United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Follow-up Competitions

2003 – 2019 Collection

August 18, 2020

Comments and suggestions to 89272376@QQ.com .

IMC Follow-up 5

Year 11 Olympiad - Maclaurin

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

1. A train leaves K for L at 09:30 while another train leaves L for K at 10:00. The first train arrives in L 40 minutes after the trains pass each other. The second train arrives in K 1 hour and 40 minutes after the trains pass.

Each train travels at a constant speed.

At what time did the trains pass each other?

2. A right-angled triangle has area $150\,\text{cm}^2$ and the length of its perimeter is $60\,\text{cm}$.

What are the lengths of its sides?

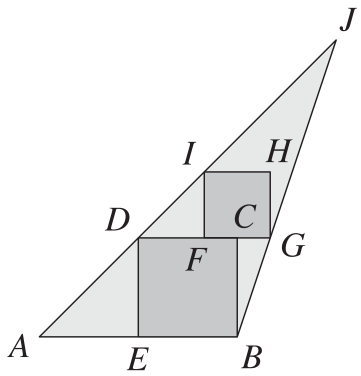
3. Two numbers are such that the sum of their reciprocals is equal to 1. Each of these numbers is then reduced by 1 to give two new numbers.

Prove that these two new numbers are reciprocals of each other.

[*The* reciprocal *of a non-zero number x is the number* $\frac{1}{x}$.]

4. The diagram shows the two squares $BCDE$ and $FGHI$ inside the triangle ABJ, where E is the midpoint of AB and C is the midpoint of FG.

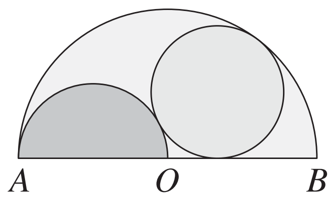
What is the ratio of the area of the square $BCDE$ to the area of the triangle ABJ?



5. A semicircle of radius 1 is drawn inside a semicircle of radius 2, as shown in the diagram, where $OA = OB = 2$.

A circle is drawn so that it touches each of the semicircles and their common diameter, as shown.

What is the radius of the circle?



6. A tiling of an $n \times n$ square grid is formed using 4×1 tiles.

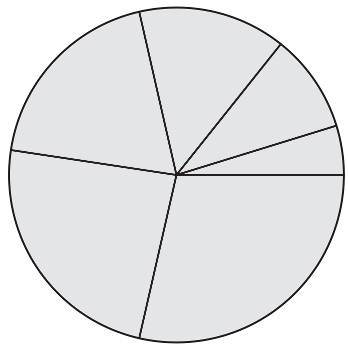
What are the possible values of n?

[*A* tiling *has no gaps or overlaps, and no tile goes outside the region being tiled.*]

M1. The sum of the squares of two real numbers is equal to fifteen times their sum. The difference of the squares of the same two numbers is equal to three times their difference.

Find all possible pairs of numbers that satisfy the above criteria.

M2. The diagram shows a circle that has been divided into six sectors of different sizes.



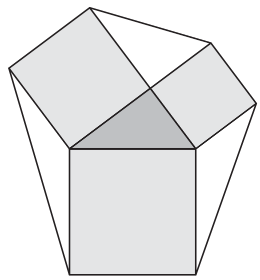
Two of the sectors are to be painted red, two of them are to be painted blue, and two of them are to be painted yellow. Any two sectors which share an edge are to be painted in different colours.

In how many ways can the circle be painted?

M3. Three positive integers have sum 25 and product 360.

Find all possible triples of these integers.

M4. The squares on each side of a right-angled scalene triangle are constructed and three further line segments drawn from the corners of the squares to create a hexagon, as shown. The squares on these three further line segments are then constructed (outside the hexagon).



The combined area of the two equal-sized squares is 2018 cm^2.

What is the total area of the six squares?

M5. For which integers n is $\dfrac{16\left(n^2 - n - 1\right)^2}{2n - 1}$ also an integer?

M6. The diagram shows a triangle ABC and points T, U on the edge AB, points P, Q on BC, and R, S on CA, where:

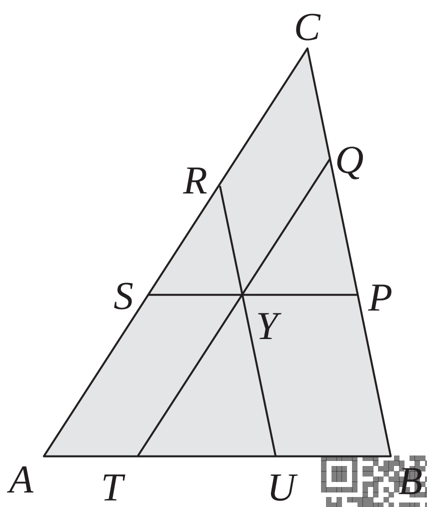
(i) SP and AB are parallel, UR and BC are parallel, and QT and CA are parallel;

(ii) SP, UR and QT all pass through a point Y; and

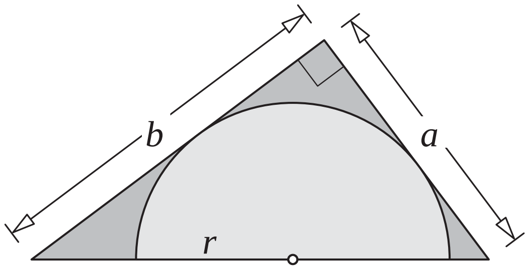
(iii) $PQ = RS = TU$.

