Question Number	Scheme	Marks
	There is no credit for attempts that do not use exponential definitions	
1(a) Working from LHS to RHS	$(2\cosh 5x\cosh x) = 2\dfrac{e^{5x}+e^{-5x}}{2}\times\dfrac{e^{x}+e^{-x}}{2} = \dfrac{e^{6x}+e^{4x}+e^{-4x}+e^{-6x}}{2}$	M1
	$= \dfrac{e^{6x}+e^{-6x}}{2}+\dfrac{e^{4x}+e^{-4x}}{2} = \cosh 6x+\cosh 4x\,*$	A1*
		(2)
ALT Working from RHS to LHS	$\cosh 6x+\cosh 4x = \dfrac{e^{6x}+e^{-6x}}{2}+\dfrac{e^{4x}+e^{-4x}}{2} = \dfrac{e^{6x}+e^{4x}+e^{-4x}+e^{-6x}}{2} = 2\dfrac{e^{5x}+e^{-5x}}{2}\times\dfrac{e^{x}+e^{-x}}{2}$	M1
	$= 2\dfrac{e^{5x}+e^{-5x}}{2}\times\dfrac{e^{x}+e^{-x}}{2} = 2\cosh 5x\cosh x$	A1*
(b)	$\cosh 6x+\cosh 4x = 8\cosh x \Rightarrow 2\cosh 5x\cosh x = 8\cosh x$ $\Rightarrow 2\cosh x(\cosh 5x-4)=0 \Rightarrow \cosh 5x = \ldots$ or $2\cosh 5x = \ldots$	M1
	$\cosh 5x = 4$ or $2\cosh 5x = 8$	A1
	$\cosh 5x = 4 \Rightarrow 5x = \ln\left(4+\sqrt{4^2-1}\right)$ or $\cosh 5x = 4 \Rightarrow \dfrac{e^{5x}+e^{-5x}}{2} = 4 \Rightarrow e^{10x}-8e^{5x}+1=0 \Rightarrow e^{5x} = \dfrac{8\pm\sqrt{60}}{2}$	M1
	$x = \pm\dfrac{1}{5}\ln\left(4+\sqrt{15}\right)$ or $x = \dfrac{1}{5}\ln\left(4\pm\sqrt{15}\right)$	A1
		(4)
		Total 6

Notes:

(a)

M1: Expresses $\cosh 5x$ and $\cosh x$ correctly in terms of exponentials and attempts to multiply to obtain an expression involving four terms of the form $e^{\pm 6x}$ and $e^{\pm 4x}$ only.

A1*: Separates terms correctly from a correct expansion and achieves the required result with no errors. Must see one further line isolating the correct terms in exponential form before the stated result. Must clearly link the two exponential expressions to cosh 6*x* and cosh 4*x* so e.g. $= \frac{e^{6x}+e^{-6x}}{2}+\frac{e^{4x}+e^{-4x}}{2}$ and then stop is insufficient unless cosh 6*x* and cosh 4*x* have been correctly defined elsewhere in the work, or e.g. "=RHS" is seen

Note: One of the two's may be shown cancelled, but the correct exponential forms must be seen before any cancellation so e.g. $2\cosh 5x\cosh x = e^{5x}+e^{-5x}\times\frac{e^{x}+e^{-x}}{2}=\frac{e^{6x}+e^{4x}+e^{-4x}+e^{-6x}}{2}$ is M0A0

Missing '*h*' is A0

ALT

M1: Substitutes the correct exponential definitions of cosh 5*x* and cosh *x* into the RHS, combines over a common denominator and re-writes as "2 x a product involving $e^{\pm 5x}$ and $e^{\pm x}$ terms only

A1* Re-writes the expression in the correct form and achieves the required result with no errors. Must clearly link the two exponential expressions to cosh 6*x* and cosh 4*x* so e.g. $=2\frac{e^{5x}+e^{-5x}}{2}\times\frac{e^{x}+e^{-x}}{2}$ and then stop is insufficient unless cosh 5*x* and cosh *x* have been correctly defined elsewhere in the work, or e.g. "=LHS" is seen

Note: One of the two's may be shown cancelled, but the correct exponential forms must be seen before any cancellation so e.g. $\frac{e^{6x}+e^{4x}+e^{-4x}+e^{-6x}}{2}=e^{5x}+e^{-5x}\times\frac{e^{x}+e^{-x}}{2}=2\cosh 5x\cosh x$ is M0A0

Missing '*h*' is A0

(b)

M1: Uses the result from part (a) to obtain a value for $\cosh 5x$ or $2\cosh 5x$

A1: For $\cosh 5x = 4$ or $2\cosh 5x = 8$ (No need to discount $\cosh x = 0$)

M1: Uses the correct logarithmic form for arcosh *x* to find a value for 5*x* (which could be implied by a correct value of *x*) or uses the correct exponential form for $\cosh 5x$ then forms and solves (usual rules) a 3TQ in e^{5x} (may be implied by a correct value for e^{5x})

A1: Both correct values in either correct form and no extras. Condone modulus signs instead of brackets around the $4\pm\sqrt{15}$. Condone equivalent duplicate solutions.

Question Number	Scheme	Marks
In all parts, accept alternative notation for arsinh and arcosh such as $\sinh^{-1}$ and $\cosh^{-1}$ etc		
2(i)	$4x^2+8x+9=4(x+1)^2+5$ or $4x^2+8x+9=(2x+2)^2+5$	B1
	$\int \frac{1}{\sqrt{4x^2+8x+9}}\,dx=\int \frac{1}{\sqrt{4(x+1)^2+5}}\,dx=\frac{1}{2}\text{ar}\sinh\left(\frac{2(x+1)}{\sqrt{5}}\right)(+c)$ o.e.	M1A1
		(3)
(ii)	$\int \text{arcosh}3x\,dx=x\text{arcosh}3x-\int \frac{3x}{\sqrt{9x^2-1}}\,dx$	M1 A1
	$=\ldots-\frac{1}{3}(9x^2-1)^{\frac{1}{2}}$ or $\ldots-\frac{1}{3}\sinh(\text{arcosh}3x)$	M1
	$\int \text{arcosh}3x\,dx=x\text{arcosh}3x-\frac{1}{3}(9x^2-1)^{\frac{1}{2}}(+c)$ or $x\text{arcosh}3x-\left(x^2-\frac{1}{9}\right)^{\frac{1}{2}}(+c)$ or $\int \text{arcosh}3x\,dx=x\text{arcosh}3x-\frac{1}{3}\sinh(\text{arcosh}3x)(+c)$	A1
		(4)
ALT	$u=\text{ar}\cosh 3x\Rightarrow x=\frac{1}{3}\cosh u\Rightarrow \frac{dx}{du}=\frac{1}{3}\sinh u$ $\int \text{arcosh}3x\,dx=\int \frac{1}{3}u\sinh u\,du=\frac{u}{3}\cosh u-\int \frac{1}{3}\cosh u\,du$	M1A1
	$\frac{u}{3}\cosh u-\frac{1}{3}\sinh u\ (+c)$	M1
	$x\text{ar}\cosh 3x-\frac{1}{3}\sinh(\text{ar}\cosh 3x)(+c)$ or $x\text{arcosh}3x-\frac{1}{3}(9x^2-1)^{\frac{1}{2}}(+c)$ or $x\text{arcosh}3x-\left(x^2-\frac{1}{9}\right)^{\frac{1}{2}}(+c)$	A1
		(4)
		Total 7

Notes:

(i)

B1: Correct completion of the square. May be implied by an integral of the form $\frac{1}{2}\int \frac{1}{\sqrt{(x+1)^2+\frac{5}{4}}}\,\mathrm{d}x$

M1: Obtains $k\,\mathrm{arsinh}\,\mathrm{f}(x)$ $(k \in \square)$ or other equivalent form e.g.

