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Sixth Term Examination Paper

02-S3

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Section A: Pure Mathematics

1 **(i)** Find the area of the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$. What happens to this area as a tends to infinity?

(ii) Find the volume of the solid obtained when the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$, is rotated through 2π radians about the x-axis. What happens to this volume as a tends to infinity?

2 **(i)** Prove that $\arctan a + \arctan b = \arctan\left(\dfrac{a+b}{1-ab}\right)$ when $0 < a < 1$ and $0 < b < 1$.

(ii) Prove by induction that, for $n \geqslant 1$,

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{r^2+r+1}\right) = \arctan\left(\frac{n}{n+2}\right)$$

and hence find

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2+r+1}\right).$$

(iii) Hence prove that

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2-r+1}\right) = \frac{\pi}{2}.$$

3 Let

$$\mathrm{f}(x) = a\sqrt{x} - \sqrt{x-b}\,,$$

where $x \geqslant b > 0$ and $a > 1$. Sketch the graph of $\mathrm{f}(x)$. Hence show that the equation $\mathrm{f}(x) = c$, where $c > 0$, has no solution when $c^2 < b\,(a^2-1)$. Find conditions on c^2 in terms of a and b for the equation to have exactly one or exactly two solutions.

Solve the equations

(i) $3\sqrt{x} - \sqrt{x-2} = 4$,

(ii) $3\sqrt{x} - \sqrt{x-3} = 5$.

4 **(i)** Show that if x and y are positive and $x^3 + x^2 = y^3 - y^2$ then $x < y$.

(ii) Show further that if $0 < x \leqslant y - 1$, then $x^3 + x^2 < y^3 - y^2$.

(iii) Prove that there does not exist a pair of *positive* integers such that the difference of their cubes is equal to the sum of their squares.

(iv) Find all the pairs of integers such that the difference of their cubes is equal to the sum of their squares.

5 **(i)** Give a condition that must be satisfied by p, q and r for it to be possible to write the quadratic polynomial $px^2 + qx + r$ in the form $p\,(x+h)^2$, for some h.

(ii) Obtain an equation, which you need not simplify, that must be satisfied by t if it is possible to write

$$\left(x^2 + \tfrac{1}{2}bx + t\right)^2 - \left(x^4 + bx^3 + cx^2 + dx + e\right)$$

in the form $k\,(x+h)^2$, for some k and h.

(iii) Hence, or otherwise, write $x^4 + 6x^3 + 9x^2 - 2x - 7$ as a product of two quadratic factors.

6 Find all the solution curves of the differential equation

$$y^4 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^4 = \left(y^2 - 1\right)^2$$

that pass through either of the points

(i) $\left(0,\, \tfrac{1}{2}\sqrt{3}\right)$,

(ii) $\left(0,\, \tfrac{1}{2}\sqrt{5}\right)$.

Show also that $y = 1$ and $y = -1$ are solutions of the differential equation. Sketch all these solution curves on a single set of axes.

7 **(i)** Given that α and β are acute angles, show that $\alpha + \beta = \frac{1}{2}\pi$ if and only if $\cos^2\alpha + \cos^2\beta = 1$.

(ii) In the x–y plane, the point A has coordinates $(0, s)$ and the point C has coordinates $(s, 0)$, where $s > 0$. The point B lies in the first quadrant ($x > 0$, $y > 0$). The lengths of AB, OB and CB are respectively a, b and c.

Show that

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4s^2b^2$$

and hence that

$$(2s^2 - a^2 - c^2)^2 + (2b^2 - a^2 - c^2)^2 = 4a^2c^2 .$$

(iii) Deduce that

$$(a - c)^2 \leqslant 2b^2 \leqslant (a + c)^2 \ .$$

8 Four complex numbers u_1, u_2, u_3 and u_4 have unit modulus, and arguments θ_1, θ_2, θ_3 and θ_4, respectively, with $-\pi < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \pi$.

(i) Show that

$$\arg(u_1 - u_2) = \tfrac{1}{2}(\theta_1 + \theta_2 - \pi) + 2n\pi$$

where $n = 0$ or 1.

(ii) Deduce that

$$\arg\left((u_1 - u_2)(u_4 - u_3)\right) = \arg\left((u_1 - u_4)(u_3 - u_2)\right) + 2n\pi$$

for some integer n.

(iii) Prove that

$$|\,(u_1 - u_2)(u_4 - u_3)\,| + |\,(u_1 - u_4)(u_3 - u_2)\,| = |\,(u_1 - u_3)(u_4 - u_2)\,| \ .$$

Section B: Mechanics

9 A tall container made of light material of negligible thickness has the form of a prism, with a square base of area a^2. It contains a volume ka^3 of fluid of uniform density. The container is held so that it stands on a rough plane, which is inclined at angle θ to the horizontal, with two of the edges of the base of the container horizontal.

