STEP

SIXTH TERM EXAM PAPER MATHEMATICS

Dear Mathematicians,

This is free material compiled from internet resources. I hope you like this free copy.

I am currently LaTeX ing the solutions, so that we have a neater and nicer copy of answers too. Some drafts are already available on TSR under replies of the post.

The work is/will be free too. If you wish to join me voluntarily, please contact me at `89272376@QQ.com`. I need someone who are good at LaTeX, drawing vector graphics, compiling and sorting answers. Or any other help you can think of. I'd appreciate any help with this long term project.

Best wishes and good luck everyone.
Fight with COVID19!

Section A: Pure Mathematics

1 The line $y = a^2x$ and the curve $y = x(b-x)^2$, where $0 < a < b$, intersect at the origin O and at points P and Q. The x-coordinate of P is less than the x-coordinate of Q.

(i) Find the coordinates of P and Q, and sketch the line and the curve on the same axes.

(ii) Show that the equation of the tangent to the curve at P is

$$y = a(3a-2b)x + 2a(b-a)^2.$$

(iii) This tangent meets the y-axis at R. The area of the region between the curve and the line segment OP is denoted by S. Show that

$$S = \frac{1}{12}(b-a)^3(3a+b)\,.$$

(iv) The area of triangle OPR is denoted by T. Show that $S > \frac{1}{3}T$.

2 If $x = \log_b(c)$, express c in terms of b and x and prove that $\dfrac{\log_a(c)}{\log_a(b)} = \log_b(c)$.

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2(\pi)} + \frac{1}{\log_5(\pi)} > 2\,.$$

(ii) Given that $\log_2\left(\dfrac{\pi}{\mathrm{e}}\right) > \dfrac{1}{5}$ and that $\mathrm{e}^2 < 8$, prove that $\ln\pi > \frac{17}{15}$.

(iii) Given that $\mathrm{e}^3 > 20$, $\pi^2 < 10$ and $\log_{10}(\pi) > \frac{3}{10}$, prove that $\ln(\pi) < \frac{15}{13}$.

3 The points R and S have coordinates $(-a\,,\,0)$ and $(2a\,,\,0)$, respectively, where $a > 0$. The point P has coordinates $(x\,,\,y)$ where $y > 0$ and $x < 2a$. Let $\angle PRS = \alpha$ and $\angle PSR = \beta$.

(i) Show that, if $\beta = 2\alpha$, then P lies on the curve $y^2 = 3(x^2 - a^2)$.

(ii) Find the possible relationships between α and β when $0 < \alpha < \pi$ and P lies on the curve $y^2 = 3(x^2 - a^2)$.

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{x\ln(x)}\left(1-(\ln(x))^2\right)^2 \qquad (x>0, \quad x\neq 1)\,.$$

Show that, when $(\ln x)^2 = 1$, both $\mathrm{f}(x) = 0$ and $\mathrm{f}'(x) = 0$.

The function F is defined by

$$\mathrm{F}(x) = \begin{cases} \displaystyle\int_{1/\mathrm{e}}^{x} \mathrm{f}(t)\,\mathrm{d}t & \text{for } 0<x<1\,, \\ \displaystyle\int_{\mathrm{e}}^{x} \mathrm{f}(t)\,\mathrm{d}t & \text{for } x>1\,. \end{cases}$$

(i) Find $\mathrm{F}(x)$ explicitly and hence show that $\mathrm{F}(x^{-1}) = \mathrm{F}(x)$.

(ii) Sketch the curve with equation $y = \mathrm{F}(x)$.

5 **(i)** Write down the most general polynomial of degree 4 that leaves a remainder of 1 when divided by any of $x-1$, $x-2$, $x-3$ or $x-4$.

(ii) The polynomial $\mathrm{P}(x)$ has degree N, where $N \geqslant 1$, and satisfies

$$\mathrm{P}(1) = \mathrm{P}(2) = \cdots = \mathrm{P}(N) = 1\,.$$

Show that $\mathrm{P}(N+1) \neq 1$.

