Problem Solving for Undergraduate Admission Interviews in Mathematics

CC BY NC ND

Last updated: May 3, 2023

Question 1: www.CasperYC.club

How many ways can you arrange each of the digits 1 to 6 to create distinct 6 digit numbers?

How many of these contain the digits $1, 2$ and 3 next to each other and in that order?

In how many arrangements does 5 occur before 1?

How many distinct 6 digit numbers are there in which all of the digits 1 to 5 appear?

Question 2: www.CasperYC.club

The Power Set is formed from all subsets of a given set.

If a set contains n elements what is the cardinality of its Power Set?

How many subsets contain a given element x_1?

Question 3: www.CasperYC.club

If a round table has n people sitting around it, what is the probability of person A sitting exactly k seats away from person B?

Question 4: www.CasperYC.club

How can you maximise the number of regions n straight lines will divide the plane in to?

What is this maximum in terms of n?

What if we replace lines with circles?

What does this tell you about Venn Diagrams?

Question 5: www.CasperYC.club

How many vertices and edges does a line segment have? A square? A cube? A tesseract?

Can you conjecture formulae for the number of edges and vertices of an n dimensional hypercube?

Can you give the coordinates of the vertices of a tesseract (where 4 edges coincide with the coordinate axes)?

What would the longest length between two vertices be?

Question 6: www.CasperYC.club

In a football competition where every round played is a knockout match (i.e. a draw leading to a replay is not an option), how many matches will be played in the competition in total if there are n teams?

Question 7: `www.CasperYC.club`

In how many ways can $2n$ opponents be paired in the first round of a tennis competition?

Can you come up with a more succinct expression for your previous solution (i.e. if I gave you a large value for n you could then use your calculator to calculate the answer quickly)?

Question 8: www.CasperYC.club

How many distinct tessellations of the plane use only one regular polygon?

Why are there only five platonic solids?

Using Euler's Polyhedron Formula $V - E + F = 2$ show that a platonic solid made of triangles must have $4, 8$ or 20 faces.

Question 9: www.CasperYC.club

The numbers 1 to 1000 are written on a blackboard.
You randomly choose two numbers a and b from among them and replace them with their difference.
You continue this process until you are left with a single number on the board, is it possible for you to be left with the number 1?

Question 10: `www.CasperYC.club`

If I colour three faces of a cube red and the other faces blue, how many distinguishable colourings are there?

Question 11: www.CasperYC.club

Every subset of the set $(1, 2, 3, \ldots, n)$ either contains the element 1 or it does not.
By considering these two possibilities, show that

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

Explain why

$$\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}.$$

Question 12: www.CasperYC.club

10 distinct points lie within a unit square, prove at least two of the points lie within $\frac{\sqrt{2}}{3}$ units of each other.

Question 13: `www.CasperYC.club`

Consider an infinite chessboard, the squares of which have been filled with positive integers. Each of these integers is the arithmetic mean of four of its neighbours (above, below, left, right). Show that all the integers are equal to each other.

Question 14: www.CasperYC.club

Two full decks of cards are shuffled and placed side by side.
I take the top card from each pile and pair them up.
What is the probability I have:

i) 52 matching pairs?

ii) 51 matching pairs?

iii) 50 matching pairs?

iv) 49 matching pairs?

v) k matching pairs?

Question 15: www.CasperYC.club

If n points are distributed around the circumference of a circle and each point is joined to every other point by a chord of the circle (assuming that no three chords intersect at a point inside the circle) into how many regions is the circle divided?

Question 16: `www.CasperYC.club`

$2n$ points are chosen in the plane such that no 3 are collinear, n are coloured blue and n are coloured red.

Prove that it is always possible to join the n red points to the n blue points by line segments, such that no two line segments cross.

Question 17: www.CasperYC.club

For $n > 1$, the integers from 1 to n^2 are placed in the cells of an $n \times n$ chessboard.
Show that there is a pair of horizontally, vertically, or diagonally adjacent cells whose value differs by at least $n + 1$.

Question 18:

Say you have finitely many red and blue points on a plane with the interesting property:
every line segment that joins two points of the same colour contains a point of the other colour.
Prove that all the points lie on a single straight line.

