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Sixth Term Examination Paper

10-S3

Compiled by: Dr Yu 郁博士

www.CasperYC.club

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Suggestions to DrYuFromShanghai@QQ.com

Section A: Pure Mathematics

1 Let $x_1, x_2, \ldots, x_n$ and x_{n+1} be any fixed real numbers. The numbers A and B are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k \quad \text{and} \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2\,,$$

and the numbers C and D are defined by

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k \quad \text{and} \quad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2\,.$$

(i) Express C in terms of A, x_{n+1} and n.

(ii) Show that $B = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k^2 - A^2\,.$

(iii) Express D in terms of B, A, x_{n+1} and n.

Hence show that $(n+1)D \geqslant nB$ for all values of x_{n+1}, but that $D < B$ if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}\,.$$

2 In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,\mathrm{d}x = \frac{a}{2\sinh a}\,.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,\mathrm{d}x$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,\mathrm{d}x\,.$$

3 For any given positive integer n, a number a (which may be complex) is said to be a *primitive* n*th root of unity* if $a^n = 1$ and there is no integer m such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive nth roots of unity, the coefficient of the highest power of x is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

(i) Find $\mathrm{C}_1(x), \mathrm{C}_2(x), \mathrm{C}_3(x), \mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of n for which $\mathrm{C}_n(x) = x^4 + 1$.

(iii) Given that p is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers q, r and s such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.

4 **(i)** The number α is a common root of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ (that is, α satisfies both equations). Given that $a \neq c$, show that

$$\alpha = -\frac{b-d}{a-c}.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

Does this result still hold if the condition $a \neq c$ is not imposed?

(ii) Show that the equations $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0.$$

Hence, or otherwise, find the values of b for which the equations $2x^2 + 5x + 2b = 0$ and $2x^3 + 7x^2 + 5x + 1 = 0$ have at least one common root.

5 The vertices A, B, C and D of a square have coordinates $(0,0), (a,0), (a,a)$ and $(0,a)$, respectively. The points P and Q have coordinates $(an,0)$ and $(0,am)$ respectively, where $0 < m < n < 1$. The line CP produced meets DA produced at R and the line CQ produced meets BA produced at S. The line PQ produced meets the line RS produced at T.

(i) Show that TA is perpendicular to AC.

(ii) Explain how, given a square of area a^2, a square of area $2a^2$ may be constructed using only a straight-edge.

[**Note**: a straight-edge is a ruler with no markings on it; no measurements (and no use of compasses) are allowed in the construction.]

6 The points P, Q and R lie on a sphere of unit radius centred at the origin, O, which is fixed. Initially, P is at $P_0(1,0,0)$, Q is at $Q_0(0,1,0)$ and R is at $R_0(0,0,1)$.

(i) The sphere is then rotated about the z-axis, so that the line OP turns directly towards the positive y-axis through an angle ϕ. The position of P after this rotation is denoted by P_1. Write down the coordinates of P_1.

(ii) The sphere is now rotated about the line in the x-y plane perpendicular to OP_1, so that the line OP turns directly towards the positive z-axis through an angle λ. The position of P after this rotation is denoted by P_2. Find the coordinates of P_2. Find also the coordinates of the points Q_2 and R_2, which are the positions of Q and R after the two rotations.

(iii) The sphere is now rotated for a third time, so that P returns from P_2 to its original position P_0. During the rotation, P remains in the plane containing P_0, P_2 and O. Show that the angle of this rotation, θ, satisfies

$$\cos\theta = \cos\phi\cos\lambda\,,$$

and find a vector in the direction of the axis about which this rotation takes place.

7 **(i)** Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2 y = 0\,.$$

(ii) Obtain a similar equation relating $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4}$, $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

(iii) Conjecture and prove a relation between $\dfrac{\mathrm{d}^{n+2} y}{\mathrm{d}x^{n+2}}$, $\dfrac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}}$ and $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

(iv) Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2 \sin^2\theta}{2!} + \frac{m^2(m^2 - 2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

(v) State the degree of the polynomial.

8 Given that $\mathrm{P}(x) = \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$, write down an expression for

$$\int \frac{\mathrm{P}(x)}{\big(\mathrm{Q}(x)\big)^2}\,\mathrm{d}x\,.$$

(i) By choosing the function $\mathrm{R}(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x\,.$$

Show that the choice of $\mathrm{R}(x)$ is not unique and, by comparing the two functions $\mathrm{R}(x)$ corresponding to two different values of a, explain how the different choices are related.