$\frac{1}{2}\ln\left[(x+1)+\sqrt{(x+1)^2+\frac{5}{4}}\right]$ or $\frac{1}{2}\ln\left[(2x+2)+\sqrt{4x^2+8x+9}\right]$ etc

A1: Correct integration (condone omission of + c). Missing 'h' is A0 so arsin is A0 but condone arcsinh x.

(ii)

M1: Applies integration by parts to obtain an expression of the form $ax\mathrm{arcosh}3x - b\int \frac{x}{\sqrt{cx^2-1}}\,\mathrm{d}x$

A1: Correct expression (condone missing 'dx' etc)

M1: Integrates $b\int \frac{x}{\sqrt{\pm(cx^2-1)}} \to k\sqrt{\pm(cx^2-1)}$ or $b\int \frac{x}{\sqrt{cx^2-1}} \to k\sinh(\mathrm{arcosh}3x)$

A1: Fully correct integration (condone omission of + c)

ALT

M1: Applies integration by parts (after substitution of $u = \mathrm{arcosh}\,3x$) to obtain an expression of the form $au\cosh u - \int b\cosh u\,\mathrm{d}u$

A1: Correct expression (condone missing 'du' etc)

M1: Integrates $\int b\cosh u\,\mathrm{d}u \to c\sinh u$

A1: Replaces x and obtains a fully correct integration (condone omission of + c)

Note: attempts that re-write arcosh3x in ln form then attempt to integrate score zero.

Question Number	Scheme	Marks
	Accept vectors in **i, j, k** form or column vector form throughout	
3(a)	$\lambda = (1\times 1)+(-1\times 2)+(4\times 1)=...$	M1
	$\lambda = 3$	A1
		(2)
(b)	$\begin{pmatrix}1 & -1 & 4\\ 3 & a & b\\ a & 1 & b\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}"\lambda"\\ "2\lambda"\\ "\lambda"\end{pmatrix}\Rightarrow \begin{matrix}3+2a+b="2\lambda"\\ a+2+b="\lambda"\end{matrix}\Rightarrow a=...,\ b=...$	M1
	$a=2,\ b=-1$	A1
		(2)
(c)(i)	$\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & b\\ a & 1 & -1-\lambda\end{vmatrix}$ $=(1-\lambda)\left[(\lambda-2)(\lambda+1)+1\right]+3(-1-\lambda)+2+4\left[3-2(2-\lambda)\right]$ (via first row) $=(1-\lambda)\left[(\lambda-2)(\lambda+1)+1\right]-3\left[(1+\lambda)-4\right]+2\left[1-4(2-\lambda)\right]$ (via first column) $=(1-\lambda)(2-\lambda)(-1-\lambda)+2+12-8(2-\lambda)+(1-\lambda)+3(-1-\lambda)$ (Sarrus)	M1
	$\lambda^3-2\lambda^2-5\lambda+6=0\Rightarrow(\lambda-3)(\lambda^2+\lambda-2)=0\Rightarrow\lambda=...$	dM1
	$\lambda=1,\ -2,(3)$	A1
		(3)
(ii)	$\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow\begin{matrix}x-y+4z=x\\ 3x+2y-z=y\\ 2x+y-z=z\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\cdots\left(\begin{pmatrix}-1\\4\\1\end{pmatrix}\right)$ or $\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-2\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow\begin{matrix}x-y+4z=-2x\\ 3x+2y-z=-2y\\ 2x+y-z=-2z\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\cdots\left(\begin{pmatrix}-1\\1\\1\end{pmatrix}\right)$	M1
	$\lambda=1\to\begin{pmatrix}-1\\4\\1\end{pmatrix}$ or $\lambda=-2\to\begin{pmatrix}-1\\1\\1\end{pmatrix}$	A1
	$\lambda=1\to\begin{pmatrix}-1\\4\\1\end{pmatrix}$ and $\lambda=-2\to\begin{pmatrix}-1\\1\\1\end{pmatrix}$	A1
		(3)
		Total 10

Notes

(a)

M1: Correct strategy to obtain the eigenvalue, multiplying the first row of matrix M with the given eigenvector, equating to λ and solving. Condone one slip for this mark but need to see some attempt at multiplying elements together.

A1: Correct eigenvalue obtained. Answer of 3 with no working scores both marks.

(b)

M1: Uses their eigenvalue from part (a) to form and solve 2 equations in *a* and *b* (must obtain a value for both)

A1: Correct values

(c)(i)

M1: Attempts to expand $\det(\mathbf{M}-\lambda\mathbf{I})$. This may be along any row or down any column, or via a "shoelace" approach (rule of Sarrus). Condone sign slips, but the overall structure should be correct. Working must be shown- do not accept solutions that just state the factorised form. The expansion must be seen in some form. Accept alternative approaches. Allow this mark to be scored if the attempt is seen under other parts of the question, or if *a* and *b* are not yet found.

Note: Attempts at using $\mathbf{Mx} = \lambda\mathbf{x}$ to obtain simultaneous equations are unlikely to make much progress

dM1: Sets the characteristic polynomial (which must be a four-term cubic) equal to zero (which may be implied) and solves via any valid method, including calculator, to find two other eigenvalues (not 3). This is a dependent mark, so as a minimum an attempt at the expansion must have been seen, but the cubic may not be seen fully simplified. If no working is shown, then their values must be correct for their unfactorised cubic (or apply general rules for a 3TQ if they factorise first)

A1: Correct remaining eigenvalues obtained

(ii)

M1: Applies $\mathbf{Mx} = \lambda\mathbf{x}$ with one of their calculated eigenvalues, however found (not using 3), forms simultaneous equations and proceeds to solve to find one eigenvector, or finds the cross product of any two rows of $\mathbf{M} - \lambda\mathbf{I}$ with their eigenvalues to obtain one eigenvector. Condone $x =\ldots$, $y =\ldots$ $z =\ldots$ for this mark.

A1: One correct eigenvector. Allow any multiple of the correct vector. Must be a vector.

A1: Both correct eigenvectors. Allow any multiples of the correct vectors. Must be a vector.

Question Number	Scheme	Marks
4.	$y = \operatorname{arsinh} x + \operatorname{arsinh}\left(\frac{1}{x}\right) \qquad x > 0$	
(a)	$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+\frac{1}{x^2}}} \times -x^{-2}$	M1
	$= \frac{1}{\sqrt{1+x^2}} - \frac{1}{x^2\sqrt{1+\frac{1}{x^2}}} = \underline{\underline{\frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}}}} = \frac{x-1}{x\sqrt{1+x^2}}$ * or $= \frac{1}{\sqrt{1+x^2}} - \frac{x^{-2}}{\sqrt{1+\frac{1}{x^2}}} = \underline{\underline{\frac{\sqrt{1+\frac{1}{x^2}} - x^{-2}\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{1+\frac{1}{x^2}}}}} = \frac{\sqrt{1+x^2} - \frac{1}{x}\sqrt{1+x^2}}{1+x^2} = \frac{1-\frac{1}{x}}{\sqrt{x^2+1}} = \frac{x-1}{x\sqrt{1+x^2}}$ *	A1*
		(2)
(b)	$\frac{x-1}{x\sqrt{1+x^2}} = 0 \Rightarrow x = 1$	B1
	$y = \operatorname{arsinh}1 + \operatorname{arsinh}\left(\frac{1}{1}\right) = 2\operatorname{arsinh}1 = 2\ln\left(1+\sqrt{1^2+1}\right)$ o.e.	M1
	$y = \ln\left(3+2\sqrt{2}\right)$ or $\ln\left(1+\sqrt{2}\right)^2$ or $2\ln\left(1+\sqrt{2}\right)$	A1
		(3)
Alt for M1A1	$y = 2\operatorname{arsinh}1 \Rightarrow \sinh\frac{y}{2} = 1 \Rightarrow \frac{e^{\frac{y}{2}} - e^{-\frac{y}{2}}}{2} = 1 \Rightarrow e^y - 2e^{\frac{y}{2}} - 1 = 0$ $\Rightarrow e^{\frac{y}{2}} = 1 \pm \sqrt{2} \Rightarrow \frac{y}{2} = \ldots$	M1
	$y = \ln\left(3+2\sqrt{2}\right)$ or $\ln\left(1+\sqrt{2}\right)^2$ or $2\ln\left(1+\sqrt{2}\right)$	A1
		Total 5