Prove that

$$\frac{1}{PQ} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$



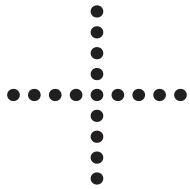
M1. The diagram shows a semicircle of radius r inside a right-angled triangle. The shorter edges of the triangle are tangents to the semicircle, and have lengths a and b. The diameter of the semicircle lies on the hypotenuse of the triangle.



Prove that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

M2. How many triangles (with non-zero area) are there with each of the three vertices at one of the dots in the diagram?

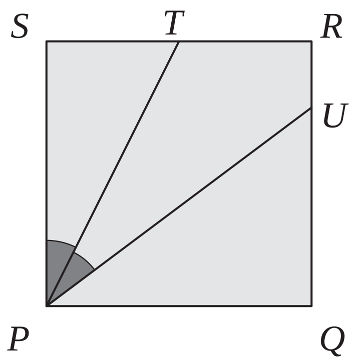


M3. How many solutions are there to the equation

$$m^4 + 8n^2 + 425 = n^4 + 42m^2,$$

where m and n are integers?

M4. The diagram shows a square $PQRS$ with sides of length 2. The point T is the midpoint of RS, and U lies on QR so that $\angle SPT = \angle TPU$.



What is the length of UR?

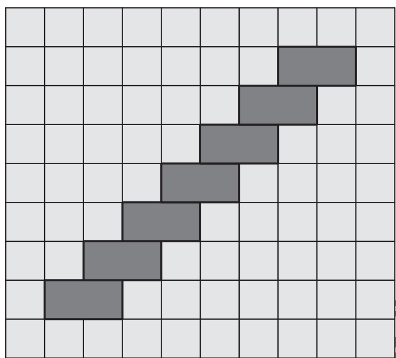
M5. Solve the pair of simultaneous equations

$$(a + b)(a^2 - b^2) = 4 \qquad \text{and}$$

$$(a - b)(a^2 + b^2) = \frac{5}{2}.$$

M6. The diagram shows a 10×9 board with seven 2×1 tiles already in place.

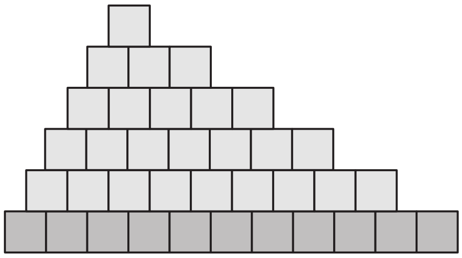
What is the largest number of additional 2×1 tiles that can be placed on the board, so that each tile covers exactly two 1×1 cells of the board, and no tiles overlap?



M1. The positive integer N has five digits.
The six-digit integer P is formed by appending the digit 2 to the front of N.
The six-digit integer Q is formed by appending the digit 2 to the end of N.

Given that $Q = 3P$, what values of N are possible?

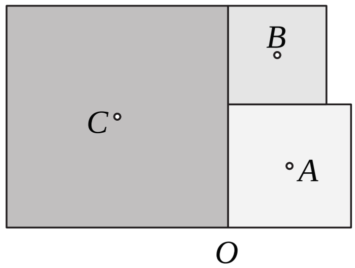
M2. A ‘stepped’ shape, such as the example shown, is made from 1×1 squares in the following way.



(i) There are no gaps or overlaps.
(ii) There are an odd number of squares in the bottom row (eleven in the example shown).
(iii) In every row apart from the bottom one, there are two fewer squares than in the row immediately below.
(iv) In every row apart from the bottom one, each square touches two squares in the row immediately below.
(v) There is one square in the top row.

Prove that $36A = (P + 2)^2$, where A is the area of the shape and P is the length of its perimeter.

M3. The diagram shows three squares with centres A, B and C. The point O is a vertex of two squares.



Prove that OB and AC are equal and perpendicular.

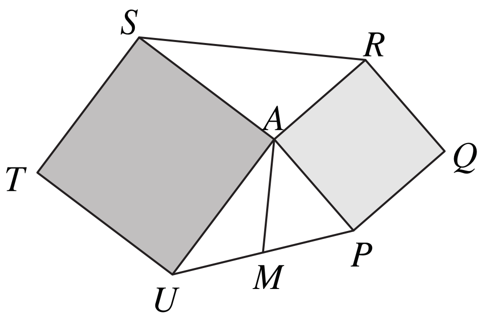
M4. What are the solutions of the simultaneous equations:

$$3x^2 + xy - 2y^2 = -5;$$
$$x^2 + 2xy + y^2 = 1\ ?$$

M5. The number of my hotel room is a three-digit integer. I thought that the same number could be obtained by multiplying together all of:
(i) one more than the first digit;
(ii) one more than the second digit;
(iii) the third digit.

