There are 14 questions in this paper.

Sixth Term Examination Paper

99-S3

Compiled by: Dr Yu 郁博士

www.CasperYC.club

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Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1 Consider the cubic equation

$$x^3 - px^2 + qx - r = 0\,,$$

where $p \neq 0$ and $r \neq 0$.

(i) If the three roots can be written in the form ak^{-1}, a and ak for some constants a and k, show that one root is q/p and that $q^3 - rp^3 = 0$.

(ii) If $r = q^3/p^3$, show that q/p is a root and that the product of the other two roots is $(q/p)^2$. Deduce that the roots are in geometric progression.

(iii) Find a necessary and sufficient condition involving p, q and r for the roots to be in arithmetic progression.

2 **(i)** Let $\mathrm{f}(x) = (1 + x^2)\mathrm{e}^x$. Show that $\mathrm{f}'(x) \geqslant 0$ and sketch the graph of $\mathrm{f}(x)$. Hence, or otherwise, show that the equation

$$(1 + x^2)\mathrm{e}^x = k,$$

where k is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leqslant 0$.

(ii) Determine the number of real roots of the equation

$$(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$$

in the cases (a) $0 < k \leqslant 2/\pi$ and (b) $2/\pi < k < 1$.

3 **(i)** Justify, by means of a sketch, the formula

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\mathrm{f}(1 + m/n)\right\} = \int_1^2 \mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Show that

$$\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right\} = \ln 2\,.$$

(iii) Evaluate

$$\lim_{n\to\infty}\left\{\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right\}.$$

4 A polyhedron is a solid bounded by F plane faces, which meet in E edges and V vertices. You may assume *Euler's formula*, that $V - E + F = 2$.

(i) In a regular polyhedron the faces are equal regular m–sided polygons, n of which meet at each vertex. Show that

$$F = \frac{4n}{h},$$

where $h = 4 - (n-2)(m-2)$.

(ii) By considering the possible values of h, or otherwise, prove that there are only five regular polyhedral, and find V, E and F for each.

5 The sequence u_0, u_1, u_2, ... is defined by

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for} \quad n \geqslant 1\,.$$

(i) Prove that

$$u_{n+2}^2 + u_{n-1}^2 = 2(u_{n+1}^2 + u_n^2)\,.$$

(ii) Using induction, or otherwise, prove the following result:

$$u_{2n} = u_n^2 + u_{n-1}^2 \quad \text{and} \quad u_{2n+1} = u_{n+1}^2 - u_{n-1}^2$$

for any positive integer n.

6 **(i)** A closed curve is given by the equation

$$x^{2/n} + y^{2/n} = a^{2/n} \qquad (*)$$

where n is an odd integer and a is a positive constant. Find a parametrization $x = x(t)$, $y = y(t)$ which describes the curve anticlockwise as t ranges from 0 to 2π.

(ii) Sketch the curve in the case $n = 3$, justifying the main features of your sketch.

(iii) The area A enclosed by such a curve is given by the formula

$$A = \frac{1}{2}\int_0^{2\pi}\left[x(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} - y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]\,\mathrm{d}t\,.$$

Use this result to find the area enclosed by $(*)$ for $n = 3$.

7 Let a be a non-zero real number and define a binary operation on the set of real numbers by

$$x * y = x + y + axy\,.$$

(i) Show that the operation $*$ is associative.

(ii) Show that $(G, *)$ is a group, where G is the set of all real numbers except for one number which you should identify.

(iii) Find a subgroup of $(G, *)$ which has exactly 2 elements.

8 The function $y(x)$ is defined for $x \geqslant 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at } x = 0.$$

When x is in the range $2(n-1)\pi < x < 2n\pi$, where n is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + n^2 y = 0.$$

Both y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0,\ 1,\ 2,\ \ldots$.

(i) Find $y(x)$ for $0 \leqslant x \leqslant 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leqslant x \leqslant 4\pi$, and find $y(x)$ for all $x \geqslant 0$.