Notes

(a)

M1: Differentiates to the form $\frac{1}{\sqrt{1+x^2}}+\frac{kx^{-2}}{\sqrt{1+\frac{1}{x^2}}}$.

Note: There may be approaches involving e.g. $y = \text{ar}\sinh x \Rightarrow x = \sinh y \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \cdots \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cdots$ for one or both parts of the expression for *y*, but the attempt must reach an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the required form before the M mark can be awarded.

A1*: Correct proof with no errors and sufficient working shown. Must have at least the underlined steps shown above for the approach taken to score this mark.

(b)

B1: Deduces $x = 1$ when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

M1: Substitutes their value of *x* into the given equation and applies the logarithmic form of arsinh

A1: Correct value or equivalent e.g. $\ln\left(1+\sqrt{2}\right)^2$, $2\ln\left(1+\sqrt{2}\right)$ but do not condone poor bracketing here.

ALT for final two marks

M1: Substitutes their value of *x* into the given equation then uses the exponential form for sinh to form and solve a quadratic in $\mathrm{e}^{\frac{y}{2}}$ (usual rules) reaching as far as a value for $\frac{y}{2}$

A1: Correct value or equivalent e.g. $\ln\left(1+\sqrt{2}\right)^2$, $2\ln\left(1+\sqrt{2}\right)$ but do not condone poor bracketing here. ISW once a correct answer of the correct form is seen.

Question Number	Scheme	Marks
5(a)	$\int_1^6 x^n(3x-2)^{-\frac{1}{2}}\,dx = \left[\frac{2}{3}x^n(3x-2)^{\frac{1}{2}}\right]_{(1)}^{(6)} - \frac{2}{3}\int_{(1)}^{(6)} nx^{n-1}(3x-2)^{\frac{1}{2}}\,dx$	M1A1
	$= \ldots - \frac{2}{3}\int_{(1)}^{(6)} nx^{n-1}(3x-2)(3x-2)^{-\frac{1}{2}}\,dx$	**d**M1
	$= \frac{2}{3}\times 6^n \times 4 - \frac{2}{3}\times 1 - \frac{2}{3}n\times 3I_n + \frac{4}{3}nI_{n-1}$ $\left(I_n = \frac{8}{3}\times 6^n - \frac{2}{3} - 2nI_n + \frac{4}{3}nI_{n-1}\right)$	**dd**M1
	$3I_n = 8\times 6^n - 2 - 6nI_n + 4nI_{n-1}$ $\Rightarrow (3+6n)I_n = 4nI_{n-1} + 8\times 6^n - 2\,*$	A1*
		(5)
(b)	$I_0 = \int_1^6 (3x-2)^{-\frac{1}{2}}\,dx = \left[\frac{2}{3}(3x-2)^{\frac{1}{2}}\right]_1^6 = \frac{8}{3} - \frac{2}{3} = 2$	B1
	$21I_3 = 12I_2 + 1728 - 2$ $\left(\text{Implied by } 21I_3 = \frac{9966}{5} \text{ or } I_3 = \frac{3322}{35}\right)$ Or $15I_2 = 8I_1 + 288 - 2$ $\left(\text{Implied by } 15I_2 = 334 \text{ or } I_2 = \frac{334}{15}\right)$ Or $9I_1 = 4I_0 + 48 - 2$ $\left(\text{Implied by } 9I_1 = 54\right)$	M1
	$21I_3 = 12\left(\frac{8}{15}I_1 + \frac{286}{15}\right) + 1728 - 2 = 12\left(\frac{8}{15}\left(\frac{4}{9}I_0 + \frac{46}{9}\right) + \frac{286}{15}\right) + 1726 = \cdots\left(\frac{9966}{5}\right)$ or $= \frac{12}{15}(8I_1 + 288 - 2) + 1726 = \frac{32}{45}(4I_0 + 48 - 2) + \frac{9774}{5} = \cdots\left(\frac{9966}{5}\right)$	M1
	$I_3 = \frac{3322}{35}$	A1
		(4)
		Total 9

Notes

(a)

M1: Integrates by parts to obtain $\left[\alpha x^n (3x-2)^{\frac{1}{2}}\right]_{(1)}^{(6)} - \beta\int_{(1)}^{(6)} x^{n-1}(3x-2)^{\frac{1}{2}}\,\mathrm{d}x$ with or without limits

A1: Correct expression (limits not required on either part, and ‘dx’ may be missing)

dM1: Writes $(3x-2)^{\frac{1}{2}}$ as $(3x-2)(3x-2)^{-\frac{1}{2}}$ **Requires previous M mark.**

ddM1: Splits the integral, applies the given limits to the first expression and introduces I_n and I_{n-1}
Requires both previous M marks

A1*: Completes the proof with no mathematical errors seen to obtain the printed answer. Allow recovery of poor bracketing. Limits need not be shown on the integrals. Loss and subsequent recovery of the occasional ‘n’ and/or ‘dx’ loses this mark.

(b)

B1: Correct value for I_0 This may be implied by a correct value for I_1, I_2 or I_3

M1: Applies the reduction formula **at least once** to obtain I_3 in terms of I_2 or I_2 in terms of I_1 or I_1 in terms of I_0
Allow slips but must have the correct number of terms. May be embedded in another expression for I_1, I_2 or I_3
This could be implied by the exact numerical values seen coupled with an obvious attempt. Allow a multiple of I_3, I_2, I_1 e.g. $21I_3 = \ldots$

M1: Completes the process to obtain I_3 in terms of I_0 (or I_1 if I_1 is found by integration) and substitutes for I_0 (or I_1) to find a value for kI_3 or I_3

A1: For $\dfrac{3322}{35}$ or exact equivalent

Note: It is possible for I_1 to be found using integration, and a correct value would imply B1, but method marks are for using the reduction formula.