Question 19: `www.CasperYC.club`

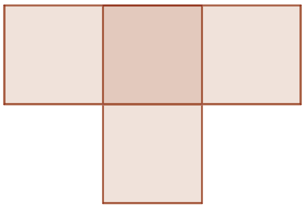
n students are standing in a field such that the distance between each pair is distinct.
Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student.
Prove that there is a pair of students that throw their balls to each other.

Question 20: www.CasperYC.club

A longevity chain is a sequence of consecutive integers, whose digit sums are never a multiple of 9. What is the longest possible length of a longevity chain?

Question 21: www.CasperYC.club

The T-tetromino is the shape made by joining four 1×1 squares edge to edge, as shown.



A rectangle R has dimensions $(2a) \times (2b)$ where a and b are integers. The expression 'R can be tiled by T' means that R can be covered exactly by copies of T without gaps or overlaps.

i) Can R be tiled by T when both a and b are even?

ii) Can R be tiled by T when both a and b are odd?

Question 22: www.CasperYC.club

I have a large supply of counters which I place in each of the 1×1 squares of an 8×8 chessboard (1 counter on each square).

Each counter is red, white or blue. A particular pattern of coloured counters is called an arrangement.

Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters.

Question 23: www.CasperYC.club

Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal (the perimeter of a cuboid is the sum of the lengths of all its twelve edges).

Question 24: www.CasperYC.club

A two player game is played on a 5×5 grid. A token starts in the bottom left corner of the grid. On each turn, a player can move the token one or two units to the right, or to the leftmost square of the above row. The last player who is able to move wins. Determine which positions of the token are winning positions and which are losing. Generalize this problem to larger grids. How many winning positions are there on an $m \times n$ grid?

Question 25:

Two people play a game:

There are n sweets in a pile and they each take it in turns to remove at least one sweet from the pile whilst ensuring they take no more than half of what remains.

The person who removes the last sweet is the loser.

Are there values of n for which the second player has a winning strategy?

Question 26: www.CasperYC.club

Show that the family of concentric circles which have centre $(\frac{1}{3}, \sqrt{2})$ are such that each circle has exactly 1 lattice point on its boundary, and each lattice point is on a circle.

Question 27: www.CasperYC.club

There are 6 ropes in a bag.

In each step, two rope ends are picked at random, tied together and put back into a bag.

The process is repeated until there are no free ends.

What is the expected number of loops e_n at the end of the process?

(Hint: Find a formula linking e_n and e_{n-1})

Question 28: www.CasperYC.club

For each non-empty subset of integers $(1, 2, 3, \ldots, n)$ consider the reciprocal of the product of the elements. Let S_n denote the sum of these products. Conjecture and prove a formula for S_n.

Question 29: www.CasperYC.club

A thin rod is broken into three pieces.

What is the probability that a triangle can be formed from the three pieces?

Question 30: www.CasperYC.club

Given n consecutive positive integers, show that $n!$ is a factor of their product.

Question 31: www.CasperYC.club

The lengths of the sides of a triangle are in geometric progression with common ratio r.
Prove that

$$\frac{2}{1+\sqrt{5}} < r < \frac{1+\sqrt{5}}{2}.$$

Question 32: www.CasperYC.club

Find the number of integer solutions the equation $|x| + |y| \leq 100$.

Question 33: www.CasperYC.club

Are there any integer solutions to the equation $x^2 + y^2 = 3z^2$ where x, y, z are co-prime?
Are there any integer solutions at all?

Question 34: `www.CasperYC.club`

Sketch $x^2 - ny^2 = 0$ where n is a natural number.

Find all natural solution pairs (x, y) in the case $n = 9$.

Find all natural solution pairs (x, y) in the case $n = 10$.

Question 35: www.CasperYC.club

How many natural number solutions are there to the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ where $a < b < c$?

Question 36: www.CasperYC.club

Find all positive integer solutions (x, y) to $x^2 + y^2 = 2015$.

Will the following equation have any positive integer solutions $x^2 + 33y^2 = 555555555$?

Question 37: www.CasperYC.club

If n, x, y, z are all positive integers, find all solutions of the equation $n^x + n^y = n^z$.

Question 38: `www.CasperYC.club`

Use algebraic techniques to determine whether the following equation has any real solutions:

$$x^4 + 2x^3 + 3x^2 + 2x + 1 = 0.$$

Question 39:

How many solutions are there to the equation $|x| + |x - 1| = 0$?

