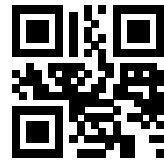


THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

## Sixth Term Examination Paper

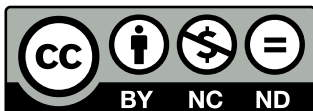
14-S3



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## Section A: Pure Mathematics

1 Let  $a$ ,  $b$  and  $c$  be real numbers such that  $a + b + c = 0$  and let

$$(1 + ax)(1 + bx)(1 + cx) = 1 + qx^2 + rx^3$$

for all real  $x$ . Show that  $q = bc + ca + ab$  and  $r = abc$ .

(i) Show that the coefficient of  $x^n$  in the series expansion (in ascending powers of  $x$ ) of  $\ln(1 + qx^2 + rx^3)$  is  $(-1)^{n+1}S_n$  where

$$S_n = \frac{a^n + b^n + c^n}{n}, \quad (n \geq 1).$$

(ii) Find, in terms of  $q$  and  $r$ , the coefficients of  $x^2$ ,  $x^3$  and  $x^5$  in the series expansion (in ascending powers of  $x$ ) of  $\ln(1 + qx^2 + rx^3)$  and hence show that  $S_2S_3 = S_5$ .

(iii) Show that  $S_2S_5 = S_7$ .

(iv) Give a proof of, or find a counterexample to, the claim that  $S_2S_7 = S_9$ .

2 (i) Show, by means of the substitution  $u = \cosh x$ , that

$$\int \frac{\sinh x}{\cosh 2x} dx = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} \cosh x - 1}{\sqrt{2} \cosh x + 1} \right| + C.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x} dx.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1 + u^4} du = \frac{\pi + 2 \ln(\sqrt{2} + 1)}{4\sqrt{2}}.$$



- 3 (i) The line  $L$  has equation  $y = mx + c$ , where  $m > 0$  and  $c > 0$ . Show that, in the case  $mc > a > 0$ , the shortest distance between  $L$  and the parabola  $y^2 = 4ax$  is

$$\frac{mc - a}{m\sqrt{m^2 + 1}}.$$

What is the shortest distance in the case that  $mc \leq a$ ?

- (ii) Find the shortest distance between the point  $(p, 0)$ , where  $p > 0$ , and the parabola  $y^2 = 4ax$ , where  $a > 0$ , in the different cases that arise according to the value of  $p/a$ . [ You may wish to use the parametric coordinates  $(at^2, 2at)$  of points on the parabola. ]

Hence find the shortest distance between the circle  $(x - p)^2 + y^2 = b^2$ , where  $p > 0$  and  $b > 0$ , and the parabola  $y^2 = 4ax$ , where  $a > 0$ , in the different cases that arise according to the values of  $p$ ,  $a$  and  $b$ .

- 4 (i) Let

$$I = \int_0^1 ((y')^2 - y^2) dx \quad \text{and} \quad I_1 = \int_0^1 (y' + y \tan x)^2 dx,$$

where  $y$  is a given function of  $x$  satisfying  $y = 0$  at  $x = 1$ . Show that  $I - I_1 = 0$  and deduce that  $I \geq 0$ . Show further that  $I = 0$  only if  $y = 0$  for all  $x$  ( $0 \leq x \leq 1$ ).

- (ii) Let

$$J = \int_0^1 ((y')^2 - a^2 y^2) dx,$$

where  $a$  is a given positive constant and  $y$  is a given function of  $x$ , not identically zero, satisfying  $y = 0$  at  $x = 1$ . By considering an integral of the form

$$\int_0^1 (y' + ay \tan bx)^2 dx,$$

where  $b$  is suitably chosen, show that  $J \geq 0$ . You should state the range of values of  $a$ , in the form  $a < k$ , for which your proof is valid.

In the case  $a = k$ , find a function  $y$  (not everywhere zero) such that  $J = 0$ .



- 5** A quadrilateral drawn in the complex plane has vertices  $A, B, C$  and  $D$ , labelled anticlockwise. These vertices are represented, respectively, by the complex numbers  $a, b, c$  and  $d$ . Show that  $ABCD$  is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if  $a + c = b + d$ . Show further that, in this case,  $ABCD$  is a square if and only if  $i(a - c) = b - d$ .

Let  $PQRS$  be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than  $180^\circ$ . Squares with centres  $X, Y, Z$  and  $T$  are constructed externally to the quadrilateral on the sides  $PQ, QR, RS$  and  $SP$ , respectively.