Question Number	Scheme	Marks
6(a)	$\mathbf{A}=\begin{pmatrix}1 & k & 2\\ 5 & 3 & -2\\ 6 & -1 & 4\end{pmatrix}$	
	$\lvert\mathbf{A}\rvert=12-2-k(20+12)+2(-5-18)$ (via first row) $\lvert\mathbf{A}\rvert=12-2-5(4k+2)+6(-2k-6)$ (via first column) $\lvert\mathbf{A}\rvert=12-12k-10-36-2-20k$ (Sarrus)	M1
	$\lvert\mathbf{A}\rvert=0\Rightarrow 10-32k-46=0\Rightarrow k=...$	M1
	$k=-\dfrac{9}{8}$	A1
		(3)
(b)	$\begin{pmatrix}1 & k & 2\\ 5 & 3 & -2\\ 6 & -1 & 4\end{pmatrix}\to\begin{pmatrix}10 & 32 & -23\\ 4k+2 & -8 & -1-6k\\ -2k-6 & -12 & 3-5k\end{pmatrix}\to\begin{pmatrix}10 & -32 & -23\\ -4k-2 & -8 & 6k+1\\ -2k-6 & 12 & 3-5k\end{pmatrix}$	M1A1
	$\begin{pmatrix}10 & -32 & -23\\ -4k-2 & -8 & 6k+1\\ -2k-6 & 12 & 3-5k\end{pmatrix}\to\begin{pmatrix}10 & -4k-2 & -2k-6\\ -32 & -8 & 12\\ -23 & 6k+1 & 3-5k\end{pmatrix}$ $\mathbf{A}^{-1}=\dfrac{-1}{32k+36}\begin{pmatrix}10 & -4k-2 & -2k-6\\ -32 & -8 & 12\\ -23 & 6k+1 & 3-5k\end{pmatrix}$	**d**M1A1
		(4)
		Total 7

Notes

(a)

M1: Correct determinant attempt. This may be along any row or down any column, or via a “shoelace” approach (rule of Sarrus). Condone sign slips, but the overall structure should be correct.

M1: Sets their linear expression for the determinant = 0 and solves for k

A1: Correct value

(b)

M1: Applies the correct method to reach at least a matrix of cofactors. Two correct rows or two correct columns needed. This may be implied by the transpose of their cofactor matrix, if the cofactor matrix is not seen, with at least two correct rows or two correct columns.

A1: Correct cofactor matrix

dM1: Transposes their cofactor matrix and divides by their determinant. **Depends on previous method mark**. If the candidate miscopies one element from their cofactor matrix, then allow this mark.

A1: Correct matrix. Allow any equivalent correct simplified matrix.

Note: a correct matrix of minors followed by a fully correct answer scores M1A1dM1A1

Question Number	Scheme	Marks
7(a)	$y = \cos 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin 2x$ and then also: $S = 2\pi\int_{(0)}^{\left(\frac{\pi}{4}\right)} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,(\mathrm{d}x)$ and $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \ldots \Rightarrow S = \cdots$ or $S = 2\pi\int_{(0)}^{\left(\frac{\pi}{4}\right)} \cos 2x\sqrt{1+(-2\sin 2x)^2}\,(\mathrm{d}x) \Rightarrow S = \cdots$	M1
	$S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,\mathrm{d}x$ *	A1*
		(2)
(b)	$2\sin 2x = \sinh\theta \Rightarrow 4\cos 2x\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$ o.e.	B1
	$S = 2\pi\int \cos 2x\sqrt{1+4\sin^2 2x}\,\mathrm{d}x = 2\pi\int \cancel{\cos 2x}\sqrt{1+\sinh^2\theta}\,\frac{\cosh\theta}{4\cancel{\cos 2x}}\mathrm{d}\theta$	M1
	$= \frac{\pi}{2}\int \cosh\theta\sqrt{1+\sinh^2\theta}\,\mathrm{d}\theta = \frac{\pi}{2}\int \cosh^2\theta\,\mathrm{d}\theta$	A1
ALT for first 3 marks	$x = \frac{1}{2}\arcsin\left(\frac{1}{2}\sinh\theta\right) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1}{2}\frac{\frac{1}{2}\cosh\theta}{\sqrt{1-\frac{1}{4}\sinh^2\theta}}$ or $\theta = \operatorname{ar}\sinh(2\sin 2x) \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{4\cos 2x}{\sqrt{1+4\sin^2 2x}}$	B1
	$S = 2\pi\int \cos 2x\sqrt{1+4\sin^2 2x}\,\mathrm{d}x$ $= 2\pi\int \sqrt{\cancel{1-\frac{1}{4}\sinh^2\theta}}\,\sqrt{1+\sinh^2\theta}\,\frac{1}{2}\frac{\frac{1}{2}\cosh\theta}{\sqrt{\cancel{1-\frac{1}{4}\sinh^2\theta}}}\mathrm{d}\theta$ or $S = 2\pi\int \cos 2x\sqrt{1+4\sin^2 2x}\,\mathrm{d}x$ $= 2\pi\int \cancel{\cos 2x}\,\sqrt{1+4\sin^2 2x}\,\frac{\sqrt{1+4\sin^2 2x}}{4\cancel{\cos 2x}}\mathrm{d}\theta$ $= 2\pi\int \cancel{\cos 2x}\,\frac{1+\sinh^2\theta}{4\cancel{\cos 2x}}\mathrm{d}\theta$	M1
	$= \frac{\pi}{2}\int \cosh^2\theta\,\mathrm{d}\theta$	A1
	$= \frac{\pi}{2}\int\left(\frac{1}{2}+\frac{1}{2}\cosh 2\theta\right)\mathrm{d}\theta$ **or** $= \frac{\pi}{2}\int\left(\frac{\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}}{2}\right)^2\mathrm{d}\theta = \frac{\pi}{2}\int\left(\frac{\mathrm{e}^{2\theta}+2+\mathrm{e}^{-2\theta}}{4}\right)\mathrm{d}\theta$	M1
	$= \frac{\pi}{4}\left[\theta+\frac{1}{2}\sinh 2\theta\right]$ **or** $\frac{\pi}{2}\left[\frac{1}{8}\mathrm{e}^{2\theta}+\frac{1}{2}\theta-\frac{1}{8}\mathrm{e}^{-2\theta}\right]$	A1
	$\frac{\pi}{4}\left[\theta+\frac{1}{2}\sinh 2\theta\right]_0^{\operatorname{arsinh}2} = \frac{\pi}{4}\left(\operatorname{arsinh}2+\frac{1}{2}\sinh(2\operatorname{arsinh}2)\right)$ **or** $\frac{\pi}{2}\left[\frac{1}{8}\mathrm{e}^{2\theta}+\frac{1}{2}\theta-\frac{1}{8}\mathrm{e}^{-2\theta}\right]_0^{\operatorname{ar}\sinh 2} = \frac{\pi}{2}\left(\frac{1}{8}\mathrm{e}^{2\ln(2+\sqrt{5})}+\frac{1}{2}\ln(2+\sqrt{5})-\frac{1}{8}\mathrm{e}^{-2\ln(2+\sqrt{5})}\right)$	M1
	$= \frac{\pi}{4}\left(\ln(2+\sqrt{5})+2\sqrt{5}\right)$	A1
		(7)
		Total 9

Notes

(a)

Note: for part (a) there are two elements that must be combined to score the M mark

- Differentiation of y to the correct form
- <u>Showing</u> the substitution of their derivative and y into a correct formula (which could be implied by a derivative of the correct form, a correct $\left(\frac{dy}{dx}\right)^2$ for their derivative and a correct SA formula quoted).

M1: Differentiates y to obtain $y = \cos 2x \Rightarrow \frac{dy}{dx} = k\sin 2x$ and substitutes into the correct surface area formula.

Must see evidence of using the correct formula, which could be implied by a correct substitution with their derivative and y substituted- simply stating the result after differentiation (even if correct differentiation) is M0A0 without some evidence of application of their derivative to the correct SA formula. Differentiation must be seen, and the 2π but condone the omission of the limits and/or the dx for this mark.

