

THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

11-S3



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Section A: Pure Mathematics

- 1 (i) Find the general solution of the differential equation

$$\frac{du}{dx} - \left(\frac{x+2}{x+1} \right) u = 0.$$

- (ii) Show that substituting $y = ze^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \quad (*)$$

leads to a first order differential equation for $\frac{dz}{dx}$. Find z and hence show that the general solution of (*) is

$$y = Ax + Be^{-x},$$

where A and B are arbitrary constants.

- (iii) Find the general solution of the differential equation

$$(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (x+1)^2.$$

- 2 The polynomial $f(x)$ is defined by

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where $n \geq 2$ and the coefficients $a_0, \dots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $f(x) = 0$ has a rational root p/q , where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}f(p/q)$, find the value of q and deduce that any rational root of the equation $f(x) = 0$ must be an integer.

- (i) Show that the n th root of 2 is irrational for $n \geq 2$.

- (ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

- (iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geq 2$.

- 3 (i) Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \quad (p \neq 0, q \neq 0)$$

can be written in the form

$$a(x - \alpha)^3 + b(x - \beta)^3,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β .

- (ii) Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2 - 2^{\frac{1}{3}})}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω , where $\omega = e^{2\pi i/3}$.

- (iii) Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r .

- 4 The following result applies to any function f which is continuous, has positive gradient and satisfies $f(0) = 0$:

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy, \quad (*)$$

where f^{-1} denotes the inverse function of f , and $a \geq 0$ and $b \geq 0$.

- (i) By considering the graph of $y = f(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

- (ii) By taking $f(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Verify that equality holds under the condition you stated above.

- (iii) Show that, for $0 \leq a \leq \frac{1}{2}\pi$ and $0 \leq b \leq 1$,

$$ab \leq b \arcsin b + \sqrt{1 - b^2} - \cos a.$$

Deduce that, for $t \geq 1$,

$$\arcsin(t^{-1}) \geq t - \sqrt{t^2 - 1}.$$

- 5 A movable point P has cartesian coordinates (x, y) , where x and y are functions of t . The polar coordinates of P with respect to the origin O are r and θ .

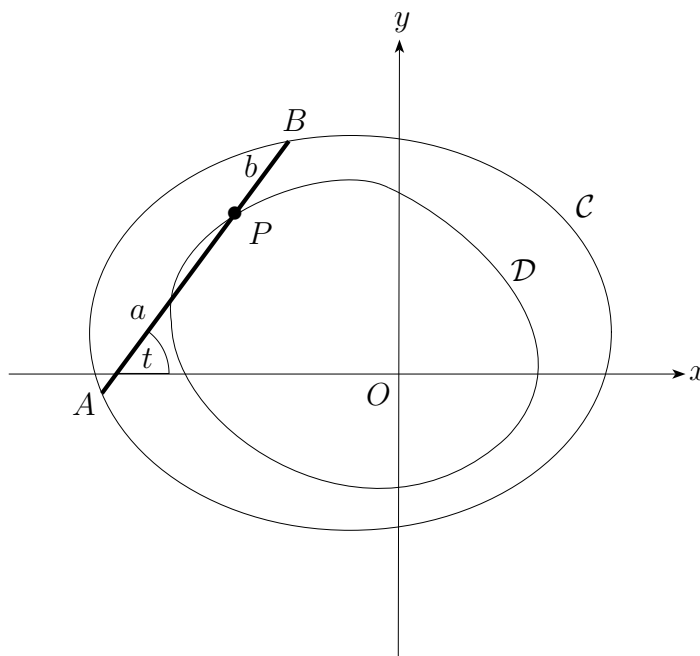
(i) Starting with the expression

$$\frac{1}{2} \int r^2 d\theta$$

for the area swept out by OP , obtain the equivalent expression

$$\frac{1}{2} \int \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt. \quad (*)$$

- (ii) The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B . The angle between AB and the positive x direction is t . As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P . Write down the coordinates of A and B in terms of a, b, x, y and t .

- (iii) The areas swept out by OA, OB and OP are denoted by $[A], [B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \frac{1}{2} \int_0^{2\pi} \left(\left(x + \frac{dy}{dt} \right) \cos t + \left(y - \frac{dx}{dt} \right) \sin t \right) dt.$$

Obtain a corresponding expression for $[B]$ involving b . Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab .

- 6 The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} dt,$$

$$U = \int_{\ln 2}^{\ln 3} \frac{u}{2 \sinh u} du,$$

$$V = - \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1 - v^2} dv,$$

$$X = \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \ln(\coth x) dx.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

- 7 Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a} \right)^n,$$

where n is a positive integer and a is any given positive integer.

- (i) In the case when n is even, show by induction that T_n can be written in the form

$$A_n + B_n \sqrt{a(a+1)},$$

where A_n and B_n are integers (depending on a and n) and $A_n^2 = a(a+1)B_n^2 + 1$.

- (ii) In the case when n is odd, show by considering $(\sqrt{a+1} + \sqrt{a})T_m$ where m is even, or otherwise, that T_n can be written in the form

$$C_n \sqrt{a+1} + D_n \sqrt{a},$$

where C_n and D_n are integers (depending on a and n) and $(a+1)C_n^2 = aD_n^2 + 1$.

- (iii) Deduce that, for each n , T_n can be written as the sum of the square roots of two consecutive integers.

- 8 The complex numbers z and w are related by

$$w = \frac{1 + iz}{i + z}.$$

Let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real. Express u and v in terms of x and y .

- (i) By setting $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$ with one point omitted.
- (ii) Find the locus of w when the locus of z is the line segment $y = 0$, $-1 < x < 1$.
- (iii) Find the locus of w when the locus of z is the line segment $x = 0$, $-1 < y < 1$.
- (iv) Find the locus of w when the locus of z is the line $y = 1$, $-\infty < x < \infty$.

Section B: Mechanics

- 9** Particles P and Q have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius a which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at O . Initially, the particles are in equilibrium.

Equilibrium is slightly disturbed and Q begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle θ between OQ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g \cos \theta + 6g \sin \theta = 10g.$$

- (i) Given that Q loses contact with the cylinder first, show that it does so when $\theta = \beta$, where β satisfies

$$15 \cos \beta + 6 \sin \beta = 10.$$

- (ii) Show also that while P and Q are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin \theta + \cos \theta)$.

- 10** Particles P and Q , each of mass m , lie initially at rest a distance a apart on a smooth horizontal plane. They are connected by a light elastic string of natural length a and modulus of elasticity $\frac{1}{2}ma\omega^2$, where ω is a constant.

- (i) Then P receives an impulse which gives it a velocity u directly away from Q . Show that when the string next returns to length a , the particles have travelled a distance $\frac{1}{2}\pi u/\omega$, and find the speed of each particle.
- (ii) Find also the total time between the impulse and the subsequent collision of the particles.

- 11** A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b ; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc.

- (i) Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin \phi = 2a \sin \frac{1}{2}\theta.$$

- (ii) Show further that the magnitude of the couple is

$$\frac{mga^2 \sin \theta}{\sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}}.$$

- (iii) The disc is now released from rest. Show that its angular speed, ω , when the strings are vertical is given by

$$\frac{a^2 \omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}.$$

Section C: Probability and Statistics

- 12** The random variable N takes positive integer values and has pgf (probability generating function) $G(t)$. The random variables X_i , where $i = 1, 2, 3, \dots$, are independently and identically distributed, each with pgf $H(t)$. The random variables X_i are also independent of N . The random variable Y is defined by

$$Y = \sum_{i=1}^N X_i.$$

- (i) Given that the pgf of Y is $G(H(t))$, show that

$$E(Y) = E(N)E(X_i) \quad \text{and} \quad \text{Var}(Y) = \text{Var}(N)(E(X_i))^2 + E(N) \text{Var}(X_i).$$

- (ii) A fair coin is tossed until a head occurs. The total number of tosses is N . The coin is then tossed a further N times and the total number of heads in these N tosses is Y . Find in this particular case the pgf of Y , $E(Y)$, $\text{Var}(Y)$ and $P(Y = r)$.

- 13** In this question, the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x , so for example $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

- (i) A bag contains n balls, of which b are black. A sample of k balls is drawn, one after another, at random *with* replacement. The random variable X denotes the number of black balls in the sample. By considering

$$\frac{P(X = r + 1)}{P(X = r)},$$

show that, in the case that it is unique, the most probable number of black balls in the sample is

$$\left\lfloor \frac{(k+1)b}{n} \right\rfloor.$$

Under what circumstances is the answer not unique?

- (ii) A bag contains n balls, of which b are black. A sample of k balls (where $k \leq b$) is drawn, one after another, at random *without* replacement. Find, in the case that it is unique, the most probable number of black balls in the sample.

Under what circumstances is the answer not unique?