

THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

14-S2



Compiled by: Dr Yu 郁博士

www.CasperYC.club

Last updated: May 8, 2025



SUGGESTIONS TO DRYUFROMSHANGHAI@QQ.COM

Section A: Pure Mathematics

1 In the triangle ABC , the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leq \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB .

(i) Show that $x \cos \theta = 1 - x \cos \alpha - x \cos \beta$, and obtain an expression for $x \sin \theta$ in terms of x , α and β . Hence show that

$$(1 + 2 \cos(\alpha + \beta))x^2 - 2(\cos \alpha + \cos \beta)x + 1 = 0. \quad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P .]

(ii) State the condition on α and β for $(*)$ to be linear in x . If this condition does not hold (but the condition $0 < \alpha \leq \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45^\circ$ and (b) $\alpha = 30^\circ$, $\beta = 90^\circ$, and illustrate each case with a sketch.

2 This question concerns the inequality

$$\int_0^\pi (f(x))^2 dx \leq \int_0^\pi (f'(x))^2 dx. \quad (*)$$

(i) Show that $(*)$ is satisfied in the case $f(x) = \sin nx$, where n is a positive integer. Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $f(0) \neq 0$ or $f(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $f(0) = f(\pi) = 0$. By setting $f(x) = ax^2 + bx + c$, where a , b and c are suitably chosen, show that $\pi^2 \leq 10$. By setting $f(x) = p \sin \frac{1}{2}x + q \cos \frac{1}{2}x + r$, where p , q and r are suitably chosen, obtain another inequality for π . Which of these inequalities leads to a better estimate for π^2 ?

- 3 (i) Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geq 0$, is $c(m^2 + 1)^{-\frac{1}{2}}$.
- (ii) The curve C lies in the x - y plane. Let the line L be tangent to C at a point P on C , and let a be the shortest distance between the origin and L . The curve C has the property that the distance a is the same for all points P on C . Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2(1 + (y')^2). \quad (*)$$

By first differentiating $(*)$ with respect to x , show that either $y = mx \pm a(1 + m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

- (iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.
- 4 (i) By using the substitution $u = 1/x$, show that for $b > 0$

$$\int_{1/b}^b \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)} dx = 0.$$

- (ii) By using the substitution $u = 1/x$, show that for $b > 0$,

$$\int_{1/b}^b \frac{\arctan x}{x} dx = \frac{\pi \ln b}{2}.$$

- (iii) By using the result $\int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = k/x$, for suitable k , show that

$$\int_0^\infty \frac{1}{(a^2 + x^2)^2} dx = \frac{\pi}{4a^3} \quad (a > 0).$$

5 Given that $y = xu$, where u is a function of x , write down an expression for $\frac{dy}{dx}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{dy}{dx} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$. Give your answer in the form of a quadratic in x and y .

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b , or otherwise, solve

$$\frac{dy}{dx} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

6 By simplifying $\sin(r + \frac{1}{2})x - \sin(r - \frac{1}{2})x$ or otherwise show that, for $\sin \frac{1}{2}x \neq 0$,

$$\cos x + \cos 2x + \dots + \cos nx = \frac{\sin(n + \frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}.$$

The functions S_n , for $n = 1, 2, \dots$, are defined by

$$S_n(x) = \sum_{r=1}^n \frac{1}{r} \sin rx \quad (0 \leq x \leq \pi).$$

(i) Find the stationary points of $S_2(x)$ for $0 \leq x \leq \pi$, and sketch this function.

(ii) Show that if $S_n(x)$ has a stationary point at $x = x_0$, where $0 < x_0 < \pi$, then

$$\sin nx_0 = (1 - \cos nx_0) \tan \frac{1}{2}x_0$$

and hence that $S_n(x_0) \geq S_{n-1}(x_0)$. Deduce that if $S_{n-1}(x) > 0$ for all x in the interval $0 < x < \pi$, then $S_n(x) > 0$ for all x in this interval.

(iii) Prove that $S_n(x) \geq 0$ for $n \geq 1$ and $0 \leq x \leq \pi$.

- 7 (i) The function f is defined by $f(x) = |x - a| + |x - b|$, where $a < b$. Sketch the graph of $f(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$.

Sketch on the same diagram the graph of $g(x)$, where $g(x) = |2x - a - b|$. What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $(b, f(b))$ and $(a, f(a))$?