(iii) Show that

$$\int_0^\infty y^2\,\mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

Section B: Mechanics

9 The gravitational force between two point particles of masses m and m' is mutually attractive and has magnitude

$$\frac{Gmm'}{r^2},$$

where G is a constant and r is the distance between them.

A particle of unit mass lies on the axis of a thin uniform circular ring of radius r and mass m, at a distance x from its centre.

(i) Explain why the net force on the particle is directed towards the centre of the ring and show that its magnitude is

$$\frac{Gmx}{(x^2+r^2)^{3/2}}.$$

(ii) The particle now lies inside a thin hollow spherical shell of uniform density, mass M and radius a, at a distance b from its centre. Show that the particle experiences no gravitational force due to the shell.

10 **(i)** A chain of mass m and length l is composed of n small smooth links. It is suspended vertically over a horizontal table with its end just touching the table, and released so that it collapses inelastically onto the table. Calculate the change in momentum of the $(k+1)$th link from the bottom of the chain as it falls onto the table.

(ii) Write down an expression for the total impulse sustained by the table in this way from the whole chain. By approximating the sum by an integral, show that this total impulse is approximately

$$\tfrac{2}{3}m\sqrt{2gl}$$

when n is large.

11 **(i)** Calculate the moment of inertia of a uniform thin circular hoop of mass m and radius a about an axis perpendicular to the plane of the hoop through a point on its circumference.

(ii) The hoop, which is rough, rolls with speed v on a rough horizontal table straight towards the edge and rolls over the edge without initially losing contact with the edge. Show that the hoop will lose contact with the edge when it has rotated about the edge of the table through an angle θ, where

$$\cos\theta = \frac{1}{2} + \frac{v^2}{2ag}.$$

Section C: Probability and Statistics

12 In the game of endless cricket the scores X and Y of the two sides are such that

$$\mathrm{P}(X = j,\ Y = k) = \mathrm{e}^{-1}\frac{(j+k)\lambda^{j+k}}{j!k!},$$

for some positive constant λ, where $j, k = 0,\ 1,\ 2,\ \ldots$.

(i) Find $\mathrm{P}(X + Y = n)$ for each $n > 0$.

(ii) Show that $2\lambda\mathrm{e}^{2\lambda-1} = 1$.

(iii) Show that $2x\mathrm{e}^{2x-1}$ is an increasing function of x for $x > 0$ and deduce that the equation in (ii) has at most one solution and hence determine λ.

(iv) Calculate the expectation $\mathrm{E}(2^{X+Y})$.

13 The cakes in our canteen each contain exactly four currants, each currant being randomly placed in the cake. I take a proportion X of a cake where X is a random variable with density function

$$\mathrm{f}(x) = Ax$$

for $0 \leqslant x \leqslant 1$ where A is a constant.

(i) What is the expected number of currants in my portion?

(ii) If I find all four currants in my portion, what is the probability that I took more than half the cake?

14 In the basic version of Horizons (H1) the player has a maximum of n turns, where $n \geqslant 1$. At each turn, she has a probability p of success, where $0 < p < 1$. If her first success is at the rth turn, where $1 \leqslant r \leqslant n$, she collects r pounds and then withdraws from the game. Otherwise, her winnings are nil.

(i) Show that in H1, her expected winnings are

$$p^{-1}\left[1 + nq^{n+1} - (n+1)q^n\right] \quad \text{pounds,}$$

where $q = 1 - p$.

(ii) The rules of H2 are the same as those of H1, except that n is randomly selected from a Poisson distribution with parameter λ. If $n = 0$ her winnings are nil. Otherwise she plays H1 with the selected n. Show that in H2, her expected winnings are

$$\frac{1}{p}\left(1 - \mathrm{e}^{-\lambda p}\right) - \lambda q\mathrm{e}^{-\lambda p} \quad \text{pounds.}$$