Prove that I was mistaken.

M6. The diagram shows two squares $APQR$ and $ASTU$, which have vertex A in common. The point M is the midpoint of PU.



Prove that $AM = \frac{1}{2}RS$.

1. Consider the sequence 5, 55, 555, 5555, 55 555, ….

Are any of the numbers in this sequence divisible by 495; if so, what is the smallest such number?

2. Two real numbers x and y satisfy the equation $x^2 + y^2 + 3xy = 2015$.

What is the maximum possible value of xy ?

3. Two integers are *relatively prime* if their highest common factor is 1.

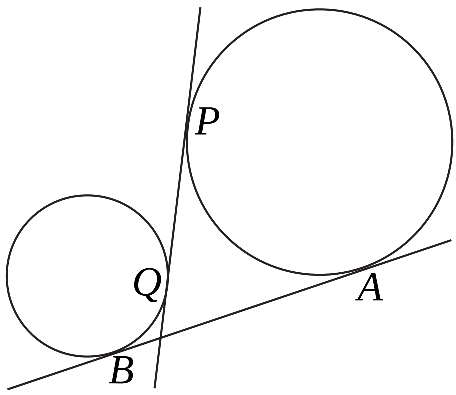
I choose six different integers between 90 and 99 inclusive.

(a) Prove that two of my chosen integers are relatively prime.

(b) Is it also true that two are *not* relatively prime?

4. The diagram shows two circles with radii a and b and two common tangents AB and PQ. The length of PQ is 14 and the length of AB is 16.

Prove that $ab = 15$.



5. Consider equations of the form $ax^2 + bx + c = 0$, where a, b, c are all single-digit prime numbers.

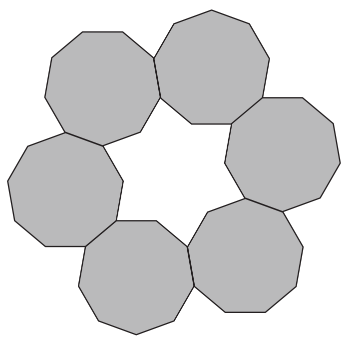
How many of these equations have at least one solution for x that is an integer?

6. A symmetrical ring of m identical regular n-sided polygons is formed according to the rules:

(i) each polygon in the ring meets exactly two others;

(ii) two adjacent polygons have only an edge in common; and

(iii) the perimeter of the inner region–enclosed by the ring–consists of exactly two edges of each polygon.



The example in the figure shows a ring with $m = 6$ and $n = 9$.

For how many different values of n is such a ring possible?

1. What is the largest three-digit prime number 'abc' whose digits a, b and c are different prime numbers?

2. Nine buns cost £11 + a pence and 13 buns cost £15 + b pence, where $0 < a < 100$ and $0 < b < 100$.

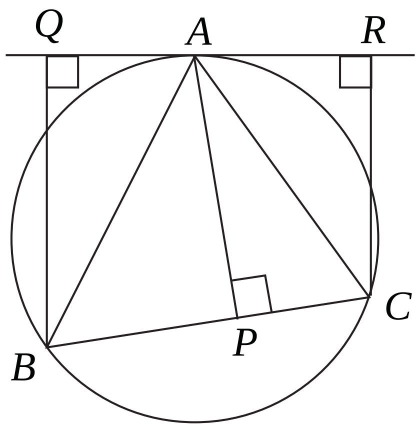
What is the cost of a bun?

3. A regular hexagon, with sides of length 2 cm, is cut into two pieces by a straight line parallel to one of its sides. The ratio of the area of the smaller piece to the area of the larger piece is 1 : 5.

What is the length of the cut?

4. In the diagram, RAQ is the tangent at A to the circle ABC, and $\angle AQB$, $\angle CRA$ and $\angle APC$ are all right angles.

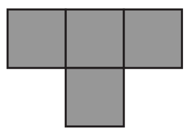
Prove that $BQ \times CR = AP^2$.



5. Kim and Oli played nine games of chess, playing alternately with the white and black pieces. Exactly five games were won by whoever was playing with the black pieces, Kim won exactly six games, and no game was drawn.

With which colour pieces did Kim play the first game?

6. The T-tetromino T is the shape made by joining four 1×1 squares edge to edge, as shown. The rectangle R has dimensions $2a \times 2b$, where a and b are integers. The expression 'R can be tiled by T' means that R can be covered exactly with identical copies of T without gaps or overlaps.



(a) Prove that R can be tiled by T when both a and b are even.

(b) Prove that R cannot be tiled by T when both a and b are odd.

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