(i) In the case $k > \frac{1}{2}\tan\theta$, show that the centre of mass of the fluid is at a distance x from the lower side of the container and at a distance y from the base of the container, where

$$\frac{x}{a} = \frac{1}{2} - \frac{\tan\theta}{12k}\,, \quad \frac{y}{a} = \frac{k}{2} + \frac{\tan^2\theta}{24k}\,.$$

(ii) Determine the corresponding coordinates in the case $k < \frac{1}{2}\tan\theta$.

(iii) The container is now released. Given that $k < \frac{1}{2}$, show that the container will topple if $\theta > 45°$.

10 A light hollow cylinder of radius a can rotate freely about its axis of symmetry, which is fixed and horizontal. A particle of mass m is fixed to the cylinder, and a second particle, also of mass m, moves on the rough inside surface of the cylinder. Initially, the cylinder is at rest, with the fixed particle on the same horizontal level as its axis and the second particle at rest vertically below this axis. The system is then released.

(i) Show that, if θ is the angle through which the cylinder has rotated, then

$$\ddot{\theta} = \frac{g}{2a}\left(\cos\theta - \sin\theta\right)\,,$$

provided that the second particle does not slip.

(ii) Given that the coefficient of friction is $(3+\sqrt{3})/6$, show that the second particle starts to slip when the cylinder has rotated through $60°$.

11 A particle moves on a smooth triangular horizontal surface AOB with angle $AOB = 30°$. The surface is bounded by two vertical walls OA and OB and the coefficient of restitution between the particle and the walls is e, where $e < 1$. The particle, which is initially at point P on the surface and moving with velocity u_1, strikes the wall OA at M_1, with angle $PM_1A = \theta$, and rebounds, with velocity v_1, to strike the wall OB at N_1, with angle $M_1N_1B = \theta$. Find e and $\dfrac{v_1}{u_1}$ in terms of θ.

The motion continues, with the particle striking side OA at M_2, M_3, ... and striking side OB at N_2, N_3, ... Show that, if $\theta < 60°$, the particle reaches O in a finite time.

Section C: Probability and Statistics

12 In a game, a player tosses a biased coin repeatedly until two successive tails occur, when the game terminates. For each head which occurs the player wins £1.

(i) If E is the expected number of tosses of the coin in the course of a game, and p is the probability of a head, explain why

$$E = p\,(1+E) + (1-p)\,p\,(2+E) + 2\,(1-p)^2\ ,$$

and hence determine E in terms of p.

(ii) Find also, in terms of p, the expected winnings in the course of a game.

(iii) A second game is played, with the same rules, except that the player continues to toss the coin until r successive tails occur. Show that the expected number of tosses in the course of a game is given by the expression $\dfrac{1-q^r}{pq^r}$, where $q = 1-p$.

13 **(i)** A continuous random variable is said to have an exponential distribution with parameter λ if its density function is $\mathrm{f}(t) = \lambda \mathrm{e}^{-\lambda t}\ (0 \leqslant t < \infty)$. If X_1 and X_2, which are independent random variables, have exponential distributions with parameters λ_1 and λ_2 respectively, find an expression for the probability that either X_1 or X_2 (or both) is less than x.

(ii) Prove that if X is the random variable whose value is the lesser of the values of X_1 and X_2, then X also has an exponential distribution.

(iii) Route A and Route B buses run from my house to my college. The time between buses on each route has an exponential distribution and the mean time between buses is 15 minutes for Route A and 30 minutes for Route B. The timings of the buses on the two routes are independent. If I emerge from my house one day to see a Route A bus and a Route B bus just leaving the stop, show that the median wait for the next bus to my college will be approximately 7 minutes.

14 Prove that, for any two discrete random variables X and Y,

$$\mathrm{Var}\,(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y),$$

where $\mathrm{Var}(X)$ is the variance of X and $\mathrm{Cov}(X,Y)$ is the covariance of X and Y.

When a Grandmaster plays a sequence of m games of chess, she is, independently, equally likely to win, lose or draw each game. If the values of the random variables W, L and D are the numbers of her wins, losses and draws respectively, justify briefly the following claims:

(i) $W + L + D$ has variance 0;

(ii) $W + L$ has a binomial distribution.

Find the value of $\dfrac{\mathrm{Cov}(W,L)}{\sqrt{\mathrm{Var}(W)\mathrm{Var}(L)}}$.