Given that $\mathrm{P}(N+1) = 2$, find $\mathrm{P}(N+r)$ where r is a positive integer. Find a positive integer r, independent of N, such that $\mathrm{P}(N+r) = N+r$.

(iii) The polynomial $\mathrm{S}(x)$ has degree 4. It has integer coefficients and the coefficient of x^4 is 1. It satisfies

$$\mathrm{S}(a) = \mathrm{S}(b) = \mathrm{S}(c) = \mathrm{S}(d) = 2001\,,$$

where a, b, c and d are distinct (not necessarily positive) integers.

(a) Show that there is no integer e such that $\mathrm{S}(e) = 2018$.

(b) Find the number of ways the (distinct) integers a, b, c and d can be chosen such that $\mathrm{S}(0) = 2017$ and $a<b<c<d$.

6 Use the identity

$$2\sin P\,\sin Q = \cos(Q-P) - \cos(Q+P)$$

to show that

$$2\sin\theta\left(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta\right) = 1 - \cos(2n\theta)\ .$$

(i) Let A_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using n rectangular strips each of width $\frac{\pi}{n}$, such that the midpoint of the top of each strip lies on the curve. Show that

$$A_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\,.$$

(ii) Let B_n be the approximation to the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, using the trapezium rule with n strips each of width $\frac{\pi}{n}$. Show that

$$B_n \sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{n}\cos\left(\frac{\pi}{2n}\right).$$

(iii) Show that

$$\frac{1}{2}(A_n + B_n) = B_{2n}\,,$$

and that

$$A_n B_{2n} = A_{2n}^2\,.$$

7 **(i)** In the cubic equation $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, use the substitution

$$x = \frac{pz+q}{z+1}$$

to show that the equation reduces to $az^3 + b = 0$, where a and b are to be expressed in terms of p and q.

(ii) Show further that the equation $x^3 - 3cx + d = 0$, where c and d are non-zero real numbers, can be written in the form $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, provided $d^2 > 4c^3$.

(iii) Find the real root of the cubic equation $x^3 + 6x - 2 = 0$.

(iv) Find the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$, and hence show how the equation $x^3 - 3cx + d = 0$ can be solved in the case $d^2 = 4c^3$.

8 The functions s and c satisfy $\mathrm{s}(0) = 0$, $\mathrm{c}(0) = 1$ and

$$\mathrm{s}'(x) = \mathrm{c}(x)^2,$$

$$\mathrm{c}'(x) = -\mathrm{s}(x)^2.$$

You may assume that s and c are uniquely defined by these conditions.

(i) Show that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3$ is constant, and deduce that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3 = 1$.

(ii) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{s}(x)\mathrm{c}(x)\Big) = 2\mathrm{c}(x)^3 - 1$$

and find (and simplify) an expression in terms of $\mathrm{c}(x)$ for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{s}(x)}{\mathrm{c}(x)}\right)$.

(iii) Find the integrals

$$\int \mathrm{s}(x)^2 \,\mathrm{d}x \qquad \text{and} \qquad \int \mathrm{s}(x)^5 \,\mathrm{d}x\,.$$

(iv) Given that s has an inverse function, s^{-1}, use the substitution $u = \mathrm{s}(x)$ to show that

$$\int \frac{1}{(1-u^3)^{\frac{2}{3}}}\,\mathrm{d}u = \mathrm{s}^{-1}(u) + \text{constant.}$$

(v) Find, in terms of u, the integrals

$$\int \frac{1}{(1-u^3)^{\frac{4}{3}}}\,\mathrm{d}u \qquad \text{and} \qquad \int (1-u^3)^{\frac{1}{3}}\,\mathrm{d}u\,.$$

Section B: Mechanics

9 A straight road leading to my house consists of two sections. The first section is inclined downwards at a constant angle α to the horizontal and ends in traffic lights; the second section is inclined upwards at an angle β to the horizontal and ends at my house. The distance between the traffic lights and my house is d.