How many solutions are there to the equation $|27x^2 - 48| + |6x^2 - 5x - 4| = 0$?

What are the solutions of the equation $|\sin(2x)| + |\cos(0.5x)| = 0$?

Question 40: www.CasperYC.club

If three positive real numbers a, b, c satisfy the following equations show that at least one of them must be 1 and hence deduce all solutions:

$$abc = 1 \quad \text{and} \quad a + b + c = 3 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Question 41: www.CasperYC.club

Prove that the only solution to the equation $x^2 + y^2 + z^2 = 2xyz$ for integers x, y, z is $x = y = z = 0$.

Question 42:

Show that no three real numbers a, b, c satisfy the equations $a + b + c = 0 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Question 43: www.CasperYC.club

If x and y are positive integers, find all solutions of the equation $2xy - 4x^2 + 12x - 5y = 11$.

Question 44: www.CasperYC.club

A right angled triangle has all of its sides an integer length.

If the length of the perimeter equals the area, find all such triangles.

Question 45: www.CasperYC.club

Prove that $n^2(n^2-1)(n^2-4)$ is divisible by 360 whenever n is a natural number.

Question 46: www.CasperYC.club

Find the last two digits of 99^n.

Question 47: www.CasperYC.club

Which will be larger as $n \to \infty$; $2^{2^{2^n}}$ or 100^{100^n}?

Question 48: www.CasperYC.club

Do you know a solution to the equation $5^x = 4^x + 3^x$? Are there any more? Prove it.

Question 49: www.CasperYC.club

Prove that $n^2 - 1$ is divisible by 8 when n is odd.

Prove that $n^5 - n$ is divisible by 6 whenever n is a natural number.

Prove that $n^5 - n$ is divisible by 30 whenever n is a natural number.

Question 50: www.CasperYC.club

Simplify

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2n-1)^2 - (2n)^2.$$

Find

$$21^2 - 22^2 + 23^2 - 24^2 + \ldots + 39^2 - 40^2.$$

Question 51: www.CasperYC.club

Construct a counter example to the statement: When written in decimal notation, every square number has at most 1000 digits that are not 0 or 1.

Question 52: `www.CasperYC.club`

Prove that $4^n - 1$ is divisible by 3 whenever n is a natural number.

Question 53: www.CasperYC.club

A natural number from 1 to 1000000 is selected at random, what is the probability its cube ends in 11?

Question 54: www.CasperYC.club

Given that $8 < \pi^2 < 10$, show that

$$\frac{1}{\log_2(\pi)} + \frac{1}{\log_5(\pi)} > 2 \quad \text{and} \quad \frac{1}{\log_2(\pi)} + \frac{1}{\log_\pi(2)} > 2.$$

Question 55: www.CasperYC.club

Prove that there are infinitely many primes.

Question 56: `www.CasperYC.club`

Prove that there are infinitely many primes of the form $4n + 3$.

Question 57: www.CasperYC.club

Is $\log_2(3)$ rational? Prove it.

Question 58: `www.CasperYC.club`

Prove that $14^n + 11$ is never prime.

Question 59: www.CasperYC.club

Let n be a natural number. Suppose $a^n - 1$ is prime.

Show that $a = 2$ and that n must be prime (*Mersenne Primes*).

Comment on primes of the form $2^n + 1$ (*Fermat Numbers*).

Question 60: `www.CasperYC.club`

Find all prime numbers p such that $2p-1$ and $2p+1$ are also prime.

Question 61: www.CasperYC.club

Show that $3 < \pi < 4$.

Question 62: www.CasperYC.club

If the ratio of consecutive *Fibonacci numbers* approaches a limit what must this limit be?

Question 63: www.CasperYC.club

Find the exact value of $\cos^2(1^\circ) + \cos^2(2^\circ) + \cos^2(3^\circ) + \ldots + \cos^2(89^\circ)$.

Question 64: www.CasperYC.club

If a natural number n has N digits how many digits can n^2 have? What about n^n?
How would you write a formula for the number of digits of n?

Question 65: www.CasperYC.club

Given that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6},$$

find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

Question 66: www.CasperYC.club

Prove that if a, b, c are all odd then the quadratic equation $ax^2 + bx + c = 0$ cannot have rational roots.