(ii) Find the general solution of

$$(1 + \cos x + 2\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x\,.$$

Section B: Mechanics

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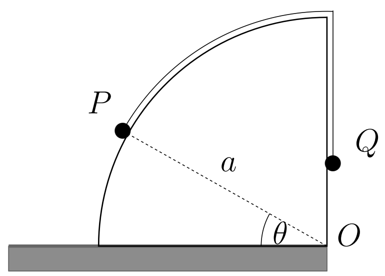

The diagram shows two particles, P and Q, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre O, which is at the edge of the table, and radius a. The angle between OP and the table is θ. The masses of P and Q are m and M, respectively, where $m < M$.

Initially, P is held at rest on the table and in contact with the block, Q is vertically above O, and the string is taut. Then P is released. Given that, in the subsequent motion, P remains in contact with the block as θ increases from 0 to $\frac{1}{2}\pi$, find an expression, in terms of m, M, θ and g, for the normal reaction of the block on P and show that

$$\frac{m}{M} \geqslant \frac{\pi - 1}{3}\,.$$

10 A small bead B, of mass m, slides without friction on a fixed horizontal ring of radius a. The centre of the ring is at O. The bead is attached by a light elastic string to a fixed point P in the plane of the ring such that $OP = b$, where $b > a$. The natural length of the elastic string is c, where $c < b - a$, and its modulus of elasticity is λ.

(i) Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta + \phi)\,,$$

where $\theta = \angle BPO$ and $\phi = \angle BOP$.

(ii) Given that θ and ϕ are small, show that $a(\theta + \phi) \approx b\theta$.

(iii) Hence find the period of small oscillations about the equilibrium position $\theta = \phi = 0$.

11 A bullet of mass m is fired horizontally with speed u into a wooden block of mass M at rest on a horizontal surface. The coefficient of friction between the block and the surface is μ. While the bullet is moving through the block, it experiences a constant force of resistance to its motion of magnitude R, where $R > (M+m)\mu g$. The bullet moves horizontally in the block and does not emerge from the other side of the block.

(i) Show that the magnitude, a, of the deceleration of the bullet relative to the block while the bullet is moving through the block is given by

$$a = \frac{R}{m} + \frac{R - (M+m)\mu g}{M}\,.$$

(ii) Show that the common speed, v, of the block and bullet when the bullet stops moving through the block satisfies

$$av = \frac{Ru - (M+m)\mu g u}{M}\,.$$

(iii) Obtain an expression, in terms of u, v and a, for the distance moved by the block while the bullet is moving through the block.

(iv) Show that the total distance moved by the block is

$$\frac{muv}{2(M+m)\mu g}\,.$$

Describe briefly what happens if $R < (M+m)\mu g$.

Section C: Probability and Statistics

12 **(i)** The infinite series S is given by

$$S = 1 + (1+d)r + (1+2d)r^2 + \cdots + (1+nd)r^n + \cdots ,$$

for $|r| < 1$. By considering $S - rS$, or otherwise, prove that

$$S = \frac{1}{1-r} + \frac{rd}{(1-r)^2} .$$

(ii) Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is a; the probability that an arrow shot by Boadicea hits the target is b. Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is $1/a$.

(iii) Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is α and the probability that Boadicea wins is β. Show that

$$\alpha = \frac{a}{1 - a'b'} ,$$

where $a' = 1 - a$ and $b' = 1 - b$, and find β.

(iv) Show that the expected number of shots in the contest is $\dfrac{\alpha}{a} + \dfrac{\beta}{b}$.

13 In this question, $\text{Corr}(U, V)$ denotes the product moment correlation coefficient between the random variables U and V, defined by

$$\text{Corr}(U, V) \equiv \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\,\text{Var}(V)}} .$$

(i) The independent random variables Z_1, Z_2 and Z_3 each have expectation 0 and variance 1. What is the value of $\text{Corr}(Z_1, Z_2)$?

(ii) Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12} Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}} Z_2 ,$$

where ρ_{12} is a given constant with $-1 < \rho_{12} < 1$. Find $\text{E}(Y_2), \text{Var}(Y_2)$ and $\text{Corr}(Y_1, Y_2)$.

(iii) Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where a, b and c are real constants and $c \geqslant 0$. Given that $\text{E}(Y_3) = 0, \text{Var}(Y_3) = 1, \text{Corr}(Y_1, Y_3) = \rho_{13}$ and $\text{Corr}(Y_2, Y_3) = \rho_{23}$, express a, b and c in terms of ρ_{23}, ρ_{13} and ρ_{12}.

(iv) Given constants μ_i and σ_i, for $i = 1, 2$ and 3, give expressions in terms of the Y_i for random variables X_i such that $\text{E}(X_i) = \mu_i, \text{Var}(X_i) = \sigma_i^2$ and $\text{Corr}(X_i, X_j) = \rho_{ij}$.