I have a go-kart which I start from rest, pointing downhill, a distance x from the traffic lights on the downward-sloping section. The go-kart is not powered in any way, all resistance forces are negligible, and there is no sudden change of speed as I pass the traffic lights.

(i) Given that I reach my house, show that $x \sin\alpha \geqslant d \sin\beta$.

(ii) Let T be the total time taken to reach my house. Show that

$$\left(\frac{g\sin\alpha}{2}\right)^{\frac{1}{2}} T = (1+k)\sqrt{x} - \sqrt{k^2x - kd}\ ,$$

where $k = \dfrac{\sin\alpha}{\sin\beta}$.

(iii) Hence determine, in terms of d and k, the value of x which minimises T.

[You need not justify the fact that the stationary value is a minimum.]

10 A train is made up of two engines, each of mass M, and n carriages, each of mass m. One of the engines is at the front of the train, and the other is coupled between the kth and $(k+1)$th carriages. When the train is accelerating along a straight, horizontal track, the resistance to the motion of each carriage is R and the driving force on each engine is D, where $2D > nR$. The tension in the coupling between the engine at the front and the first carriage is T.

(i) Show that

$$T = \frac{n(mD + MR)}{nm + 2M}\,.$$

(ii) Show that T is greater than the tension in any other coupling provided that $k > \frac{1}{2}n$.

(iii) Show also that, if $k > \frac{1}{2}n$, then at least one of the couplings is in compression (that is, there is a negative tension in the coupling).

11 The point O lies on a rough plane that is inclined at an angle α to the horizontal, where $\alpha < 45°$. The point A lies on the plane a distance d from O up the line L of greatest slope through O. The point B, which is not on the rough plane, lies in the same vertical plane as O and A, and AB is horizontal. The distance from O to B is d.

A particle P of mass m rests on L between O and A. One end of a light inelastic string is attached to P. The string passes over a smooth light pulley fixed at B and its other end is attached to a freely hanging particle of mass λm.

(i) Show that the acute angle, θ, between the string and the line L satisfies $\alpha \leqslant \theta \leqslant 2\alpha$.

(ii) Given that P can rest in equilibrium at every point on L between O and A, show that $2\lambda \sin\alpha \leqslant 1$.

(iii) The coefficient of friction between P and the plane is μ, and the acute angle β is given by $\mu = \tan\beta$. Show that if $\beta \geqslant 2\alpha$, then a necessary condition for equilibrium to be possible for every position of P on L between O and A is

$$\lambda \leqslant \frac{\sin(\beta - \alpha)}{\cos(\beta - 2\alpha)}.$$

Obtain the corresponding result if $\alpha \leqslant \beta \leqslant 2\alpha$.

Section C: Probability and Statistics

12 A bag contains three coins. The probabilities of their showing heads when tossed are p_1, p_2 and p_3.

(i) A coin is taken at random from the bag and tossed. What is the probability that it shows a head?

(ii) A coin is taken at random from the bag (containing three coins) and tossed; the coin is returned to the bag and again a coin is taken at random from the bag and tossed. Let N_1 be the random variable whose value is the number of heads shown on the two tosses. Find the expectation of N_1 in terms of p, where $p = \frac{1}{3}(p_1 + p_2 + p_3)$, and show that $\text{Var}(N_1) = 2p(1-p)$.

(iii) Two of the coins are taken at random from the bag (containing three coins) and tossed. Let N_2 be the random variable whose value is the number of heads showing on the two coins. Find $\text{E}(N_2)$ and $\text{Var}(N_2)$.

(iv) Show that $\text{Var}(N_2) \leqslant \text{Var}(N_1)$, with equality if and only if $p_1 = p_2 = p_3$.