A1*: Reaches the printed answer with no errors or omissions. Fully correct working shown, including limits, and the dx etc. Must be fully shown. Some exemplars are below:

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$ **M0A0***

(Correct derivative but no substitution shown, no use of formula seen, states given answer with no justification)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+(-2\sin 2x)^2}\,dx = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$ **M1A1***

(Correct derivative, use of correct formula seen - implied by a correct substitution with their derivative and y)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = 2\sin 2x \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$ **M0A0***

(Incorrect derivative, no use of formula seen by correct substitution, states given result with no justification)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = 2\sin 2x \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+(2\sin 2x)^2}\,dx = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$ **M1A0***

(Incorrect derivative, use of correct formula implied by a correct substitution with their derivative and y)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x \text{ and } S = 2\pi\int_0^{\frac{\pi}{4}} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$ **M0A0***

(Correct derivative and formula seen but no evidence of using the correct formula to obtain the given result)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x \text{ and } \left(\frac{dy}{dx}\right)^2 = 4\sin^2 2x \Rightarrow S = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$

M0A0*

(Correct derivative and square seen but no evidence of using this in the correct formula to obtain the given result)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2\sin 2x,\ \left(\frac{dy}{dx}\right)^2 = 4\sin^2 2x,\ S = 2\pi\int_0^{\frac{\pi}{4}} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{\frac{\pi}{4}} \cos 2x\sqrt{1+4\sin^2 2x}\,dx$$

M1A1*

(Correct derivative and square seen and use of this in the correct formula is implied by the correct formula being quoted.

(b)
B1: Correct differentiation. Accept equivalent forms if the substitution is rearranged first. Must be a fully correct differentiation to score this mark.
M1: Substitutes fully into the given integral and makes progress to eliminate the *x* terms. Must substitute fully, including the $\mathrm{d}x$ and arrive at an integral involving only θ. Limits not needed for this mark and condone if they are not changed yet.
A1: Correct integral in terms of $\cosh\theta$ (can ignore limits for this mark).
M1: Applies $\cosh^2\theta = \pm\frac{1}{2}\pm\frac{1}{2}\cosh 2\theta$ or uses the correct exponential form and expands to obtain an integrable form for the integral (can ignore limits for this mark)
A1: Fully correct integration (can ignore limits for this mark)
M1: Applies the limits 0 and arsinh 2 or e.g. 0 and $\ln\left(2+\sqrt{5}\right)$ to an integrated expression of the form $a\theta + b\sinh 2\theta$ or $a\mathrm{e}^{2\theta} + b\theta + c\mathrm{e}^{-2\theta}$. Must see clear evidence of the limits being applied, but condone the omission of the lower limit as it gives 0
A1: Cao. Must be in the exact form requested.

Question Number	Scheme	Marks
	Accept $\mathbf{i},\mathbf{j},\mathbf{k}$ notation or column vector notation throughout this question	
8(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 5 & -6 \\ -3 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix}$	B1
		(1)
(b)	$\begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \ldots$ or $\begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \ldots$	M1
	$\mathbf{r}.(8\mathbf{i}+2\mathbf{j}+11\mathbf{k}) = 25$	A1
		(2)
ALT Using another point on the plane	e.g. $\lambda = 0, \mu = 1 \Rightarrow$ point $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ on the plane $\Rightarrow \begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \ldots$ or $\Rightarrow \begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \ldots$	M1
	$\mathbf{r}.(8\mathbf{i}+2\mathbf{j}+11\mathbf{k}) = 25$	A1
		(2)
(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 2 & 11 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 13 \\ 3 \\ -10 \end{pmatrix}$ Or Points on the line are (see below) $\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} \frac{52}{3} \\ 0 \\ -\frac{31}{3} \end{pmatrix}$ so direction is $\begin{pmatrix} \frac{52}{3} \\ 4 \\ -\frac{40}{3} \end{pmatrix}$ or any multiple e.g. $\begin{pmatrix} 52 \\ 12 \\ -40 \end{pmatrix}$ etc	M1
	$x = \ldots(0) \Rightarrow \begin{cases} 2y+11z = 25 \\ -y+z = 7 \end{cases} \Rightarrow z = \ldots(3), y = \ldots(-4)$ $y = \ldots(0) \Rightarrow \begin{cases} 8x+11z = 25 \\ x+z = 7 \end{cases} \Rightarrow z = \ldots\left(-\frac{31}{3}\right), x = \ldots\left(\frac{52}{3}\right)$ $z = \ldots(0) \Rightarrow \begin{cases} 8x+2y = 25 \\ x-y = 7 \end{cases} \Rightarrow x = \ldots\left(\frac{39}{10}\right), y = \ldots\left(-\frac{31}{10}\right)$ **Note**: points will have the form $\left(\frac{52+13\alpha}{3}, \alpha, \frac{-31-10\alpha}{3}\right)$	M1

	Common points $(0,-4,3)$ $\left(\frac{52}{3},0,-\frac{31}{3}\right)$ $\left(\frac{39}{10},-\frac{31}{10},0\right)$ $\left(1,-\frac{49}{13},\frac{29}{13}\right)$ $\left(\frac{65}{3},1,-\frac{41}{3}\right)$ $\left(\frac{13}{5},-\frac{17}{5},1\right)$	
	$(\mathbf{r}-(-4\mathbf{j}+3\mathbf{k}))\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ or $\left(\mathbf{r}-\left(\frac{52}{3}\mathbf{i}-\frac{31}{3}\mathbf{k}\right)\right)\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ or $\left(\mathbf{r}-\left(\frac{39}{10}\mathbf{i}-\frac{31}{10}\mathbf{j}\right)\right)\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ Or equivalent for their correct point on the line and correct direction vector	A1
		(3)
Alt to find a point and/or direction vector for either M1 mark	Combining cartesian equations together (other eliminations are possible): $\begin{cases}x-y+z=7\\8x+2y+11z=25\end{cases}\Rightarrow 3x-13y=52\Rightarrow y=\frac{3x-52}{13}\quad x=\frac{52+13y}{3}$ $z=7-x+y=\frac{-31-10y}{3}\Rightarrow y=\frac{-3z-31}{10}$ So $\frac{3x-52}{13}=y=\frac{-3z-31}{10}\Rightarrow\frac{x-\frac{52}{3}}{\frac{13}{3}}=\frac{y-0}{1}=\frac{z+\frac{31}{3}}{-\frac{10}{3}}$ so point is $\begin{pmatrix}\frac{52}{3}\\0\\-\frac{31}{3}\end{pmatrix}$ Or Direction is $\begin{pmatrix}\frac{13}{3}\\1\\-\frac{10}{3}\end{pmatrix}$ or any multiple	M1
	point is $\begin{pmatrix}\frac{52}{3}\\0\\-\frac{31}{3}\end{pmatrix}$ and direction is $\begin{pmatrix}\frac{13}{3}\\1\\-\frac{10}{3}\end{pmatrix}$ or any multiple	M1
	$(\mathbf{r}-(-4\mathbf{j}+3\mathbf{k}))\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ Or $\left(\mathbf{r}-\left(\frac{52}{3}\mathbf{i}-\frac{31}{3}\mathbf{k}\right)\right)\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ Or $\left(\mathbf{r}-\left(\frac{39}{10}\mathbf{i}-\frac{31}{10}\mathbf{j}\right)\right)\times(13\mathbf{i}+3\mathbf{j}-10\mathbf{k})=\mathbf{0}$ Or equivalent for their correct point on the line and correct direction vector	A1
		(3)