Question 67: www.CasperYC.club

Is $\tan(1^\circ)$ irrational? What about $\cos(1^\circ)$?

Question 68: www.CasperYC.club

Find
$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{(n-1)^2}{n^3}+\frac{(n-2)^2}{n^3}+\ldots+\frac{n}{n^3}\right).$$

Question 69: www.CasperYC.club

Evaluate

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right).$$

Question 70: www.CasperYC.club

Conjecture and prove a formula for $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n!$.

Question 71: www.CasperYC.club

Is 1234567891011 a square number? Is 24681012141618202224?

Question 72: www.CasperYC.club

Let $\mathrm{f}(x, y)$ be a function of two real variables which is not identically zero.
If $\mathrm{f}(x, y) = k(\mathrm{f}(y, x))$ for all values of x and y, what are the possible values of k?

Question 73: www.CasperYC.club

Let $\mathrm{h}(x) = x^3 + ax$, where a is a constant. When will an inverse to $\mathrm{h}(x)$ exist for all x?

Question 74: www.CasperYC.club

Suppose that $\mathrm{f}(0) = 0$ and that, for $x \neq 0, 0 < \dfrac{\mathrm{f}(x)}{x} < 1$.
Show that

$$-\frac{1}{2} < \int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x < \frac{1}{2}.$$

How does the above inequality change if $0 < \dfrac{\mathrm{f}(x)}{x^2} < 1$ instead?

Question 75: www.CasperYC.club

Show that $\cos(n\theta) = \mathrm{f}_n(\cos(\theta))$ for polynomials $\mathrm{f}_n(x)$ satisfying $\mathrm{f}_{n+1}(x) = 2x\mathrm{f}_n(x) - \mathrm{f}_{n-1}(x)$. Find all the roots of $\mathrm{f}_2(x) + \mathrm{f}_3(x) = 0$, and write them in the form $\cos(\phi)$ for suitable ϕ.

Question 76: www.CasperYC.club

Consider the cubic curve given by the equation

$$y = ax^3 + bx^2 + cx + d,$$

find conditions on a, b, c, d which ensure the curve has a local maximum and a local minimum. Under these conditions, show that the curve has a point of inflection midway between the turning points.

Question 77: www.CasperYC.club

Find all real valued functions which satisfy

$$(\mathrm{f}(x+y))^2 \equiv (\mathrm{f}(x))^2 + (\mathrm{f}(y))^2.$$

Question 78: www.CasperYC.club

What is the domain and range of the functions;

$$\mathrm{f}(x) = \ln(x), \mathrm{ff}(x) \quad \text{and} \quad \mathrm{fff}(x)?$$

What about $\mathrm{f}^n(x)$?

Question 79: www.CasperYC.club

Find the value of

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}$$

Question 80: www.CasperYC.club

Find the smallest $a > 1$ such that $\dfrac{a+\sin(x)}{a+\sin(y)} \leq \mathrm{e}^{y-x}$ for all $x \leq y$.

Question 81: `www.CasperYC.club`

Let $\mathrm{f}(x)$ be a non-constant function satisfying the functional equation $\mathrm{f}(x+y)=\mathrm{f}(x)\mathrm{f}(y)$.
Show that $\mathrm{f}(n)=k^n$ for all integers n and for $k=\mathrm{f}(0)$.
Show also that the same holds for all rational numbers and that $k>0$.

Question 82: www.CasperYC.club

Where in the plane is $\sin^2(x) + \cos^2(y) = 1$?

Question 83: www.CasperYC.club

Sketch $y = x\ln(x)$.

Question 84: www.CasperYC.club

Sketch $y = \dfrac{\ln(x)}{x}$ and hence find all natural solutions of the equation $a^b = b^a$.

Question 85: www.CasperYC.club

Sketch $y = x^x$ and $y = x^{\frac{1}{x}}$.

Question 86: www.CasperYC.club

Sketch $y = \dfrac{\sin(x)}{x}$ and $y = \dfrac{\sin(x)}{x-1}$.

Question 87: www.CasperYC.club

Sketch $y = \cos\left(\frac{1}{x}\right)$ and $y = \sin\left(\frac{1}{x}\right)$.