13 A multiple-choice test consists of five questions. For each question, n answers are given $(n \geqslant 2)$ only one of which is correct and candidates either attempt the question by choosing one of the n given answers or do not attempt it.

For each question attempted, candidates receive two marks for the correct answer and lose one mark for an incorrect answer. No marks are gained or lost for questions that are not attempted. The pass mark is five.

Candidates A, B and C don't understand any of the questions so, for any question which they attempt, they each choose one of the n given answers at random, independently of their choices for any other question.

(i) Candidate A chooses in advance to attempt exactly k of the five questions, where $k = 0$, 1, 2, 3, 4 or 5. Show that, in order to have the greatest probability of passing the test, she should choose $k = 4$.

(ii) Candidate B chooses at random the number of questions he will attempt, the six possibilities being equally likely. Given that Candidate B passed the test find, in terms of n, the probability that he attempted exactly four questions.

(iii) For each of the five questions Candidate C decides whether to attempt the question by tossing a biased coin. The coin has a probability of $\frac{n}{n+1}$ of showing a head, and she attempts the question if it shows a head. Find the probability, in terms of n, that Candidate C passes the test.

Section A: Pure Mathematics

1 **(i)** Use the substitution $u = x\sin x + \cos x$ to find

$$\int \frac{x}{x\tan x + 1}\,\mathrm{d}x\,.$$

Find by means of a similar substitution, or otherwise,

$$\int \frac{x}{x\cot x - 1}\,\mathrm{d}x\,.$$

(ii) Use a substitution to find

$$\int \frac{x\sec^2 x\ \tan x}{x\sec^2 x - \tan x}\,\mathrm{d}x$$

and

$$\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2}\,\mathrm{d}x\,.$$

2 **(i)** The inequality $\dfrac{1}{t} \leqslant 1$ holds for $t \geqslant 1$. By integrating both sides of this inequality over the interval $1 \leqslant t \leqslant x$, show that

$$\ln x \leqslant x - 1 \qquad (*)$$

for $x \geqslant 1$. Show similarly that $(*)$ also holds for $0 < x \leqslant 1$.

(ii) Starting from the inequality $\dfrac{1}{t^2} \leqslant \dfrac{1}{t}$ for $t \geqslant 1$, show that

$$\ln x \geqslant 1 - \frac{1}{x} \qquad (**)$$

for $x > 0$.

(iii) Show, by integrating $(*)$ and $(**)$, that

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y}$$

for $y > 0$ and $y \neq 1$.

3 The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p > 0$ and $q < 0$, lie on the curve C with equation

$$y^2 = 4ax\,,$$

where $a > 0\,$.

(i) Show that the equation of the tangent to C at P is

$$y = \frac{1}{p}x + ap\,.$$

(ii) The tangents to the curve at P and at Q meet at R. These tangents meet the y-axis at S and T respectively, and O is the origin. Prove that the area of triangle OPQ is twice the area of triangle RST.

4 **(i)** Let r be a real number with $|r| < 1$ and let

$$S = \sum_{n=0}^{\infty} r^n\,.$$

You may assume without proof that $S = \dfrac{1}{1-r}\,$.

Let $p = 1 + r + r^2$. Sketch the graph of the function $1 + r + r^2$ and deduce that $\frac{3}{4} \leqslant p < 3\,$.

Show that, if $1 < p < 3$, then the value of p determines r, and hence S, uniquely.

Show also that, if $\frac{3}{4} < p < 1$, then there are two possible values of S and these values satisfy the equation $(3 - p)S^2 - 3S + 1 = 0$.

(ii) Let r be a real number with $|r| < 1$ and let

$$T = \sum_{n=1}^{\infty} nr^{n-1}\,.$$

You may assume without proof that $T = \dfrac{1}{(1-r)^2}\,$.

Let $q = 1 + 2r + 3r^2$. Find the set of values of q that determine T uniquely.

Find the set of values of q for which T has two possible values. Find also a quadratic equation, with coefficients depending on q, that is satisfied by these two values.