(d)	$\pm\left(\begin{pmatrix}3\\0\\2\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}1\\-1\\-1\end{pmatrix}$	M1
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\13&3&-10\\1&-1&-1\end{vmatrix}=\begin{pmatrix}-13\\3\\-16\end{pmatrix}$ or $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&-1\\13&3&-10\end{vmatrix}=\begin{pmatrix}13\\-3\\16\end{pmatrix}$	M1
	$d=\left\lvert\dfrac{(-4\mathbf{j}+3\mathbf{k}-(3\mathbf{i}+2\mathbf{k}))\cdot(-13\mathbf{i}+3\mathbf{j}-16\mathbf{k})}{\sqrt{13^2+3^2+16^2}}\right\rvert=\ldots$	M1
	$d=\dfrac{11}{\sqrt{434}}$	A1
		(4)
Alt using general points on the lines	$\pm\left(\begin{pmatrix}3\\0\\2\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}1\\-1\\-1\end{pmatrix}$	M1
	Let the general points on the lines be X and Y $\overrightarrow{XY}=\begin{pmatrix}2+\lambda\\1-\lambda\\3-\lambda\end{pmatrix}-\begin{pmatrix}0+13\mu\\-4+3\mu\\3-10\mu\end{pmatrix}=\begin{pmatrix}2+\lambda-13\mu\\5-\lambda-3\mu\\-\lambda+10\mu\end{pmatrix}\Rightarrow$ $\begin{pmatrix}2+\lambda-13\mu\\5-\lambda-3\mu\\-\lambda+10\mu\end{pmatrix}\bullet\begin{pmatrix}13\\3\\-10\end{pmatrix}=0$ and $\begin{pmatrix}2+\lambda-13\mu\\5-\lambda-3\mu\\-\lambda+10\mu\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-1\end{pmatrix}=0\Rightarrow\begin{cases}41+20\lambda-278\mu=0\\-3+3\lambda-20\mu=0\end{cases}$ $\Rightarrow\lambda=\ldots\left(\dfrac{827}{217}\right),\mu=\ldots\left(\dfrac{183}{434}\right)$	M1
	$\overrightarrow{XY}=\begin{pmatrix}2+\lambda-13\mu\\5-\lambda-3\mu\\-\lambda+10\mu\end{pmatrix}=\begin{pmatrix}\frac{143}{434}\\-\frac{33}{434}\\\frac{88}{217}\end{pmatrix}$ $\lvert\overrightarrow{XY}\rvert=\sqrt{\left(\dfrac{143}{434}\right)^2+\left(\dfrac{33}{434}\right)^2+\left(\dfrac{88}{217}\right)^2}=\ldots\left(\dfrac{121}{434}\right)$	M1
	$d=\dfrac{11}{\sqrt{434}}$	A1
		(4)
		Total 10

Notes

(a)

B1: Correct vector found. ISW once a correct vector product is seen e.g. if the components are divided by 2 etc.

(b)

M1: Attempts $\mathbf{r} \bullet \mathbf{n} = p$ form for the plane by finding the scalar product between their part (a) or a multiple of their part (a) and $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$ (or another point on the line, correctly obtained by substituting values of the parameters into the line) . If the scalar product is found with a point not on the line, then M0.

A1: Correct equation

(c)

M1: Attempts the vector product between $8\mathbf{i}+2\mathbf{j}+11\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$ or e.g. finds 2 points on the line and subtracts to find the direction. If method not directly shown, then two correct components correct implies method.

M1: Attempts to find a point on the line of intersection, either by substituting a value for *x*, *y*, *z* into the cartesian equations of the planes, and solving to find the other two values or by combining the cartesian equations together. M0 if they find a correct point then multiply by a scalar.

A1: Any correct equation in the required form e.g. $\left(\mathbf{r}-\left(\frac{39}{10}\mathbf{i}-\frac{31}{10}\mathbf{j}\right)\right)\times\left(-\frac{13}{10}\mathbf{i}-\frac{3}{10}\mathbf{j}+\mathbf{k}\right)=\mathbf{0}$

ALT

M1: Combines the cartesian equations together by eliminating one variable and finding one in terms of the other two or expressing two variables in terms of the other one, and making progress to write the line in the form $\frac{x-a}{d}=\frac{y-b}{e}=\frac{z-c}{f}$ and then correctly extracting either their point or their direction correctly from their line equation. Allow minor slips if the line is in the correct form.

M1: Correctly extracts both the point and the direction from their line equation in cartesian form. Allow minor slips if the line is in the correct form.

A1: Forms the correct equation of the line in the correct form. Condone the '**0**' and/or the '***r***' without an underscore to indicate that it is a vector.

(d)

M1: Attempts the direction of l_2

M1: Attempts the vector product between their direction from part (c) and their direction of l_2. If method not directly shown, then two correct components implies the correct method.

M1: A full method to find the required distance using formula $\left|\frac{(\mathbf{a}-\mathbf{c})\bullet\mathbf{n}}{|\mathbf{n}|}\right|$ where $\mathbf{a}$ and $\mathbf{c}$ are the position vectors of individual points on the lines l_1 and l_2 and $\mathbf{n}$ is the vector product of the direction vectors for their lines. $\left(\text{e.g. } \mathbf{r}_1=\begin{pmatrix}0\\-4\\3\end{pmatrix}+\lambda\begin{pmatrix}13\\3\\-10\end{pmatrix} \text{ and } \mathbf{r}_2=\begin{pmatrix}3\\0\\2\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right)$. Note that the point *A* could also be used in the line $\mathbf{r}_2$ so that $\begin{pmatrix}3\\0\\2\end{pmatrix}$ is replaced with $\begin{pmatrix}2\\1\\3\end{pmatrix}$ (or another point). The numerator may be seen as a scalar triple product e.g. $\begin{vmatrix}-3 & -4 & 1\\13 & 3 & -10\\1 & -1 & -1\end{vmatrix}$ etc.

A1: For $\frac{11}{\sqrt{434}}$ or awrt 0.528

ALT

M1: Attempts the direction of l_2

M1: Forms the general point between the two lines, attempts the dot product between the general point and both direction vectors of the lines l_1 and l_2 to form and then solve two simultaneous equations in their parameters. Must find a value for both.

M1: Substitutes their parameters to find their $\overrightarrow{XY}$ and then finds the magnitude of their $\overrightarrow{XY}$