Question 88: www.CasperYC.club

Sketch $y = \dfrac{x + \sin(x)}{x - \sin(x)}$.

Question 89: www.CasperYC.club

Sketch $y = \cos(x + |x|)$ for $-2\pi < x < 2\pi$.

Question 90: www.CasperYC.club

Sketch $y = \sqrt{x^3 - x}$ and $y^2 = x^3 - x$.

Question 91: www.CasperYC.club

Sketch $y = \dfrac{x^4 - 7x^2 + 12}{x^4 - 4x^2 + 4}$.

Question 92: www.CasperYC.club

Sketch $y = \dfrac{x^2+1}{x^2-1}$.

Question 93: `www.CasperYC.club`

Sketch $y = |x^2 - 1|, y = x^{\frac{1}{3}}, y = x^{\frac{2}{3}}$ and comment on their derivative.

Question 94: www.CasperYC.club

Sketch $y = x^2 - x^4$ and $y^2 = x^2 - x^4$ (consider the derivative at the origin carefully).

Question 95: www.CasperYC.club

Sketch $y = \mathrm{e}^{-x^2} - \mathrm{e}^{-3x^2}$.

Question 96: www.CasperYC.club

By sketching appropriate graphs, find all solutions to the equation $x - 1 = (\mathrm{e} - 1)\ln(x)$.
Hence sketch the graph with equation $y = \mathrm{e}^x - x^{\mathrm{e}}$.

Question 97: www.CasperYC.club

Write $\dfrac{3\mathrm{e}^{x}-\mathrm{e}^{-x}}{\mathrm{e}^{x}+\mathrm{e}^{-x}}$ in the form $a+\dfrac{b}{\mathrm{e}^{2x}+1}$ and hence sketch $y=\dfrac{3\mathrm{e}^{x}-\mathrm{e}^{-x}}{\mathrm{e}^{x}+\mathrm{e}^{-x}}$.

Question 98: www.CasperYC.club

Sketch $x^{2n} + y^{2n} = 1$ for $n = 2$ and 4.

Explain what happens to the graph as $n \to \infty$.

Question 99: `www.CasperYC.club`

Sketch the curve $|3x^2+y^2-12| = |x^2-y^2+4|$.

Question 100: www.CasperYC.club

Sketch $y = \sqrt{1 - x^2} + \sqrt{4 - x^2}$.

Question 101: www.CasperYC.club

Sketch $1 = |x| + |y|, 1 = |x| - |y|$ and $1 = |y - x|$.

Question 102: `www.CasperYC.club`

What is the area of the region in the Cartesian plane whose points (x, y) satisfy $|x|+|y|+|x+y| < 2$?

Question 103: www.CasperYC.club

Find the minimum value of the expression $|x-1|+|x-2|+|x-4|+|x-6|$.

Question 104: www.CasperYC.club

Solve the differential equation for $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ subject to the initial condition $x = 0, y = 1$ and $k > 0$. Sketch the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky(1 - \frac{y}{M})$$

where M is a large constant and the same initial conditions apply (without directly finding y).

Question 105: www.CasperYC.club

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \displaystyle\int_0^x t^8 \mathrm{e}^t \,\mathrm{d}t$.

Question 106: www.CasperYC.club

Find $\mathrm{f}(x)$ if

$$\int_0^x \mathrm{f}(t)\,\mathrm{d}t = 3\mathrm{f}(x) + k$$

where k is a constant.

Question 107: www.CasperYC.club

Find explicit expressions for $\sinh^{-1}(x), \cosh^{-1}(x)$ and $\tanh^{-1}(x)$.

Question 108: www.CasperYC.club

Is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

divergent?

How about the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots ?$$

Sketch on separate axes $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ considering your sketches and by using integration justify your claims.

Question 109: www.CasperYC.club

By considering the inequality

$$\int_0^t (f(x)+\mu g(x))^2\,\mathrm{d}x \geq 0$$

where μ is a constant, prove that, for all functions f(x) and g(x):

$$\left(\int_0^t f(x)g(x)\,\mathrm{d}x\right)^2 \leq \left(\int_0^t (f(x))^2\,\mathrm{d}x\right)\left(\int_0^t (g(x))^2\,\mathrm{d}x\right) \qquad \textbf{(Cauchy–Schwarz Inequality)}$$

Hence show that

$$\int_0^1 (1+x^5)^{\frac{1}{2}}\,\mathrm{d}x \leq \sqrt{\frac{7}{6}}.$$

Question 110: www.CasperYC.club

Let

$$A = \int \frac{\sin(x)}{\sin(x)+\cos(x)}\,\mathrm{d}x \quad \text{and} \quad B = \int \frac{\cos(x)}{\sin(x)+\cos(x)}\,\mathrm{d}x,$$

find A and B.