5 A circle of radius a is centred at the origin O. A rectangle $PQRS$ lies in the minor sector OMN of this circle where M is $(a, 0)$ and N is $(a\cos\beta, a\sin\beta)$, and β is a constant with $0 < \beta < \frac{\pi}{2}$. Vertex P lies on the positive x-axis at $(x, 0)$; vertex Q lies on ON; vertex R lies on the arc of the circle between M and N; and vertex S lies on the positive x-axis at $(s, 0)$.

(i) Show that the area A of the rectangle can be written in the form

$$A = x(s - x)\tan\beta\,.$$

(ii) Obtain an expression for s in terms of a, x and β, and use it to show that

$$\frac{\mathrm{d}A}{\mathrm{d}x} = (s - 2x)\tan\beta - \frac{x^2}{s}\tan^3\beta\,.$$

(iii) Deduce that the greatest possible area of rectangle $PQRS$ occurs when $s = x(1 + \sec\beta)$ and show that this greatest area is $\frac{1}{2}a^2\tan\frac{1}{2}\beta$.

(iv) Show also that this greatest area occurs when $\angle ROS = \frac{1}{2}\beta$.

6 In this question, you may assume that, if a continuous function takes both positive and negative values in an interval, then it takes the value 0 at some point in that interval.

(i) The function f is continuous and $\mathrm{f}(x)$ is non-zero for some value of x in the interval $0 \leqslant x \leqslant 1$. Prove by contradiction, or otherwise, that if

$$\int_0^1 \mathrm{f}(x)\mathrm{d}x = 0\,,$$

then $\mathrm{f}(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$.

(ii) The function g is continuous and

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x = 1\,, \quad \int_0^1 x\mathrm{g}(x)\,\mathrm{d}x = \alpha\,, \quad \int_0^1 x^2\mathrm{g}(x)\,\mathrm{d}x = \alpha^2\,. \qquad (*)$$

Show, by considering

$$\int_0^1 (x-\alpha)^2\mathrm{g}(x)\,\mathrm{d}x\,,$$

that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

Find a function of the form $\mathrm{g}(x) = a + bx$ that satisfies the conditions $(*)$ and verify that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

(iii) The function h has a continuous derivative h' and

$$\mathrm{h}(0) = 0\,, \quad \mathrm{h}(1) = 1\,, \quad \int_0^1 \mathrm{h}(x)\,\mathrm{d}x = \beta\,, \quad \int_0^1 x\mathrm{h}(x)\,\mathrm{d}x = \tfrac{1}{2}\beta(2-\beta)\,.$$

Use the result in part **(ii)** to show that $\mathrm{h}'(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

7 The triangle ABC has side lengths $|BC| = a$, $|CA| = b$ and $|AB| = c$. Equilateral triangles BXC, CYA and AZB are erected on the sides of the triangle ABC, with X on the other side of BC from A, and similarly for Y and Z. Points L, M and N are the centres of rotational symmetry of triangles BXC, CYA and AZB respectively.

(i) Show that $|CM| = \dfrac{b}{\sqrt{3}}$ and write down the corresponding expression for $|CL|$.

(ii) Use the cosine rule to show that

$$6\,|LM|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\,\Delta\,,$$

where Δ is the area of triangle ABC. Deduce that LMN is an equilateral triangle.

Show further that the areas of triangles LMN and ABC are equal if and only if

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta\,.$$

(iii) Show that the conditions

$$(a-b)^2 = -2ab\big(1 - \cos(C - 60^\circ)\big)$$

and

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta$$

are equivalent.

Deduce that the areas of triangles LMN and ABC are equal if and only if ABC is equilateral.