- (i) If  $P$  and  $Q$  are represented by the complex numbers  $p$  and  $q$ , respectively, show that  $X$  can be represented by

$$\frac{1}{2}(p(1+i) + q(1-i)).$$

- (ii) Show that  $XYZT$  is a square if and only if  $PQRS$  is a parallelogram.

- 6** Starting from the result that

$$h(t) > 0 \text{ for } 0 < t < x \implies \int_0^x h(t) dt > 0,$$

show that, if  $f''(t) > 0$  for  $0 < t < x_0$  and  $f(0) = f'(0) = 0$ , then  $f(t) > 0$  for  $0 < t < x_0$ .

- (i) Show that, for  $0 < x < \frac{1}{2}\pi$ ,

$$\cos x \cosh x < 1.$$

- (ii) Show that, for  $0 < x < \frac{1}{2}\pi$ ,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}.$$



7 The four distinct points  $P_i$  ( $i = 1, 2, 3, 4$ ) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines  $P_1P_3$  and  $P_2P_4$  intersect at  $Q$ .

(i) By considering the triangles  $P_1QP_4$  and  $P_2QP_3$  show that  $(P_1Q)(QP_3) = (P_2Q)(QP_4)$ .

(ii) Let  $\mathbf{p}_i$  be the position vector of the point  $P_i$  ( $i = 1, 2, 3, 4$ ). Show that there exist numbers  $a_i$ , not all zero, such that

$$\sum_{i=1}^4 a_i = 0 \quad \text{and} \quad \sum_{i=1}^4 a_i \mathbf{p}_i = \mathbf{0}. \quad (*)$$

(iii) Let  $a_i$  ( $i = 1, 2, 3, 4$ ) be any numbers, not all zero, that satisfy (\*). Show that  $a_1 + a_3 \neq 0$  and that the lines  $P_1P_3$  and  $P_2P_4$  intersect at the point with position vector

$$\frac{a_1 \mathbf{p}_1 + a_3 \mathbf{p}_3}{a_1 + a_3}.$$

Deduce that  $a_1 a_3 (P_1P_3)^2 = a_2 a_4 (P_2P_4)^2$ .

8 The numbers  $f(r)$  satisfy  $f(r) > f(r+1)$  for  $r = 1, 2, \dots$ . Show that, for any non-negative integer  $n$ ,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n)$$

where  $k$  is an integer greater than 1.

(i) By taking  $f(r) = 1/r$ , show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

Deduce that the sum  $\sum_{r=1}^{\infty} \frac{1}{r}$  does not converge.

(ii) By taking  $f(r) = 1/r^3$ , show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

(iii) Let  $S(n)$  be the set of positive integers less than  $n$  which do not have a 2 in their decimal representation and let  $\sigma(n)$  be the sum of the reciprocals of the numbers in  $S(n)$ , so for example  $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$ . Show that  $S(1000)$  contains  $9^3 - 1$  distinct numbers.

Show that  $\sigma(n) < 80$  for all  $n$ .



## Section B: Mechanics

9 A particle of mass  $m$  is projected with velocity  $\mathbf{u}$ . It is acted upon by the force  $m\mathbf{g}$  due to gravity and by a resistive force  $-mk\mathbf{v}$ , where  $\mathbf{v}$  is its velocity and  $k$  is a positive constant.

(i) Given that, at time  $t$  after projection, its position  $\mathbf{r}$  relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + e^{-kt}}{k^2} \mathbf{g} + \frac{1 - e^{-kt}}{k} \mathbf{u},$$

find an expression for  $\mathbf{v}$  in terms of  $k$ ,  $t$ ,  $\mathbf{g}$  and  $\mathbf{u}$ . Verify that the equation of motion and the initial conditions are satisfied.

(ii) Let  $\mathbf{u} = u \cos \alpha \mathbf{i} + u \sin \alpha \mathbf{j}$  and  $\mathbf{g} = -g \mathbf{j}$ , where  $0 < \alpha < 90^\circ$ , and let  $T$  be the time after projection at which  $\mathbf{r} \cdot \mathbf{j} = 0$ . Show that

$$uk \sin \alpha = \left( \frac{kT}{1 - e^{-kT}} - 1 \right) g.$$

(iii) Let  $\beta$  be the acute angle between  $\mathbf{v}$  and  $\mathbf{i}$  at time  $T$ . Show that

$$\tan \beta = \frac{(e^{kT} - 1)g}{uk \cos \alpha} - \tan \alpha.$$

(iv) Show further that  $\tan \beta > \tan \alpha$  (you may assume that  $\sinh kT > kT$ ) and deduce that  $\beta > \alpha$ .