Question 111: www.CasperYC.club

Integrate $\sin^4(x)\cos(x)$ and $\sin^6(x)\cos^3(x)$, in general when will this method work?

Question 112: www.CasperYC.club

Evaluate

$$I = \int \frac{1}{x^n + x} \, \mathrm{d}x.$$

Question 113: www.CasperYC.club

Prove that for a continuous function

$$\int_b^a \mathrm{f}(x)\,\mathrm{d}x = \int_b^a \mathrm{f}(a+b-x)\,\mathrm{d}x.$$

Hence evaluate

$$I = \int_4^8 \frac{\ln(9-x)}{\ln(9-x)+\ln(x-3)}\,\mathrm{d}x \quad \text{and} \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin^{2000}(\theta)}{\sin^{2000}(\theta)+\cos^{2000}(\theta)}\,\mathrm{d}\theta$$

Question 114: www.CasperYC.club

Evaluate

$$I = \int_0^1 \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, \mathrm{d}x.$$

Question 115: www.CasperYC.club

By considering the graph of the function $\mathrm{f}(x) = x^{-s}$, show that

$$\frac{1}{s-1} < 1 + 2^{-s} + 3^{-s} + \cdots < \frac{s}{s-1}.$$

Question 116: www.CasperYC.club

Evaluate

$$I = \int \frac{1}{1-\sin(x)}\,\mathrm{d}x, \quad J = \int \mathrm{e}^{x}\sin(x)\,\mathrm{d}x, \quad K = \int \sqrt{\mathrm{e}^{2x}+1}\,\mathrm{d}x$$

Question 117:

Which of the following numbers is bigger and why?

$$\int_0^1 \sqrt[4]{1-x^7}\,\mathrm{d}x \quad \text{or} \quad \int_0^1 \sqrt[7]{1-x^4}\,\mathrm{d}x$$

Question 118: www.CasperYC.club

Show that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{(\pi - x)\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x$$

Hence find

$$I = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x.$$

Question 119: www.CasperYC.club

Show that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x.$$

Hence find

$$I = \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Question 120: www.CasperYC.club

What is the shortest distance from the point $A(3,1)$ to the curve with equation $y = x^2 + 1$?

What is the shortest distance from the line $y = x$ and the curve $y = x^2 + 1$?

What is the shortest distance between the two curves $y = x^2 + 1$ and $x = y^2 + 1$?

Question 121: `www.CasperYC.club`

The points $A(6,0)$ and $B(0,-4)$ are points on a triangle, the third point lies on the graph of $y=x^2$, find the co–ordinates of the third point which minimises the area of the triangle.

Question 122:

If I have a triangle of fixed perimeter P what will the maximum area be?

Does there exist a right-angled triangle of fixed perimeter P of smallest area?

Question 123: www.CasperYC.club

Prove, without directly calculating its value, that $11^{10} - 1$ is divisible by 100.

Question 124: www.CasperYC.club

Find the sum of the coefficients of the polynomial obtained after expanding and collecting terms of the product $(1-3x+3x^2-5x^3+5x^4)(1+3x-3x^2+5x^3-5x^4)$.

Question 125: www.CasperYC.club

Find a polynomial with integer coefficients whose roots include $\sqrt{2} + \sqrt{3}$.

Question 126: www.CasperYC.club

Prove that in the product

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$$

after multiplying out and collecting terms, there does not appear a term in x of odd degree.

Question 127:

Determine m, an integer, so that the equation $x^4 - (3m+2)x^2 + m^2 = 0$ has four real solutions for x that form an arithmetic progression.

Question 128: www.CasperYC.club

If the rational quantity $\frac{p}{q}$ (in lowest terms) is a root of

$$\mathrm{f}(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

prove that $p | a_0$ and $q | a_n$.

