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Sixth Term Examination Paper

19-S2

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Lasted updated: May 8, 2025

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Section A: Pure Mathematics

1 Let $\mathrm{f}(x) = (x-p)\mathrm{g}(x)$, where g is a polynomial. Show that the tangent to the curve $y = \mathrm{f}(x)$ at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$ if and only if $\mathrm{g}'(a) = 0$.

The curve C has equation

$$y = A(x-p)(x-q)(x-r),$$

where p, q and r are constants with $p < q < r$, and A is a non- zero constant.

(i) The tangent to C at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$. Show that $2a = q + r$ and find an expression for the gradient of this tangent in terms of A, q and r.

(ii) The tangent to C at the point with $x = c$, where $c \neq r$, passes through the point $(r, 0)$. Show that this tangent is parallel to the tangent in part (i) if and only if the tangent to C at the point with $x = q$ does not meet the curve again.

2 The function f satisfies $\mathrm{f}(0) = 0$ and $\mathrm{f}'(t) > 0$ for $t > 0$. Show by means of a sketch that, for $x > 0$,

$$\int_0^x \mathrm{f}(t)\,\mathrm{d}t + \int_0^{\mathrm{f}(x)} \mathrm{f}^{-1}(y)\,\mathrm{d}y = x\mathrm{f}(x).$$

(i) The (real) function g is defined, for all t, by

$$(\mathrm{g}(t))^3 + \mathrm{g}(t) = t.$$

Prove that $\mathrm{g}(0) = 0$, and that $\mathrm{g}'(t) > 0$ for all t.

Evaluate $\displaystyle\int_0^2 \mathrm{g}(t)\,\mathrm{d}t$.

(ii) The (real) function h is defined, for all t, by

$$(\mathrm{h}(t))^3 + \mathrm{h}(t) = t + 2.$$

Evaluate $\displaystyle\int_0^8 \mathrm{h}(t)\,\mathrm{d}t$.

3 For any two real numbers x_1 and x_2, show that

$$|x_1 + x_2| \leqslant |x_1| + |x_2|.$$

Show further that, for any real numbers $x_1, x_2, \cdots, x_n$,

$$|x_1 + x_2 + \cdots + x_n| \leqslant |x_1| + |x_2| + \cdots + |x_n|.$$

(i) The polynomial f is defined by

$$\mathrm{f}(x) = 1 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$$

where the coefficients are real and satisfy $|a_i| \leqslant A$ for $i = 1, 2, \cdots, n-1$, where $A \geqslant 1$.

(a) If $|x| < 1$, show that

$$|\mathrm{f}(x) - 1| \leqslant \frac{A|x|}{1-|x|}.$$

(b) Let ω be a real root of f, so that $\mathrm{f}(\omega) = 0$. In the case $|\omega| < 1$, show that

$$\frac{1}{1+A} \leqslant |\omega| \leqslant 1 + A. \qquad (*)$$

(c) Show further that the inequalities (*) also hold if $|\omega| \geqslant 1$.

(ii) Find the integer root or roots of the quintic equation

$$135x^5 - 135x^4 - 100x^3 - 91x^2 - 126^x + 135 = 0.$$

4 You are not required to consider issues of convergence in this question.

For any sequence of numbers $a_1, a_2, \ldots, a_m, \ldots, a_n$, the notation $\prod_{i=m}^{n} a_i$ denotes the product $a_m a_{m+1} \cdots a_n$.

(i) Use the identity $2\cos(x)\sin(x) = \sin(2x)$ to evaluate the product

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right).$$

(ii) Simplify the expression

$$\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right) \qquad \left(0 < x < \frac{1}{2}\pi\right).$$

Using differentiation, or otherwise, show that, for $0 < x < \frac{1}{2}\pi$,

$$\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - 2\cot(2x).$$

(iii) Using the results

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1,$$

show that

$$\prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right) = \frac{\sin(x)}{x}$$

and evaluate

$$\sum_{j=2}^{\infty}\frac{1}{2^{j-2}}\tan\left(\frac{\pi}{2^j}\right).$$

5 The sequence $u_0, u_1, \ldots$ is said to be a *constant sequence* if $u_n = u_{n+1}$ for $n = 0, 1, 2, \ldots$. The sequence is said to be a *sequence of period* 2 if $u_n = u_{n+2}$ for $n = 0, 1, 2, \ldots$ and the sequence is not constant.

(i) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = \mathrm{f}(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$\mathrm{f}(x) = p + (x - p)x,$$

and p is a given real number.

Find the values of a for which the sequence is constant.

Show that the sequence has period 2 for some value of a if and only if $p > 3$ or $p < -1$.

(ii) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = \mathrm{f}(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$\mathrm{f}(x) = q + (x - p)x,$$

and p and q are given real numbers.

Show that there is no value of a for which the sequence is constant if and only if $\mathrm{f}(x) > x$ for all x.

Deduce that, if there is no value of a for which the sequence is constant, then there is no value of a for which the sequence has period 2.

Is it true that, if there is no value of a for which the sequence has period 2, then there is no value of a for which the sequence is constant?

6 **Note:** You may assume that if the functions $y_1(x)$ and $y_2(x)$ both satisfy one of the differential equations in this question, then the curves $y = y_1(x)$ and $y = y_2(x)$ do not intersect.

(i) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + x + 1$$

that has the form $y = mx + c$, where m and c are constants.

Let $y_3(x)$ be the solution of this differential equation with $y_3(0) = k$. Show that any stationary point on the curve $y = y_3(x)$ lies on the line $y = -x - 1$. Deduce that solution curves with $k < -2$ cannot have any stationary points.

