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Sixth Term Examination Paper

15-S2

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Section A: Pure Mathematics

1 **(i)** By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)\,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2\,.$$

2 **(i)** In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

(ii) The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

(iii) The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

3 Three rods have lengths a, b and c, where $a < b < c$. The three rods can be made into a triangle (possibly of zero area) if $a + b \geqslant c$.

(i) Let T_n be the number of triangles that can be made with three rods chosen from n rods of lengths $1, 2, 3, \ldots, n$ (where $n \geqslant 3$).

(ii) Show that $T_8 - T_7 = 2 + 4 + 6$ and evaluate $T_8 - T_6$. Write down expressions for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

(iii) Prove by induction that $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$, and find the corresponding result for an odd number of rods.

4 **(i)** The continuous function f is defined by

$$\tan \mathrm{f}(x) = x \qquad (-\infty < x < \infty)$$

and $\mathrm{f}(0) = \pi$. Sketch the curve $y = \mathrm{f}(x)$.

(ii) The continuous function g is defined by

$$\tan \mathrm{g}(x) = \frac{x}{1+x^2} \qquad (-\infty < x < \infty)$$

and $\mathrm{g}(0) = \pi$. Sketch the curves $y = \dfrac{x}{1+x^2}$ and $y = \mathrm{g}(x)$.

(iii) The continuous function h is defined by $\mathrm{h}(0) = \pi$ and

$$\tan \mathrm{h}(x) = \frac{x}{1-x^2} \qquad (x \neq \pm 1).$$

(The values of $\mathrm{h}(x)$ at $x = \pm 1$ are such that $\mathrm{h}(x)$ is continuous at these points.) Sketch the curves $y = \dfrac{x}{1-x^2}$ and $y = \mathrm{h}(x)$.

5 In this question, the arctan function satisfies $0 \leqslant \arctan x < \frac{1}{2}\pi$ for $x \geqslant 0$.

(i) Let

$$S_n = \sum_{m=1}^{n} \arctan\left(\frac{1}{2m^2}\right),$$

for $n = 1, 2, 3, \ldots$. Prove by induction that

$$\tan S_n = \frac{n}{n+1}.$$

Prove also that

$$S_n = \arctan \frac{n}{n+1}.$$

(ii) In a triangle ABC, the lengths of the sides AB and BC are $4n^2$ and $4n^4 - 1$, respectively, and the angle at B is a right angle. Let $\angle BCA = 2\alpha_n$. Show that

$$\sum_{n=1}^{\infty} \alpha_n = \tfrac{1}{4}\pi.$$

6 **(i)** Show that

$$\sec^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \frac{2}{1+\sin x}\,.$$

Hence integrate $\dfrac{1}{1+\sin x}$ with respect to x.

(ii) By means of the substitution $y = \pi - x$, show that

$$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x,$$

where f is any function for which these integrals exist. Hence evaluate

$$\int_0^\pi \frac{x}{1+\sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x.$$

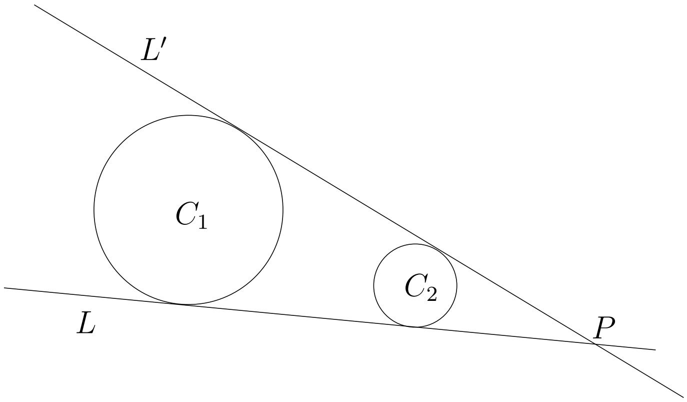
7 A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

(i) Let C be the circle of radius a in the x-y plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \leqslant a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\dfrac{a_2^2 - a_1^2}{4d}$. Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \geqslant \left(4d^2 + (a_2 - a_1)^2\right)\left(4d^2 + (a_2 + a_1)^2\right).$$

8



The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

Section B: Mechanics

9 An equilateral triangle ABC is made of three light rods each of length a. It is free to rotate in a vertical plane about a horizontal axis through A. Particles of mass $3m$ and $5m$ are attached to B and C respectively. Initially, the system hangs in equilibrium with BC below A.

