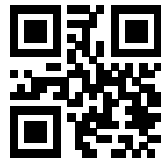


THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

13-S2



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Section A: Pure Mathematics

- 1 (i) Find the value of m for which the line $y = mx$ touches the curve $y = \ln x$. If instead the line intersects the curve when $x = a$ and $x = b$, where $a < b$, show that $a^b = b^a$. Show by means of a sketch that $a < e < b$.
- (ii) The line $y = mx + c$, where $c > 0$, intersects the curve $y = \ln x$ when $x = p$ and $x = q$, where $p < q$. Show by means of a sketch, or otherwise, that $p^q > q^p$.
- (iii) Show by means of a sketch that the straight line through the points $(p, \ln p)$ and $(q, \ln q)$, where $e \leq p < q$, intersects the y -axis at a positive value of y . Which is greater, π^e or e^π ?
- (iv) Show, using a sketch or otherwise, that if $0 < p < q$ and $\frac{\ln q - \ln p}{q - p} = e^{-1}$, then $q^p > p^q$.

- 2 For $n \geq 0$, let

$$I_n = \int_0^1 x^n (1 - x)^n dx.$$

- (i) For $n \geq 1$, show by means of a substitution that

$$\int_0^1 x^{n-1} (1 - x)^n dx = \int_0^1 x^n (1 - x)^{n-1} dx$$

and deduce that

$$2 \int_0^1 x^{n-1} (1 - x)^n dx = I_{n-1}.$$

Show also, for $n \geq 1$, that

$$I_n = \frac{n}{n+1} \int_0^1 x^{n-1} (1 - x)^{n+1} dx$$

and hence that $I_n = \frac{n}{2(2n+1)} I_{n-1}$.

- (ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!}.$$

- (iii) Use the substitution $x = \sin^2 \theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.

- 3 (i) Given that the cubic equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and $c < 0$, show with the help of sketches that either exactly one of the roots is positive or all three of the roots are positive.

- (ii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real positive roots show that

$$a^2 > b > 0, \quad a < 0, \quad c < 0. \quad (*)$$

[Hint: Consider the turning points.]

- (iii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and that

$$ab < 0, \quad c > 0,$$

determine, with the help of sketches, the signs of the roots.

- (iv) Show by means of an explicit example (giving values for a , b and c) that it is possible for the conditions $(*)$ to be satisfied even though the corresponding cubic equation has only one real root.

- 4 The line passing through the point $(a, 0)$ with gradient b intersects the circle of unit radius centred at the origin at P and Q , and M is the midpoint of the chord PQ . Find the coordinates of M in terms of a and b .

- (i) Suppose b is fixed and positive. As a varies, M traces out a curve (the *locus* of M). Show that $x = -by$ on this curve. Given that a varies with $-1 \leq a \leq 1$, show that the locus is a line segment of length $2b/(1 + b^2)^{\frac{1}{2}}$. Give a sketch showing the locus and the unit circle.

- (ii) Find the locus of M in the following cases, giving in each case its cartesian equation, describing it geometrically and sketching it in relation to the unit circle:

(a) a is fixed with $0 < a < 1$, and b varies with $-\infty < b < \infty$;

(b) $ab = 1$, and b varies with $0 < b \leq 1$.

- 5 (i) A function $f(x)$ satisfies $f(x) = f(1 - x)$ for all x . Show, by differentiating with respect to x , that $f'(\frac{1}{2}) = 0$. If, in addition, $f(x) = f(\frac{1}{x})$ for all (non-zero) x , show that $f'(-1) = 0$ and that $f'(2) = 0$.
- (ii) The function f is defined, for $x \neq 0$ and $x \neq 1$, by

$$f(x) = \frac{(x^2 - x + 1)^3}{(x^2 - x)^2}.$$

Show that $f(x) = f(\frac{1}{x})$ and $f(x) = f(1 - x)$. Given that it has exactly three stationary points, sketch the curve $y = f(x)$.