Hence show that the nth root of an integer is either an integer itself or irrational.

Question 129: www.CasperYC.club

Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear then their average x–coordinate is some constant k. Find k.

Question 130: www.CasperYC.club

Prove that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5.

Question 131: www.CasperYC.club

Show that if n is a positive integer greater than one then $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ is not an integer.

Question 132: www.CasperYC.club

Consider the sequence $0, 1, 1, 2, 2, \ldots, r, r, r+1, r+1, \ldots,$ deduce the sum of the first n terms $S(n)$. Prove that $S(s+t) - S(s-t) = st$ where s and t are positive integers and $s > t$.

Question 133: www.CasperYC.club

Prove that for n a positive integer

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 3.$$

Question 134: www.CasperYC.club

Let $P(x, y)$ be a polynomial in x and y such that:
$P(x, y) \equiv P(y, x)$ and $(x - y)$ is a factor of $P(x, y)$.
Deduce that $(x - y)^2$ is a factor of $P(x, y)$.

Question 135: www.CasperYC.club

If $2\log(x-2y)=\log(x)+\log(y)$ find $\dfrac{x}{y}$.

Question 136: `www.CasperYC.club`

Let a be the integer consisting of m digit 1's and b be the integer consisting of a digit 1 at the start, a digit 5 at the end and with $(m-1)$ digit 0's in between.
Show that $(ab+1)$ is a perfect square and find its square root.

Question 137: www.CasperYC.club

Prove that 10201 is composite in any base.

Question 138: www.CasperYC.club

How many integers from 1 to 10^{30} inclusive are perfect squares, cubes or fifth powers?

Question 139: www.CasperYC.club

Prove that for any positive integer n and any real number x, $\left[\frac{[nx]}{n}\right]$ where $[z]$ denotes the largest integer value less than or equal to z.

Question 140: `www.CasperYC.club`

If you pick 3 cards from a randomly shuffled pack of cards are you more likely to see a face card than not?

Question 141: www.CasperYC.club

Evaluate, without the use of a calculator, $\log_3(169) \cdot \log_{13}(243)$.

Question 142:

A three–dimensional version of noughts and crosses can be played with a 4×4 cube, the winner is the first player to get four noughts (or crosses) in a straight line.
How many winning lines are there?

Question 143: `www.CasperYC.club`

Two positive numbers, a and b, with distinct first digits are multiplied together. Is it possible for the first digit of the product to fall strictly between the first digits of the two numbers?

Question 144: www.CasperYC.club

A gambler played a game with his friend, he bet half of his money on the toss of coin; he won on heads and lost on tails. The game was repeated over and over and at the end the gambler had lost as many times as he had won. Did he make money, lose money or break even?

Question 145: www.CasperYC.club

Alice and Bob play a fair game repeatedly for £1 a game. If originally Alice has £a and Bob has £b, what is Alice's chance of winning all of Bob's money, assuming that play continues until one person has lost all of his or her money?

Question 146: www.CasperYC.club

Determine the function $F(x)$ which satisfies the functional equation $x^2F(x)+F(1-x)=2x-x^4$.

Question 147: www.CasperYC.club

There are $n!$ permutations $(s_1, s_2, s_3, \ldots, s_n)$ of $(1, 2, 3, ..., n)$.
How many of them satisfy $s_k > k - 3$ for $k = 1, 2, 3, \ldots, n$?

Question 148: www.CasperYC.club

Is $\frac{1}{n+1}\binom{2n}{n}$ always integer valued when n is a positive integer?

Question 149: www.CasperYC.club

If you are faced with a corridor of width m and another corridor of width n, which is perpendicular to the first, what is the maximum length of ladder you can carry through the corridors?

You may model the ladder as a one–dimensional rod.

Question 150: `www.CasperYC.club`

A bracelet is made up of a combination of 11 red, yellow or blue beads.

How many distinct bracelets can be made if you have at least 11 beads of each colour and if rotations are considered the same but reflections are not?

Question 151: www.CasperYC.club

i) A regular fair dice is rolled twelve times, what is the probability of getting two of each number?

ii) A fair ten–sided dice is rolled four times, what is the probability that your sequence of rolls is increasing?

[THIS PAGE IS INTENTIONALLY LEFT BLANK.]