(i) Show that, initially, the angle θ that BC makes with the horizontal is given by $\sin\theta = \frac{1}{7}$.

(ii) The triangle receives an impulse that imparts a speed v to the particle B. Find the minimum speed v_0 such that the system will perform complete rotations if $v > v_0$.

10 A particle of mass m is pulled along the floor of a room in a straight line by a light string which is pulled at constant speed V through a hole in the ceiling. The floor is smooth and horizontal, and the height of the room is h.

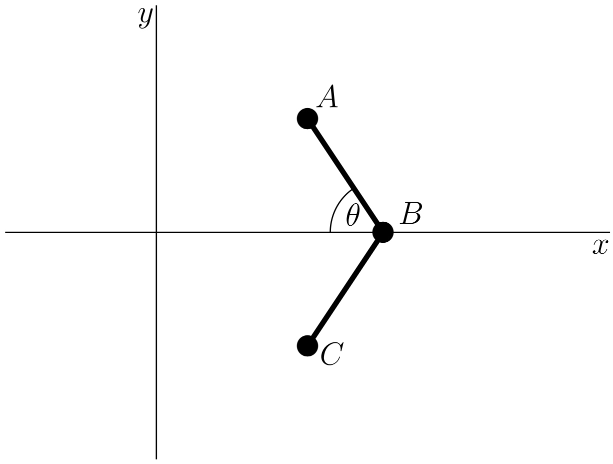
(i) Find, in terms of V and θ, the speed of the particle when the string makes an angle of θ with the vertical (and the particle is still in contact with the floor).

(ii) Find also the acceleration, in terms of V, h and θ.

(iii) Find the tension in the string and hence show that the particle will leave the floor when

$$\tan^4\theta = \frac{V^2}{gh}\,.$$

11 Three particles, A, B and C, each of mass m, lie on a smooth horizontal table. Particles A and C are attached to the two ends of a light inextensible string of length $2a$ and particle B is attached to the midpoint of the string. Initially, A, B and C are at rest at points $(0, a), (0, 0)$ and $(0, -a)$, respectively. An impulse is delivered to B, imparting to it a speed u in the positive x direction. The string remains taut throughout the subsequent motion.



(i) At time t, the angle between the x-axis and the string joining A and B is θ, as shown in the diagram, and B is at $(x, 0)$. Write down the coordinates of A in terms of x, a and θ. Given that the velocity of B is $(v, 0)$, show that the velocity of A is $(\dot{x} + a\sin\theta\,\dot{\theta}\,,\ a\cos\theta\,\dot{\theta})$, where the dot denotes differentiation with respect to time.

(ii) Show that, before particles A and C first collide,

$$3\dot{x} + 2a\dot{\theta}\sin\theta = v \quad \text{and} \quad \dot{\theta}^2 = \frac{v^2}{a^2(3 - 2\sin^2\theta)}\,.$$

(iii) When A and C collide, the collision is elastic (no energy is lost). At what value of θ does the second collision between particles A and C occur? (You should justify your answer.)

(iv) When $v = 0$, what are the possible values of θ? Is $v = 0$ whenever θ takes these values?

Section C: Probability and Statistics

12 Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows: Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT. The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT? Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$. Find the probability that C wins.

13 The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$\mathrm{f}(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{for } x \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant. It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geqslant 0$. If $X \leqslant y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$\mathrm{E}(C) = ky + \frac{a}{\lambda}\mathrm{e}^{-\lambda y}\,.$$

How should y be chosen in order to minimise $\mathrm{E}(C)$, in the different cases that arise according to the value of a/k?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.