- (iii) Hence, or otherwise, find all the roots of the equation $f(x) = \frac{27}{4}$ and state the ranges of values of x for which $f(x) > \frac{27}{4}$. Find also all the roots of the equation $f(x) = \frac{343}{36}$ and state the ranges of values of x for which $f(x) > \frac{343}{36}$.

- 6 In this question, the following theorem may be used. *Let $u_1, u_2, \dots$ be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leq b$ for all n , where b is some fixed number) and increasing (that is, $u_n \geq u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).* The sequence $u_1, u_2, \dots$ is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \quad (n \geq 1). \quad (*)$$

- (i) Show that, for $n \geq 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}.$$

- (ii) Prove, by induction or otherwise, that $1 \leq u_n \leq 2$ for all n .
- (iii) Show that the sequence $u_1, u_3, u_5, \dots$ tends to a limit, and that the sequence $u_2, u_4, u_6, \dots$ tends to a limit. Find these limits and deduce that the sequence $u_1, u_2, u_3, \dots$ tends to a limit. Would this conclusion change if the sequence were defined by (*) and $u_1 = 3$?

- 7 (i) Write down a solution of the equation

$$x^2 - 2y^2 = 1, \quad (*)$$

for which x and y are non-negative integers. Show that, if $x = p$, $y = q$ is a solution of $(*)$, then so also is $x = 3p + 4q$, $y = 2p + 3q$. Hence find two solutions of $(*)$ for which x is a positive odd integer and y is a positive even integer.

- (ii) Show that, if x is an odd integer and y is an even integer, $(*)$ can be written in the form

$$n^2 = \frac{1}{2}m(m+1),$$

where m and n are integers.

- (iii) The positive integers a , b and c satisfy

$$b^3 = c^4 - a^2,$$

where b is a prime number. Express a and c^2 in terms of b in the two cases that arise. Find a solution of $a^2 + b^3 = c^4$, where a , b and c are positive integers but b is not prime.

- 8 The function f satisfies $f(x) > 0$ for $x \geq 0$ and is strictly decreasing (which means that $f(b) < f(a)$ for $b > a$).

- (i) For $t \geq 0$, let $A_0(t)$ be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve $y = f(x)$, the y -axis and the line $y = f(t)$. Show that $A_0(t)$ can be written in the form

$$A_0(t) = x_0 (f(x_0) - f(t)),$$

where x_0 satisfies $x_0 f'(x_0) + f(x_0) = f(t)$.

- (ii) The function g is defined, for $t > 0$, by

$$g(t) = \frac{1}{t} \int_0^t f(x) dx.$$

Show that $tg'(t) = f(t) - g(t)$. Making use of a sketch show that, for $t > 0$,

$$\int_0^t (f(x) - f(t)) dx > A_0(t)$$

and deduce that $-t^2 g'(t) > A_0(t)$.

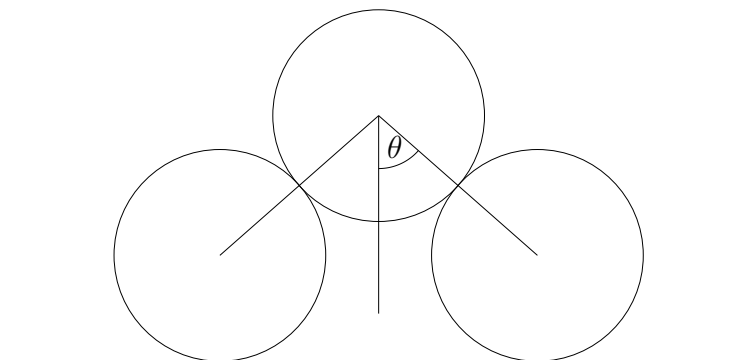
- (iii) In the case $f(x) = \frac{1}{1+x}$, use the above to establish the inequality

$$\ln \sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}},$$

for $t > 0$.

