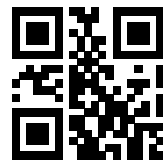


THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

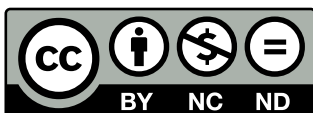
15-S3



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SUGGESTIONS TO DRYUFROMSHANGHAI@QQ.COM

Section A: Pure Mathematics

1 (i) Let

$$I_n = \int_0^{\infty} \frac{1}{(1+u^2)^n} du,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)! \pi}{2^{2n+1} (n!)^2}.$$

(ii) Let

$$J = \int_0^{\infty} f((x-x^{-1})^2) dx,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^{\infty} x^{-2} f((x-x^{-1})^2) dx = \frac{1}{2} \int_0^{\infty} (1+x^{-2}) f((x-x^{-1})^2) dx = \int_0^{\infty} f(u^2) du.$$

(iii) Hence evaluate

$$\int_0^{\infty} \frac{x^{2n-2}}{(x^4-x^2+1)^n} dx,$$

where n is a positive integer.



2 If s_1, s_2, s_3, \dots and t_1, t_2, t_3, \dots are sequences of positive numbers, we write

$$(s_n) \leq (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leq t_n$ whenever $n \geq m$ ".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m ; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \leq (n^2)$.

(ii) If it is not the case that $(s_n) \leq (t_n)$, then it is the case that $(t_n) \leq (s_n)$.

(iii) If $(s_n) \leq (t_n)$ and $(t_n) \leq (u_n)$, then $(s_n) \leq (u_n)$.

(iv) $(n^2) \leq (2^n)$.

3 In this question, r and θ are polar coordinates with $r \geq 0$ and $-\pi < \theta \leq \pi$, and a and b are positive constants.

Let L be a fixed line and let A be a fixed point not lying on L . Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a \sec \theta| = b, \quad (*)$$

where $a > b$, then $\sec \theta > 0$. Show that all points with coordinates satisfying (*) lie on a certain conchoid of Nicomedes (you should identify L , d and A). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec \theta < 0$) given by

$$|r - a \sec \theta| = b.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C$.]



- 4 (i) If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \quad (*)$$

has at least one real root.

- (ii) Let

$$S_1 = z_1 + z_2 + z_3, \quad S_2 = z_1^2 + z_2^2 + z_3^2, \quad S_3 = z_1^3 + z_2^3 + z_3^3,$$

where z_1, z_2 and z_3 are the roots of the equation (*). Express a and b in terms of S_1 and S_2 , and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3.$$

- (iii) The six real numbers r_k and θ_k ($k = 1, 2, 3$), where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^3 r_k \sin(\theta_k) = 0, \quad \sum_{k=1}^3 r_k^2 \sin(2\theta_k) = 0, \quad \sum_{k=1}^3 r_k^3 \sin(3\theta_k) = 0.$$

Show that $\theta_k = 0$ for at least one value of k .

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

- 5 (i) In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set S to be the set of positive integers with the following property:

n is in S if and only if $n\sqrt{2}$ is an integer.

3. Show that the set S contains at least one positive integer.
4. Define the integer k to be the smallest positive integer in S .
5. Show that $(\sqrt{2} - 1)k$ is in S .
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

- (ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.



6 (i) Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leq 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{aligned}w + z - u &= 0 \\w^2 + z^2 - u^2 &= -\frac{2}{3} \\w^3 + z^3 - \lambda u &= -\lambda\end{aligned}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u , all real.

Are w and z necessarily real? Give a proof or counterexample.

7 An operator D is defined, for any function f , by

$$Df(x) = x \frac{df(x)}{dx}.$$

The notation D^n means that D is applied n times; for example

$$D^2f(x) = x \frac{d}{dx} \left(x \frac{df(x)}{dx} \right).$$

Show that, for any constant a , $D^2x^a = a^2x^a$.

(i) Show that if $P(x)$ is a polynomial of degree r (where $r \geq 1$) then, for any positive integer n , $D^n P(x)$ is also a polynomial of degree r .

(ii) Show that if n and m are positive integers with $n < m$, then $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^m (-1)^r \binom{m}{r} r^n = 0.$$



- 8 (i) Show that under the changes of variable $x = r \cos \theta$ and $y = r \sin \theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y + x) \frac{dy}{dx} = y - x$$

becomes

$$\frac{dr}{d\theta} + r = 0.$$

Sketch a solution in the $x - y$ plane.

- (ii) Show that the solutions of

$$(y + x - x(x^2 + y^2)) \frac{dy}{dx} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + Ae^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A .