8 Two sequences are defined by $a_1 = 1$ and $b_1 = 2$ and, for $n \geqslant 1$,

$$a_{n+1} = a_n + 2b_n\,,$$
$$b_{n+1} = 2a_n + 5b_n\,.$$

(i) Prove by induction that, for all $n \geqslant 1$,

$$a_n^2 + 2a_nb_n - b_n^2 = 1\,. \qquad (*)$$

(ii) Let $c_n = \dfrac{a_n}{b_n}$. Show that $b_n \geqslant 2 \times 5^{n-1}$ and use $(*)$ to show that

$$c_n \to \sqrt{2} - 1 \text{ as } n \to \infty\,.$$

(iii) Show also that $c_n > \sqrt{2} - 1$ and hence that $\dfrac{2}{c_n + 1} < \sqrt{2} < c_n + 1\,.$

Deduce that $\dfrac{140}{99} < \sqrt{2} < \dfrac{99}{70}\,.$

Section B: Mechanics

9 A particle is projected at speed u from a point O on a horizontal plane. It passes through a fixed point P which is at a horizontal distance d from O and at a height $d\tan\beta$ above the plane, where $d > 0$ and β is an acute angle. The angle of projection α is chosen so that u is as small as possible.

(i) Show that $u^2 = gd\tan\alpha$ and $2\alpha = \beta + 90°$.

(ii) At what angle to the horizontal is the particle travelling when it passes through P? Express your answer in terms of α in its simplest form.

10 Particles P_1, P_2, $\ldots$ are at rest on the x-axis, and the x-coordinate of P_n is n. The mass of P_n is $\lambda^n m$. Particle P, of mass m, is projected from the origin at speed u towards P_1. A series of collisions takes place, and the coefficient of restitution at each collision is e, where $0 < e < 1$. The speed of P_n immediately after its first collision is u_n and the speed of P_n immediately after its second collision is v_n. No external forces act on the particles.

(i) Show that $u_1 = \dfrac{1+e}{1+\lambda}\,u$ and find expressions for u_n and v_n in terms of e, λ, u and n.

(ii) Show that, if $e > \lambda$, then each particle (except P) is involved in exactly two collisions.

(iii) Describe what happens if $e = \lambda$ and show that, in this case, the fraction of the initial kinetic energy lost approaches e as the number of collisions increases.

(iv) Describe what happens if $\lambda e = 1$. What fraction of the initial kinetic energy is eventually lost in this case?

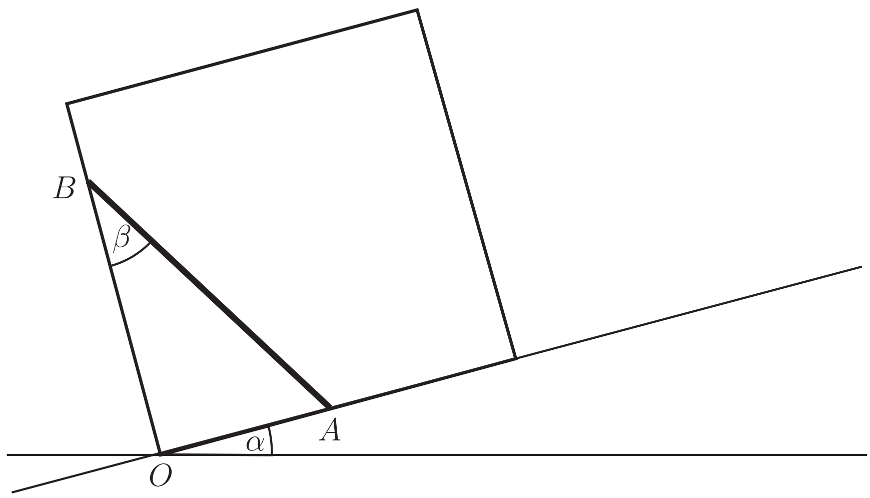
11 A plane makes an acute angle α with the horizontal. A box in the shape of a cube is fixed onto the plane in such a way that four of its edges are horizontal and two of its sides are vertical.