Show further that any stationary point on the solution curve is a local minimum.

Use the substitution $Y = y + x$ to solve the differential equation, and sketch, on the same axes, the solutions with $k = 0, k = -2$ and $k = -3$.

(ii) Find the two solutions of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 - 2xy - 4x + 4y + 3$$

that have the form $y = mx + c$.

Let $y_4(x)$ be the solution of this differential equation with $y_4(0) = -2$. (Do not attempt to find this solution.)

Show that any stationary point on the curve $y = y_4(x)$ lies on one of two lines that you should identify. What can be said about the gradient of the curve at points between these lines?

Sketch the curve $y = y_4(x)$. You should include on your sketch the two straight line solutions and the two lines of stationary points.

7 **(i)** The points A, B and C have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, respectively. Each of these vectors is a unit vector (so $\mathbf{a}\cdot\mathbf{a} = 1$, for example) and

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}.$$

Show that $\mathbf{a}\cdot\mathbf{b} = -\frac{1}{2}$. What can be said about the triangle ABC? You should justify your answer.

(ii) The four distinct points $A_i \quad (i = 1, 2, 3, 4)$ have unit position vectors $\mathbf{a}_i$ and

$$\sum_{i=1}^{4} \mathbf{a}_i = \mathbf{0}.$$

Show that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_3 \cdot \mathbf{a}_4.$$

(a) Given that the four points lie in a plane, determine the shape of the quadrilateral with vertices A_1, A_2, A_3 and A_4.

(b) Given instead that the four points are the vertices of a regular tetrahedron, find the length of the sides of this tetrahedron.

8 The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $\mathrm{f}(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that

$$\mathrm{f}(\mathbf{MN}) = \mathrm{f}(\mathbf{M})\mathrm{f}(\mathbf{N})$$

for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

(i) You are given that there is a matrix $\mathbf{M}$ such that $\mathrm{f}(\mathbf{M}) \neq \mathbf{0}$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $\mathrm{f}(\mathbf{MI})$, show that $\mathrm{f}(\mathbf{I}) = 1$.

(ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $\mathrm{f}(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $\mathrm{f}(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d,

$$\mathrm{f}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -\mathrm{f}\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = \mathrm{f}\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right).$$

(iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $\mathrm{f}(\mathbf{A}) = \mathbf{0}$.

(iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2, \mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{PK}$ for suitable values of k, evaluate $\mathrm{f}(\mathbf{P})$.

Section B: Mechanics

9 A particle P is projected from a point O on horizontal ground with speed u and angle of projection α, where $0 < \alpha < \frac{1}{2}\pi$.

(i) Show that if $\sin\alpha < \frac{2\sqrt{2}}{3}$, then the distance OP is increasing throughout the flight.

Show also that if $\sin\alpha > \frac{2\sqrt{2}}{3}$, then OP will be decreasing at some time before the particle lands.

(ii) At the same time as P is projected, a particle Q is projected horizontally from O with speed v along the ground in the opposite direction from the trajectory of P. The ground is smooth. Show that if

$$2\sqrt{2}v > \left(\sin\alpha - 2\sqrt{2}\cos\alpha\right)u$$

then QP is increasing throughout the flight of P.

10 A small light ring is attached to the end A of a uniform rod AB of weight W and length $2a$. The ring can slide on a rough horizontal rail.

One end of a light inextensible string of length $2a$ is attached to the rod at B and the other end is attached to a point C on the rail so that the rod makes an angle of θ with the rail, where $0 < \theta < 90°$. The rod hangs in the same vertical plane as the rail.

A force of kW acts vertically downwards on the rod at B and the rod is in equilibrium.

(i) You are given that the string will break if the tension T is greater than W. Show that (assuming that the ring does not slip) the string will break if

$$2k + 1 > 4\sin\theta.$$

(ii) Show that (assuming that the string does not break) the ring will slip if

$$2k + 1 > (2k + 3)\mu\tan\theta,$$

where μ is the coefficient of friction between the rail and the ring.

(iii) You are now given that $\mu\tan\theta < 1$.

Show that, when k is increased gradually from zero, the ring will slip before the string breaks if

$$\mu < \frac{2\cos\theta}{1 + 2\sin\theta}.$$

Section C: Probability and Statistics

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n + 1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

(ii) You have N rods, of lengths $1, 2, 3, \ldots, N$ (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n + 1$ where n is a positive integer, the probability that these three rods can form a triangle (of non- zero area) is

$$\frac{n-1}{2n-1}.$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

(iii) You have $2M + 1$ rods, of lengths $1, 2, 3, \ldots, 2M + 1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non- zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

Note:

$$\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1).$$

12 The random variable X has the probability density function on the interval $[0,1]$:

$$\mathrm{f}(x) = \begin{cases} nx^{n-1} & 0 \leqslant x \leqslant 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where n is an integer greater than 1.

(i) Let $\mu = E(X)$. Find an expression for μ in terms of n, and show that the variance, σ^2, of X is given by

$$\sigma^2 = \frac{n}{(n+1)^2(n+2)}.$$

(ii) In the case $n = 2$, show without using decimal approximations that the interquartile range is less than 2σ.

(iii) Write down the first three terms and the $(k+1)$th term (where $0 \leqslant k \leqslant n$) of the binomial expansion of $(1+x)^n$ in ascending powers of x.

By setting $x = \dfrac{1}{n}$, show that μ is less than the median and greater than the lower quartile.

Note: You may assume that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots < 4.$$