Section B: Mechanics

- 9 The diagram shows three identical discs in equilibrium in a vertical plane. Two discs rest, not in contact with each other, on a horizontal surface and the third disc rests on the other two. The angle at the upper vertex of the triangle joining the centres of the discs is 2θ .



The weight of each disc is W . The coefficient of friction between a disc and the horizontal surface is μ and the coefficient of friction between the discs is also μ .

- (i) Show that the normal reaction between the horizontal surface and a disc in contact with the surface is $\frac{3}{2}W$.
 - (ii) Find the normal reaction between two discs in contact and show that the magnitude of the frictional force between two discs in contact is $\frac{W \sin \theta}{2(1 + \cos \theta)}$.
 - (iii) Show that if $\mu < 2 - \sqrt{3}$ there is no value of θ for which equilibrium is possible.
- 10 A particle is projected at an angle of elevation α (where $\alpha > 0$) from a point A on horizontal ground. At a general point in its trajectory the angle of elevation of the particle from A is θ and its direction of motion is at an angle ϕ above the horizontal (with $\phi \geq 0$ for the first half of the trajectory and $\phi \leq 0$ for the second half). Let B denote the point on the trajectory at which $\theta = \frac{1}{2}\alpha$ and let C denote the point on the trajectory at which $\phi = -\frac{1}{2}\alpha$.
- (i) Show that, at a general point on the trajectory, $2 \tan \theta = \tan \alpha + \tan \phi$.
 - (ii) Show that, if B and C are the same point, then $\alpha = 60^\circ$.
 - (iii) Given that $\alpha < 60^\circ$, determine whether the particle reaches the point B first or the point C first.

- 11** Three identical particles lie, not touching one another, in a straight line on a smooth horizontal surface. One particle is projected with speed u directly towards the other two which are at rest. The coefficient of restitution in all collisions is e , where $0 < e < 1$.
- (i) Show that, after the second collision, the speeds of the particles are $\frac{1}{2}u(1 - e)$, $\frac{1}{4}u(1 - e^2)$ and $\frac{1}{4}u(1 + e)^2$. Deduce that there will be a third collision whatever the value of e .
- (ii) Show that there will be a fourth collision if and only if e is less than a particular value which you should determine.

Section C: Probability and Statistics

- 12 The random variable U has a Poisson distribution with parameter λ . The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1, 3, 5, 7, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2, 4, 6, 8, \dots \\ 0 & \text{otherwise} \end{cases}$$

- (i) Find $E(X)$ and $E(Y)$ in terms of λ , α and β , where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots$$

- (ii) Show that

$$\text{Var}(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha + \beta)^2}$$

and obtain the corresponding expression for $\text{Var}(Y)$. Are there any non-zero values of λ for which $\text{Var}(X) + \text{Var}(Y) = \text{Var}(X + Y)$?

- 13 A biased coin has probability p of showing a head and probability q of showing a tail, where $p \neq 0$, $q \neq 0$ and $p \neq q$. When the coin is tossed repeatedly, runs occur. A *straight run* of length n is a sequence of n consecutive heads or n consecutive tails. An *alternating run* of length n is a sequence of length n alternating between heads and tails. An alternating run can start with either a head or a tail. Let S be the length of the longest straight run beginning with the first toss and let A be the length of the longest alternating run beginning with the first toss.

- (i) Explain why $P(A = 1) = p^2 + q^2$ and find $P(S = 1)$. Show that $P(S = 1) < P(A = 1)$.
- (ii) Show that $P(S = 2) = P(A = 2)$ and determine the relationship between $P(S = 3)$ and $P(A = 3)$.
- (iii) Show that, for $n > 1$, $P(S = 2n) > P(A = 2n)$ and determine the corresponding relationship between $P(S = 2n + 1)$ and $P(A = 2n + 1)$. [You are advised *not* to use $p + q = 1$ in this part.]