- (ii) Show graphically that the equation

$$|x - a| + |x - b| = |x - c|,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

- (iii) For the equation

$$|x - a| + |x - b| = |x - c| + |x - d|,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a , b , c and d in each case.

- 8 For positive integers n , a and b , the integer c_r ($0 \leq r \leq n$) is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r , n , a and b . For given n , a and b , let m denote a value of r for which c_r is greatest (that is, $c_m \geq c_r$ for $0 \leq r \leq n$). Show that

$$\frac{b(n+1)}{a+b} - 1 \leq m \leq \frac{b(n+1)}{a+b}.$$

Deduce that m is either a unique integer or one of two consecutive integers. Let $G(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m .

- (i) Evaluate $G(9, 1, 3)$ and $G(9, 2, 3)$.
- (ii) For any positive integer k , find $G(2k, a, a)$ and $G(2k - 1, a, a)$ in terms of k .
- (iii) For fixed n and b , determine a value of a for which $G(n, a, b)$ is greatest.
- (iv) For fixed n , find the greatest possible value of $G(n, 1, b)$. For which values of b is this greatest value achieved?

Section B: Mechanics

9 A uniform rectangular lamina $ABCD$ rests in equilibrium in a vertical plane with the corner A in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side AB at a distance d from A . The other end of the string is attached to a point on the wall above A where it makes an acute angle θ with the downwards vertical. The side AB makes an acute angle ϕ with the upwards vertical at A . The sides BC and AB have lengths $2a$ and $2b$ respectively. The coefficient of friction between the lamina and the wall is μ .

(i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d \sin(\theta + \phi) = (\cos \theta + \mu \sin \theta)(a \cos \phi + b \sin \phi). \quad (*)$$

(ii) How should (*) be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

(iii) Find a condition on d , in terms of a , b , $\tan \theta$ and $\tan \phi$, which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if $b(2 \tan \theta + \tan \phi) < a$.

10 A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x - y plane, where the x -axis is horizontal, the y -axis is vertical and the origin is O .

(i) Obtain the Cartesian equation of the particle's trajectory in terms of u , g and λ , where $\lambda = \tan \theta$.

(ii) Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x , the coordinate y can take all values up to a maximum value, Y , which you should determine as a function of x , u and g .

(iii) Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u .

(iv) Hence find the furthest distance from O that can be achieved by such a projectile.

11 A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O , of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion. Let x be the distance between O and the ring. Taking the y -axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x , L and α .

(i) By considering the vertical component of the equation of motion of P , show that

$$km\ddot{x} \cos \alpha = T \cos \alpha - kmg,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R .

(ii) Show that $\frac{\sin \alpha}{(1 - \sin \alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k .

(iii) Show that $\ddot{x} = -g \tan \alpha$.

Section C: Probability and Statistics

- 12** The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $f(t)$ and cumulative distribution function $F(t)$. The *hazard function*, $h(t)$, is defined, for $F(t) < 1$, by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

- (i) Given that the fly lives to at least time t , show that the probability of its dying within the following δt is approximately $h(t) \delta t$ for small values of δt .
- (ii) Find the hazard function in the case $F(t) = t/a$ for $0 < t < a$. Sketch $f(t)$ and $h(t)$ in this case.
- (iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1} . Determine the probability density function for T .
- (iv) Show that $h(t)$ is constant for $t > b$ and zero otherwise if and only if $f(t) = ke^{-k(t-b)}$ for $t > b$, where k is a positive constant.
- (v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$h(t) = \left(\frac{\lambda}{\theta^\lambda} \right) t^{\lambda-1},$$

where λ and θ are positive constants. Find the probability density function for T .

13 A random number generator prints out a sequence of integers I_1, I_2, I_3, \dots . Each integer is independently equally likely to be any one of $1, 2, \dots, n$, where n is fixed. The random variable X takes the value r , where I_r is the first integer which is a repeat of some earlier integer. Write down an expression for $P(X = 4)$.

(i) Find an expression for $P(X = r)$, where $2 \leq r \leq n + 1$. Hence show that, for any positive integer n ,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right) \frac{2}{n} + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \frac{3}{n} + \dots = 1.$$

(ii) Write down an expression for $E(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $P(X \geq k)$.

(iv) Show that, for any discrete random variable Y taking the values $1, 2, \dots, N$,

$$E(Y) = \sum_{k=1}^N P(Y \geq k).$$

Hence show that, for any positive integer n ,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3^2}{n}\right) + \dots = 0.$$