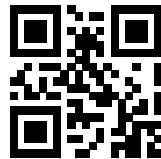


THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

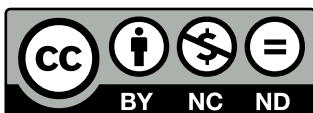
16-S2



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SUGGESTIONS TO DRYUFROMSHANGHAI@QQ.COM

Section A: Pure Mathematics

- 1** The curve C_1 has parametric equations $x = t^2, y = t^3$, where $-\infty < t < \infty$. Let O denote the point $(0, 0)$. The points P and Q on C_1 are such that $\angle POQ$ is a right angle.

- (i) Show that the tangents to C_1 at P and Q intersect on the curve C_2 with equation $4y^2 = 3x - 1$.
- (ii) Determine whether C_1 and C_2 meet, and sketch the two curves on the same axes.

- 2** Use the factor theorem to show that $a + b - c$ is a factor of

$$(a + b + c)^3 - 6(a + b + c)(a^2 + b^2 + c^2) + 8(a^3 + b^3 + c^3). \quad (*)$$

Hence factorise $(*)$ completely.

- (i) Use the result above to solve the equation

$$(x + 1)^3 - 3(x + 1)(2x^2 + 5) + 2(4x^3 + 13) = 0.$$

- (ii) By setting $d + e = c$, or otherwise, show that $(a + b - d - e)$ is a factor of

$$(a + b + d + e)^3 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x + 6)^3 - 6(x + 6)(x^2 + 14) + 8(x^3 + 36) = 0.$$

3 For each non-negative integer n , the polynomial f_n is defined by

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

(i) Show that $f'_n(x) = f_{n-1}(x)$ (for $n \geq 1$).

(ii) Show that, if a is a real root of the equation

$$f_n(x) = 0, \quad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geq 2$. Show that $f'_n(a)f'_n(b) > 0$ and use a sketch to deduce that $f_n(c) = 0$ for some number c between a and b .

Deduce that $(*)$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?

4 Let

$$y = \frac{x^2 + x \sin \theta + 1}{x^2 + x \cos \theta + 1}.$$

(i) Given that x is real, show that

$$(y \cos \theta - \sin \theta)^2 \geq 4(y - 1)^2.$$

Deduce that

$$y^2 + 1 \geq 4(y - 1)^2,$$

and hence that

$$\frac{4 - \sqrt{7}}{3} \leq y \leq \frac{4 + \sqrt{7}}{3}.$$

(ii) In the case $y = \frac{4 + \sqrt{7}}{3}$, show that

$$\sqrt{y^2 + 1} = 2(y - 1)$$

and find the corresponding values of x and $\tan \theta$.

- 5 In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \frac{p!}{q!(p-q)!} & \text{if } p \geq q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- (i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^n \binom{N+j-1}{j} = \binom{N+n}{n}.$$

- (ii) Show that, for any positive integers m, n and r with $r \leq m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^r \binom{m}{j} \binom{n}{r-j}.$$

- (iii) Show that, for any positive integers m and N ,

$$\sum_{j=0}^n (-1)^j \binom{N+m}{n-j} \binom{m+j-1}{j} = \binom{N}{n}.$$

- 6 This question concerns solutions of the differential equation

$$(1 - x^2) \left(\frac{dy}{dx} \right)^2 + k^2 y^2 = k^2 \quad (*)$$

where k is a positive integer.

For each value of k , let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k .

- (i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.
- (ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.
- (iii) Let $z(x) = 2(y_n(x))^2 - 1$. Show that

$$(1 - x^2) \left(\frac{dz}{dx} \right)^2 + 4n^2 z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

- (iv) Let $v(x) = y_n(y_m(x))$. Show that $v(x) = y_{mn}(x)$.

- 7 Show that

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx, \quad (*)$$

where f is any function for which the integrals exist.

- (i) Use $(*)$ to evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x} dx.$$

- (ii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x} dx.$$

- (iii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \ln(1 + \tan x) dx.$$

- (iv) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x (\cos x + \sin x)} dx.$$

8 Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx \quad (m > \frac{1}{2}).$$

Show by means of a sketch that

$$\sum_{r=m}^n \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2} dx, \quad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E . Use $(*)$ to obtain the following approximations for E :

$$E \approx 2; \quad E \approx \frac{5}{3}; \quad E \approx \frac{33}{20}.$$

(ii) Show that, when r is large, the error in approximating $\frac{1}{r^2}$ by $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2} dx$ is approximately $\frac{1}{4r^4}$.

Given that $E \approx 1.645$, show that $\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08$.

Section B: Mechanics

- 9 A small bullet of mass m is fired into a block of wood of mass M which is at rest. The speed of the bullet on entering the block is u . Its trajectory within the block is a horizontal straight line and the resistance to the bullet's motion is R , which is constant.
- (i) The block is fixed. The bullet travels a distance a inside the block before coming to rest. Find an expression for a in terms of m, u and R .
- (ii) Instead, the block is free to move on a smooth horizontal table. The bullet travels a distance b inside the block before coming to rest relative to the block, at which time the block has moved a distance c on the table. Find expressions for b and c in terms of M, m and a .

- 10 A thin uniform wire is bent into the shape of an isosceles triangle ABC , where AB and AC are of equal length and the angle at A is 2θ . The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ . The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geq 2 \tan \theta (1 + \sin \theta).$$

- 11 (i) Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a + ut \cos \alpha, ut \sin \alpha)$ and $(vt \cos \beta, b + vt \sin \beta)$, where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geq 0$.

Given that the two particles collide, show that

$$u \sin(\theta + \alpha) = v \sin(\theta + \beta),$$

where θ is the acute angle satisfying $\tan \theta = \frac{b}{a}$.

- (ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β . Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α , in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u \sin \alpha (u \sin \alpha - v \sin \beta) > bg.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Section C: Probability and Statistics

- 12 Starting with the result $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Write down, without proof, the corresponding result for four events A, B, C and D .

A pack of n cards, numbered $1, 2, \dots, n$, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the i th position in the row. Write down the following probabilities:

- (i) $P(E_i)$;
- (ii) $P(E_i \cap E_j)$, where $i \neq j$;
- (iii) $P(E_i \cap E_j \cap E_k)$, where $i \neq j, j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n+1} \frac{1}{n!}.$$

Find the probability that exactly one card is in the same position as the number it bears.

- 13 (i) The random variable X has a binomial distribution with parameters n and p , where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, that

$$P(X = 8) \approx \frac{1}{2\sqrt{2\pi}}.$$

- (ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n} (n!)^2}{\sqrt{n\pi}}.$$

- (iii) By considering a Poisson distribution with parameter n , show that

$$n! \approx \sqrt{2\pi n} e^{-n} n^n.$$