- 10** Two particles  $X$  and  $Y$ , of equal mass  $m$ , lie on a smooth horizontal table and are connected by a light elastic spring of natural length  $a$  and modulus of elasticity  $\lambda$ . Two more springs, identical to the first, connect  $X$  to a point  $P$  on the table and  $Y$  to a point  $Q$  on the table. The distance between  $P$  and  $Q$  is  $3a$ .

Initially, the particles are held so that  $XP = a$ ,  $YQ = \frac{1}{2}a$ , and  $PXYQ$  is a straight line. The particles are then released.

- (i) At time  $t$ , the particle  $X$  is a distance  $a + x$  from  $P$  and the particle  $Y$  is a distance  $a + y$  from  $Q$ . Show that

$$m \frac{d^2x}{dt^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving  $\frac{d^2y}{dt^2}$ .

- (ii) Deduce that

$$x - y = A \cos \omega t + B \sin \omega t$$

where  $A$  and  $B$  are constants to be determined and  $m a \omega^2 = \lambda$ . Find a similar expression for  $x + y$ .

- (iii) Show that  $Y$  will never return to its initial position.

- 11** A particle  $P$  of mass  $m$  is connected by two light inextensible strings to two fixed points  $A$  and  $B$ , with  $A$  vertically above  $B$ . The string  $AP$  has length  $x$ . The particle is rotating about the vertical through  $A$  and  $B$  with angular velocity  $\omega$ , and both strings are taut. Angles  $PAB$  and  $PBA$  are  $\alpha$  and  $\beta$ , respectively.

- (i) Find the tensions  $T_A$  and  $T_B$  in the strings  $AP$  and  $BP$  (respectively), and hence show that  $\omega^2 x \cos \alpha \geq g$ .

- (ii) Consider now the case that  $\omega^2 x \cos \alpha = g$ . Given that  $AB = h$  and  $BP = d$ , where  $h > d$ , show that  $h \cos \alpha \geq \sqrt{h^2 - d^2}$ . Show further that

$$mg < T_A \leq \frac{mgh}{\sqrt{h^2 - d^2}}.$$

- (iii) Describe the geometry of the strings when  $T_A$  attains its upper bound.



## Section C: Probability and Statistics

**12** The random variable  $X$  has probability density function  $f(x)$  (which you may assume is differentiable) and cumulative distribution function  $F(x)$  where  $-\infty < x < \infty$ . The random variable  $Y$  is defined by  $Y = e^X$ . You may assume throughout this question that  $X$  and  $Y$  have unique modes.

- (i) Find the median value  $y_m$  of  $Y$  in terms of the median value  $x_m$  of  $X$ .
- (ii) Show that the probability density function of  $Y$  is  $f(\ln y)/y$ , and deduce that the mode  $\lambda$  of  $Y$  satisfies  $f'(\ln \lambda) = f(\ln \lambda)$ .
- (iii) Suppose now that  $X \sim N(\mu, \sigma^2)$ , so that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

Explain why

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma^2)^2/(2\sigma^2)} dx = 1$$

and hence show that  $E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ .

- (iv) Show that, when  $X \sim N(\mu, \sigma^2)$ ,

$$\lambda < y_m < E(Y).$$



**13** I play a game which has repeated rounds. Before the first round, my score is 0. Each round can have three outcomes:

1. my score is unchanged and the game ends;
2. my score is unchanged and I continue to the next round;
3. my score is increased by one and I continue to the next round.

The probabilities of these outcomes are  $a$ ,  $b$  and  $c$ , respectively (the same in each round), where  $a+b+c = 1$  and  $abc \neq 0$ . The random variable  $N$  represents my score at the end of a randomly chosen game.

Let  $G(t)$  be the probability generating function of  $N$ .

- (i) Suppose in the first round, the game ends.  
Show that the probability generating function conditional on this happening is 1.
- (ii) Suppose in the first round, the game continues to the next round with no change in score.  
Show that the probability generating function conditional on this happening is  $G(t)$ .
- (iii) By comparing the coefficients of  $t^n$ , show that  $G(t) = a + bG(t) + ctG(t)$ .  
Deduce that, for  $n \geq 0$ ,

$$P(N = n) = \frac{ac^n}{(1-b)^{n+1}}.$$

- (iv) Show further that, for  $n \geq 0$ ,

$$P(N = n) = \frac{\mu^n}{(1+\mu)^{n+1}},$$

where  $\mu = E(N)$ .