Section B: Mechanics

9 A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ . The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

- (i) An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2 \left(\sqrt{x^2 + a^2} - a \right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$ and the dot denotes differentiation with respect to t .

- (ii) Find, in terms of k , a and v , the greatest value, x_0 , attained by x . Find also the acceleration of P at $x = x_0$.
- (iii) Obtain, in the form of an integral, an expression for the period of the motion. Show that in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1-u^4}} du.$$



- 10** A light rod of length $2a$ has a particle of mass m attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates (x, y) , where the x -axis is horizontal (within the plane of motion) and y is the height above a horizontal table. Initially, the rod is vertical, and at time t later it is inclined at an angle θ to the vertical.

- (i) Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta} \cos \theta \\ \dot{y} - a\dot{\theta} \sin \theta \end{pmatrix}$$

and that

$$m \begin{pmatrix} \ddot{x} + a\ddot{\theta} \cos \theta - a\dot{\theta}^2 \sin \theta \\ \ddot{y} - a\ddot{\theta} \sin \theta - a\dot{\theta}^2 \cos \theta \end{pmatrix} = -T \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} - mg \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

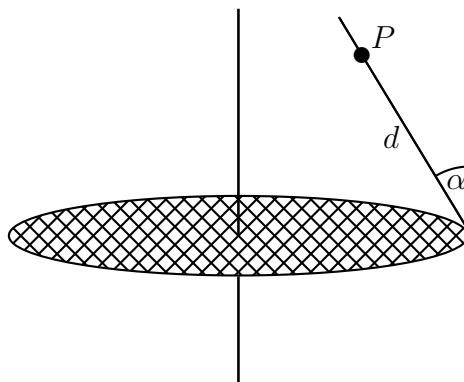
where the dots denote differentiation with respect to time t and T is the tension in the rod. Obtain the corresponding equations for the other particle.

- (ii) Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.
- (iii) Initially, the midpoint of the rod is a height h above the table, the velocity of the higher particle is $\begin{pmatrix} u \\ v \end{pmatrix}$, and the velocity of the lower particle is $\begin{pmatrix} 0 \\ v \end{pmatrix}$. Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva.$$



- 11 (i) A horizontal disc of radius r rotates about a vertical axis through its centre with angular speed ω . One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle P of mass m is attached to the rod at a distance d from the hinge. The rod makes a constant angle α with the upward vertical, as shown in the diagram, and $d \sin \alpha < r$.



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by P is parallel to the rod.

Show also that

$$r \cot \alpha = a + d \cos \alpha,$$

where $a = \frac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of m, g and α .

- (ii) The disc and rod rotate as in part (i), but two particles (instead of P) are attached to the rod. The masses of the particles are m_1 and m_2 and they are attached to the rod at distances d_1 and d_2 from the hinge, respectively. The rod makes a constant angle β with the upward vertical and $d_1 \sin \beta < d_2 \sin \beta < r$. Show that β satisfies an equation of the form

$$r \cot \beta = a + b \cos \beta,$$

where b should be expressed in terms of d_1, d_2, m_1 and m_2 .



Section C: Probability and Statistics

12 A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown n times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable S_n is the sum of the outcomes.

(i) The random variable R_n is the remainder when S_n is divided by 6. Write down the probability generating function, $G(x)$, of R_1 and show that the probability generating function of R_2 is also $G(x)$. Use a generating function to find the probability that S_n is divisible by 6.

(ii) The random variable T_n is the remainder when S_n is divided by 5. Write down the probability generating function, $G_1(x)$, of T_1 and show that $G_2(x)$, the probability generating function of T_2 , is given by

$$G_2(x) = \frac{1}{36}(x^2 + 7y)$$

where $y = 1 + x + x^2 + x^3 + x^4$.

Obtain the probability generating function of T_n and hence show that the probability that S_n is divisible by 5 is

$$\frac{1}{5} \left(1 - \frac{1}{6^n} \right)$$

if n is not divisible by 5. What is the corresponding probability if n is divisible by 5?

13 Each of the two independent random variables X and Y is uniformly distributed on the interval $[0, 1]$.

(i) By considering the lines $x + y = \text{constant}$ in the x - y plane, find the cumulative distribution function of $X + Y$.

(ii) Hence show that the probability density function f of $(X + Y)^{-1}$ is given by

$$f(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \leq t \leq 1 \\ t^{-3} & \text{for } 1 \leq t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $E\left(\frac{1}{X + Y}\right)$.

(iii) Find the cumulative distribution function of Y/X and use this result to find the probability density function of $\frac{X}{X + Y}$.

Write down $E\left(\frac{X}{X + Y}\right)$ and verify your result by integration.

