Solomon Practice Paper

Pure Mathematics 6F

Time allowed: 90 minutes

Centre: www.CasperYC.club

Name:

Teacher:

Question	Points	Score
1	6	
2	8	
3	11	
4	11	
5	11	
6	13	
7	15	
Total:	75	

How I can achieve better:

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Last updated: May 5, 2023

1. Prove by induction that, for all $n \in \mathbb{Z}^+$, [6]

$$\sum_{r=1}^{n} \ln\left(\frac{r+1}{r}\right) = \ln(n+1).$$

2.

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

(a) Find the eigenvalues of $\mathbf{M}$. [4]

(b) Find eigenvectors corresponding to each eigenvalue found in part (a). [4]

Total: 8

3. A transformation T from the z-plane to the w-plane is defined by

$$w = \frac{z + 2\mathbf{i}}{z - \mathbf{i}}, \quad z \neq \mathbf{i},$$

where $z = x + \mathbf{i}y, w = u + \mathbf{i}v$ and x, y, u and v are real.

(a) Show that the circle $|z| = 1$ is mapped onto a straight line in the w-plane under T and find an equation of the line. [5]

The circle $|z - (a + \mathbf{i}b)| = r$ in the z-plane is mapped under T onto the circle $|w| = 2$ in the w-plane, where a, b and r are real.

(b) Find the values of a, b and r. [6]

Total: 11

4. The points A, B and C with coordinates (x_{-1}, y_{-1}), (x_0, y_0) and (x_1, y_1) respectively lie on the curve $y = \mathrm{f}(x)$ with $x_1 - x_0 = x_0 - x_{-1} = h$.

(a) Use the first three terms of the Taylor series expansion in ascending powers of $(x - x_0)$ to show that [5]

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}.$$

The variable y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

with $y = 1$ at $x = 0$ and $y = 1.2$ at $x = 0.1$.

(b) Use the approximations [6]

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h} \quad \text{and} \quad \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

with a step length of 0.1 to estimate the value of y at $x = 0.2$.

Total: 11

5.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & q & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad q \neq 4\frac{1}{4}.$$

(a) Find $\mathbf{A}^{-1}$ in terms of q. [7]

(b) Hence, or otherwise, solve the simultaneous equations [4]

$$\begin{aligned} x - y + 3z &= 1 \\ 4x + y + z &= 2 \\ x + 2y - z &= 5 \end{aligned}$$

showing your working clearly.

Total: 11

6. Given that

$$y = \sqrt{1-x^2}\,\arccos(x),$$

(a) show that [5]

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy - x^2 + 1 = 0. \qquad (\star)$$

(b) By differentiating equation $\star$ twice, or otherwise, obtain the Maclaurin expansion of $y = \sqrt{1-x^2}\,\arccos(x)$ up to and including the term in x^3. [8]

Total: 13

7. The plane Π_1 has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

(a) Find a vector n which is normal to Π_1. [3]

(b) Hence find a vector equation of Π_1 in the form $\mathbf{r}.\mathbf{n} = p$. [2]

(c) Find the perpendicular distance between Π_1 and the point A with position vector $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$, giving your answer in the form $a\sqrt{6}$, where $a \in \mathbb{Q}$. [4]

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + b\mathbf{j}) = -4$. The angle between Π_1 and Π_2 is $30°$.

(d) Find the possible values of the constant b. [6]

Total: 15