A1: For $\frac{11}{\sqrt{434}}$ or awrt 0.528

Question Number	Scheme	Marks
9(a)	$b^2 = a^2\left(e^2 - 1\right) \Rightarrow 49 = 64\left(e^2 - 1\right) \Rightarrow e^2 = \ldots$	M1
	$49 = 64e^2 - 64 \Rightarrow e^2 = \dfrac{113}{64} \Rightarrow e = \dfrac{\sqrt{113}}{8}$ *	A1*
		(2)
(b)	$x = 8\sec t, y = 7\tan t \Rightarrow \dfrac{\text{d}y}{\text{d}x} = \dfrac{7\sec^2 t}{8\sec t\tan t} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta}\left(= \dfrac{7}{8}\operatorname{cosec}\theta\right)$ (Parametric diff.) or $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1 \Rightarrow \dfrac{x}{32} - \dfrac{2y}{49}\dfrac{\text{d}y}{\text{d}x} = 0 \Rightarrow \dfrac{\text{d}y}{\text{d}x} = \dfrac{49x}{64y} = \dfrac{7\sec\theta}{8\tan\theta}$ (Implicit diff.) or $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1 \Rightarrow y = 7\sqrt{\dfrac{x^2}{64} - 1} \Rightarrow \dfrac{\text{d}y}{\text{d}x} = \dfrac{7}{2}\left(\dfrac{x^2}{64} - 1\right)^{-\frac{1}{2}} \times \dfrac{x}{32} = \dfrac{7}{2}\left(\sec^2\theta - 1\right)^{-\frac{1}{2}} \times \dfrac{8\sec\theta}{32}$	B1
	$y - 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta}\left(x - 8\sec\theta\right)$ or $'y = mx + c' \Rightarrow 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta} \times 8\sec\theta + c \Rightarrow c = \ldots\left(-\dfrac{7}{\tan\theta}\right)$	M1
	$y - 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta}\left(x - 8\sec\theta\right) \Rightarrow 8y\tan\theta - 56\tan^2\theta = 7x\sec\theta - 56\sec^2\theta$ $\Rightarrow \dfrac{x}{8}\sec\theta - \dfrac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta$ $\Rightarrow \dfrac{x}{8}\sec\theta - \dfrac{y}{7}\tan\theta = 1$ *	A1*
		(3)
(c)	$\left(0, -7\cot\theta\right)$ o.e.	B1
		(1)
(d)	$\left(0, \dfrac{113}{7}\tan\theta\right)$ o.e.	B1
		(1)
(e)	Centre is $\left(0, \dfrac{113}{14}\tan\theta - \dfrac{7}{2}\cot\theta\right)$	B1
	(Radius is) $\dfrac{113}{14}\tan\theta + \dfrac{7}{2}\cot\theta$	B1
	$x^2 + \left(y - \left(\dfrac{113}{14}\tan\theta - \dfrac{7}{2}\cot\theta\right)\right)^2 = \left(\dfrac{113}{14}\tan\theta + \dfrac{7}{2}\cot\theta\right)^2$	M1
	$y = 0 \Rightarrow x^2 + \left(\dfrac{113}{14}\tan\theta - \dfrac{7}{2}\cot\theta\right)^2 = \left(\dfrac{113}{14}\tan\theta + \dfrac{7}{2}\cot\theta\right)^2$ $\Rightarrow x^2 = 2 \times 2 \times \dfrac{113}{14}\tan\theta \times \dfrac{7}{2}\cot\theta = 113 \Rightarrow x = \pm\sqrt{113}$ Foci are at $\left(\pm ae, 0\right) = \left(\pm 8 \times \dfrac{\sqrt{113}}{8}, 0\right) = \left(\pm\sqrt{113}, 0\right)$	M1
	Hence the circle passes through the foci of H *	A1*
		(5)

ALT for final M1A1	Foci are at $(\pm ae, 0) = \left(\pm 8 \times \frac{\sqrt{113}}{8}, 0\right) = \left(\pm\sqrt{113}, 0\right)$ Substituting into the circle equation: $LHS = \left(\pm\sqrt{113}\right)^2 + \left(\frac{113}{14}\right)^2 \tan^2\theta + \left(\frac{7}{2}\right)^2 \cot^2\theta - 2\left(\frac{113}{14}\right)\left(\frac{7}{2}\right) = \left(\frac{113}{14}\right)^2 \tan^2\theta + \left(\frac{7}{2}\right)^2 \cot^2\theta + \frac{113}{2}$ $RHS = \left(\frac{113}{14}\right)^2 \tan^2\theta + \left(\frac{7}{2}\right)^2 \cot^2\theta + 2\left(\frac{113}{14}\right)\left(\frac{7}{2}\right) = \left(\frac{113}{14}\right)^2 \tan^2\theta + \left(\frac{7}{2}\right)^2 \cot^2\theta + \frac{113}{2}$	M1
	$LHS = RHS$ so the circle passes through the foci of H	A1*
ALT 2 Circle theorems and gradients	$m_{FR} = \dfrac{\frac{113}{7}\tan\theta - 0}{0 - \sqrt{113}} = -\dfrac{113}{7\sqrt{113}}\tan\theta$ Or $m_{FR} = \dfrac{\frac{113}{7}\tan\theta - 0}{0 + \sqrt{113}} = \dfrac{113}{7\sqrt{113}}\tan\theta$	B1
	$m_{FQ} = \dfrac{-7\cot\theta - 0}{0 - \sqrt{113}} = \dfrac{7}{\sqrt{113}}\cot\theta$ Or $m_{FQ} = \dfrac{-7\cot\theta - 0}{0 + \sqrt{113}} = -\dfrac{7}{\sqrt{113}}\cot\theta$	B1
	$m_{FR} \times m_{FQ} = -\dfrac{113}{7\sqrt{113}}\tan\theta \times \dfrac{7}{\sqrt{113}}\cot\theta = -1$ Or $m_{FR} \times m_{FQ} = \dfrac{113}{7\sqrt{113}}\tan\theta \times -\dfrac{7}{\sqrt{113}}\cot\theta = -1$	M1
	For both $m_{FR} \times m_{FQ} = -\dfrac{113}{7\sqrt{113}}\tan\theta \times \dfrac{7}{\sqrt{113}}\cot\theta = -1$ and $m_{FR} \times m_{FQ} = \dfrac{113}{7\sqrt{113}}\tan\theta \times -\dfrac{7}{\sqrt{113}}\cot\theta = -1$	M1
	$m_{FR} \times m_{FQ} = -1$ for both foci, so the circle passing through F has QR as a diameter	A1*
		(5)

ALT 3 Circle theorems and Pythagoras	$FQ=\sqrt{113+49\cot^2\theta}$ or $FQ^2=113+49\cot^2\theta$	B1
	$FR=\sqrt{113+\left(\frac{113}{7}\tan\theta\right)^2}$ or $FR^2=113+\left(\frac{113}{7}\tan\theta\right)^2$ or better	B1
	$FQ^2+FR^2=49\cot^2\theta+\left(\frac{113}{7}\tan\theta\right)^2+226$	M1
	$QR^2=\left(7\cot\theta+\frac{113}{7}\tan\theta\right)^2=49\cot^2\theta+\left(\frac{113}{7}\tan\theta\right)^2+226$	M1
	$QR^2=FR^2+FQ^2$ for both foci, so the circle passing through F has QR as a diameter	A1*
		(5)
		Total 12

Notes

(a)

M1: Uses a correct eccentricity formula with a^2 = 64 and b^2 = 49 and rearranges for e^2

A1*: Correct proof with sufficient working and no errors. Must have at least one line of correct working between their correct eccentricity formula with a^2 = 64 and b^2 = 49 substituted before the printed answer. It's not sufficient to go from a correct formula with values substituted directly to a simplified e^2 or e without some intermediate working. Do not accept $e=\pm\frac{\sqrt{113}}{8}$. Some exemplars are below

$b^2=a^2\left(e^2-1\right)\Rightarrow 49=64\left(e^2-1\right)\Rightarrow e=\frac{\sqrt{113}}{8}$ **M1A0*** (no intermediate line of working shown)

$49=64\left(e^2-1\right)\Rightarrow e^2=\frac{113}{64}\Rightarrow e=\frac{\sqrt{113}}{8}$ **M1A0*** (no intermediate line of working shown)

$49=64\left(e^2-1\right)\Rightarrow e^2=\frac{49+64}{64}\Rightarrow e=\frac{\sqrt{113}}{8}$ **M1A1*** (Intermediate line of working before a simplified e^2 or e is seen).

$7^2=8^2\left(e^2-1\right)\Rightarrow e^2-1=\frac{49}{64}\Rightarrow e=\frac{\sqrt{113}}{8}$ **M1A1*** (Intermediate line of working before a simplified e^2 or e is seen).

