRS$ if $\overrightarrow{PQ} = \overrightarrow{SR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$?

(ii) A cube with edges of unit length has opposite vertices at $(0, 0, 0)$ and $(1, 1, 1)$. The points

$$P(p,\,0,\,0),\quad Q(1,\,q,\,0),\quad R(r,\,1,\,1)\quad \text{and}\quad S(0, s, 1)$$

lie on edges of the cube. Given that the four points lie in the same plane, show that

$$rq = (1 - s)(1 - p).$$

(a) Show that $\overrightarrow{PQ} = \overrightarrow{SR}$ if and only if the centroid of the quadrilateral $PQRS$ is at the centre of the cube.
Note: the *centroid* of the quadrilateral $PQRS$ is the point with position vector

$$\frac{1}{4}\left(\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} + \overrightarrow{OS}\right),$$

where O is the origin.

(b) Given that $\overrightarrow{PQ} = \overrightarrow{SR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$, express q, r and s in terms of p.
Show that

$$\cos(\angle PQR) = \frac{4p - 1}{5 - 4p + 8p^2}.$$

Write down the values of p, q, r and s if $PQRS$ is a square, and show that the length of each side of this square is greater than $\frac{21}{20}$.

6 In both parts of this question, x is real and $0 < \theta < \pi$.

(i) By completing the square, find in terms of θ the minimum value as x varies of

$$9x^2 - 12x\cos\theta + 4.$$

Find also the maximum value as x varies of $12x^2\sin\theta - 9x^4$.

Hence determine the values of x and θ that satisfy the equation

$$9x^4 + (9 - 12\sin\theta)x^2 - 12x\cos\theta + 4 = 0.$$

(ii) Sketch the curve

$$y = \frac{x^2}{x - \theta},$$

where θ is a constant. Deduce that either

$$\frac{x^2}{x - \theta} \leqslant 0 \quad \text{or} \quad \frac{x^2}{x - \theta} \geqslant 4\theta.$$

By considering the numerator and denominator separately, or otherwise, show that

$$\frac{\sin^2(\theta)\cos^2(x)}{1 + \cos^2(\theta)\sin^2(x)} \leqslant 1.$$

Hence determine the values of x and θ that satisfy the equation

$$\frac{x^2}{4\theta(x - \theta)} = \frac{\sin^2(\theta)\cos^2(x)}{1 + \cos^2(\theta)\sin^2(x)}.$$

7 Consider the following steps in a proof that $\sqrt{2} + \sqrt{3}$ is irrational.

1. If an integer a is not divisible by 3, then $a = 3k \pm 1$, for some integer k. In both cases, a^2 is one more than a multiple of 3.

2. Suppose that $\sqrt{2} + \sqrt{3}$ is rational, and equal to $\frac{a}{b}$, where a and b are positive integers with no common factor greater than one.

3. Then $a^4 + b^4 = 10a^2b^2$.

4. So if a is divisible by 3, then b is divisible by 3.

5. Hence $\sqrt{2} + \sqrt{3}$ is irrational.

(i) Show clearly that steps **1, 3** and **4** are all valid and that the conclusion **5** follows from the previous steps of the argument.

(ii) Prove, by means of a similar method but using divisibility by 5 instead of 3, that $\sqrt{6} + \sqrt{7}$ is irrational.

Why can divisibility by 3 not be used in this case?

8 The function f is defined, for $x > 1$, by

$$\mathrm{f}(x) = \int_1^x \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t.$$

Do not attempt to evaluate this integral.

(i) Show that, for $x > 2$,

$$\int_2^x \sqrt{\frac{u-2}{u+2}}\,\mathrm{d}u = 2\mathrm{f}\left(\frac{1}{2}x\right).$$

(ii) Evaluate in terms of f, for $x > 0$,

$$\int_0^x \sqrt{\frac{u}{u+4}}\,\mathrm{d}u.$$

(iii) Evaluate in terms of f, for $x > 5$,

$$\int_5^x \sqrt{\frac{u-5}{u+1}}\,\mathrm{d}u.$$

(iv) Evaluate in terms of f

$$\int_1^2 \frac{u^2}{\sqrt{u^2+4}}\,\mathrm{d}u.$$

Section B: Applied

9 A box has the shape of a uniform solid cuboid of height h and with a square base of side b, where $h > b$. It rests on rough horizontal ground. A light ladder has its foot on the ground and rests against one of the upper horizontal edges of the box, making an acute angle of α with the ground, where $h = b\tan\alpha$. The weight of the box is W. There is no friction at the contact between ladder and box.

A painter of weight kW climbs the ladder slowly. Neither the base of the ladder nor the box slips, but the box starts to topple when the painter reaches height λh above the ground, where $\lambda < 1$.

Show that:

(i) $R = k\lambda W\cos\alpha$, where R is the magnitude of the force exerted by the box on the ladder;

(ii) $2k\lambda\cos(2\alpha) + 1 = 0$;

(iii) $\mu \geqslant \dfrac{\sin(2\alpha)}{1 - 3\cos(2\alpha)}$, where μ is the coefficient of friction between the box and the ground.

10 In this question, the x-axis is horizontal and the positive y-axis is vertically upwards.

A particle is projected from the origin with speed u at an angle α to the *vertical*. The particle passes through the fixed point $(h\tan\beta, h)$, where $0 < \beta < 90^\circ$ and $h > 0$.

(i) Show that

$$c^2 - ck\cot\beta + 1 + k\cot^2\beta = 0, \qquad (*)$$

where $c = \cot\alpha$ and $k = \dfrac{2u^2}{gh}$.

You are given that there are two distinct values of α that satisfy equation $(*)$. Let α_1 and α_2 be these values.

(a) Show that

$$\cot\alpha_1 + \cot\alpha_2 = k\cot\beta.$$

Show also that

$$\alpha_1 + \alpha_2 = \beta.$$

(b) Show that

$$k > 2(1 + \sec\beta).$$

(ii) By considering the greatest height attained by the particle, show that $k \geqslant 4\sec^2\alpha$.

11 **(i)** Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the n^{th} round is

$$(q^2 + p^2)(2pq)^{n-1}.$$

Show also that the probability that they make a decision on or before the n^{th} round is at least

$$1 - \frac{1}{2^n}$$

whatever the value of p.

(ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p.