Note: accept a verification method which shows that 'LHS = RHS' after a correct eccentricity formula with a^2 = 64 and b^2 = 49 is seen.

(b)

B1: Correct tangent gradient obtained using calculus. Need not be simplified for this mark, but it must be a fully correct derivative expression in terms of θ.

M1: Uses a correct straight-line method with their gradient and the point P. If they are using $y=mx+c$ then they must make progress to finding a value for their 'c'.

A1*: Correct proof with no errors and sufficient working shown. At least one line of intermediate working required between their correct line equation and the given answer.

(c)

B1: Correct coordinates or $x=0$ and $y=-7\cot\theta$ o.e. e.g. $\left(0,-\frac{7}{\tan\theta}\right)$ or $\left(0,-\frac{7\cos\theta}{\sin\theta}\right)$. Can award if seen under other parts of the question, but must be clearly labelled as point Q.

(d)

B1: Correct coordinates or $x = 0$ and $y = \frac{113}{7}\tan\theta$ o.e. e.g. $\left(0, \frac{113\sin\theta}{7\cos\theta}\right)$ or $\left(0, \frac{113}{7\cot\theta}\right)$. Can award if seen under other parts of the question, but must be clearly labelled as point *R*.

(e)

B1: Correct centre in any equivalent form e.g.

$\left(0, \frac{113\tan\theta - 49\cot\theta}{14}\right)$ or $\left(0, \frac{113\frac{\sin\theta}{\cos\theta} - 49\frac{\cos\theta}{\sin\theta}}{14}\right)$ or $\left(0, \frac{113\sin^2\theta - 49\cos^2\theta}{14\sin\theta\cos\theta}\right)$

B1: Correct radius in any equivalent form e.g. $\frac{113\sin\theta}{14\cos\theta} + \frac{7\cos\theta}{2\sin\theta}$ or $\frac{113\sin^2\theta + 49\cos^2\theta}{14\sin\theta\cos\theta}$ or $\frac{64\sin^2\theta + 49}{14\sin\theta\cos\theta}$

Ignore the labelling for this mark. Award this mark if the correct expression is seen.

M1: Uses their centre and radius to form the equation of the circle in the form $x^2 + (y - \cdots)^2 = r^2$

M1: Substitutes $y = 0$, solves for x and uses the result from part (a) to compare with the values of $\pm ae$

A1*: Fully correct work with conclusion

ALT for final M1A1

M1: Substitutes the correct foci into their circle equation, and makes some attempt to simplify both sides to show they are equal- may be seen with x squared which is fine.

A1: Shows that 'LHS = RHS' for their correct circle equation and gives a basic conclusion.

ALT 2 using circle theorems and gradients

B1: for finding the gradient of *FR* for either focus *F*

B1: For finding the gradient of *FQ* for either focus *F*

M1: For using $m_{FR} \times m_{FQ} = -1$ to show that *FQ* is perpendicular to *FR* for either focus

M1: For showing that *FQ* is perpendicular to *FR* for both foci

A1: Concludes that the circle passing through the focus (point *F*) has *QR* as a diameter

ALT 3 using circle theorems and converse of Pythagoras

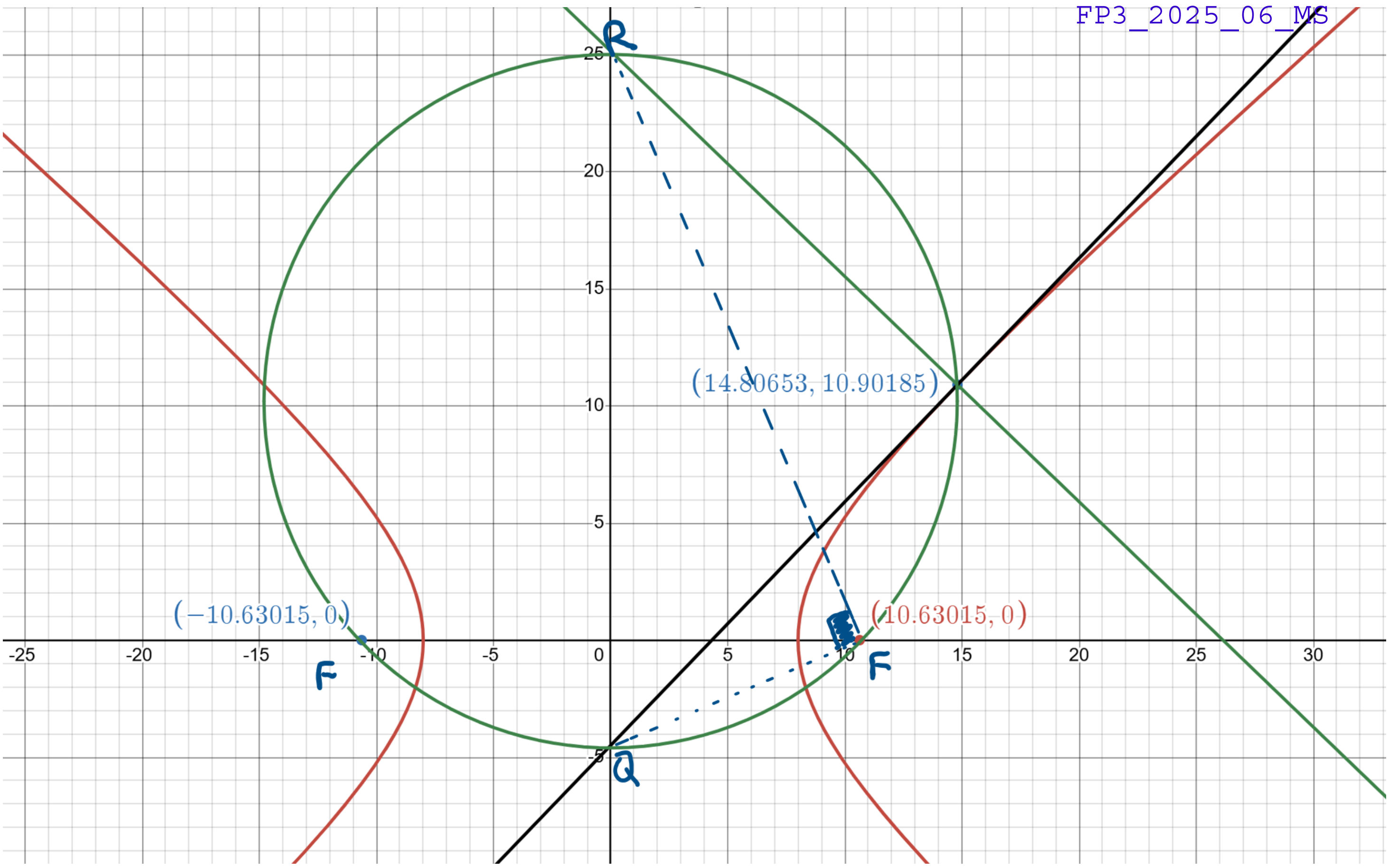
B1: For finding FQ or FQ^2

B1: FR or FR^2

M1: For finding $FQ^2 + FR^2$

M1 For showing that $QR^2 = FQ^2 + FR^2$

A1: Concludes that the circle passing through the focus (point *F*) has *QR* as a diameter


R
Q
F
F
(14.80653, 10.90185)
(−10.63015, 0)
(10.63015, 0)
-25
-20
-15
-10
-5
0
5
10
15
20
25
30
25
20
15
10
5
-5