A uniform rod of length $2L$ and weight W rests with its lower end at A on the bottom of the box and its upper end at B on a side of the box, as shown in the diagram below. The vertical plane containing the rod is parallel to the vertical sides of the box and cuts the lowest edge of the box at O. The rod makes an acute angle β with the side of the box at B.

The coefficients of friction between the rod and the box at the two points of contact are both $\tan\gamma$, where $0 < \gamma < \frac{1}{2}\pi$.

The rod is in limiting equilibrium, with the end at A on the point of slipping in the direction away from O and the end at B on the point of slipping towards O. Given that $\alpha < \beta$, show that $\beta = \alpha + 2\gamma$.

[**Hint**: You may find it helpful to take moments about the midpoint of the rod.]



Section C: Probability and Statistics

12 In a lottery, each of the N participants pays £c to the organiser and picks a number from 1 to N. The organiser picks at random the winning number from 1 to N and all those participants who picked this number receive an equal share of the prize, £J.

(i) The participants pick their numbers independently and with equal probability. Obtain an expression for the probability that no participant picks the winning number, and hence determine the organiser's expected profit.

Use the approximation

$$\left(1 - \frac{a}{N}\right)^N \approx \mathrm{e}^{-a} \qquad (*)$$

to show that if $2Nc = J$ then the organiser will expect to make a loss.

Note: $\mathrm{e} > 2$.

(ii) Instead of the numbers being equally popular, a fraction γ of the numbers are popular and the rest are unpopular. For each participant, the probability of picking any given popular number is $\dfrac{a}{N}$ and the probability of picking any given unpopular number is $\dfrac{b}{N}$.

Find a relationship between a, b and γ.

Show that, using the approximation $(*)$, the organiser's expected profit can be expressed in the form

$$A\mathrm{e}^{-a} + B\mathrm{e}^{-b} + C\,,$$

where A, B and C can be written in terms of J, c, N and γ.

In the case $\gamma = \frac{1}{8}$ and $a = 9b$, find a and b. Show that, if $2Nc = J$, then the organiser will expect to make a profit.

Note: $\mathrm{e} < 3$.

13 I have a sliced loaf which initially contains n slices of bread. Each time I finish setting a STEP question, I make myself a snack: either toast, using one slice of bread; or a sandwich, using two slices of bread. I make toast with probability p and I make a sandwich with probability q, where $p + q = 1$, unless there is only one slice left in which case I must, of course, make toast.

(i) Let s_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is the second of two slices used to make a sandwich and let t_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is used to make toast. What is the value of s_1?

(ii) Explain why the following equations hold:

$$t_r = (s_{r-1} + t_{r-1})\,p \qquad (2 \leqslant r \leqslant n-1)\,;$$
$$s_r = 1 - (s_{r-1} + t_{r-1}) \qquad (2 \leqslant r \leqslant n)\,.$$

(iii) Hence, or otherwise, show that $s_r = q(1 - s_{r-1})$ for $2 \leqslant r \leqslant n-1$.

(iv) Show further that

$$s_r = \frac{q + (-q)^r}{1+q} \qquad (1 \leqslant r \leqslant n-1)\,,$$

and find the corresponding expression for t_r.

Find also expressions for s_n and t_n in terms of q.

Section A: Pure Mathematics

1 **(i)** For $n = 1, 2, 3$ and 4, the functions p_n and q_n are defined by

$$\mathrm{p}_n(x) = (x+1)^{2n} - (2n+1)x(x^2+x+1)^{n-1}$$

and

$$\mathrm{q}_n(x) = \frac{x^{2n+1}+1}{x+1} \qquad (x \neq -1)\,.$$

Show that $\mathrm{p}_n(x) \equiv \mathrm{q}_n(x)$ (for $x \neq -1$) in the cases $n = 1$, $n = 2$ and $n = 3$.

Show also that this does not hold in the case $n = 4$.

(ii) Using results from part (i):

(a) express $\dfrac{300^3+1}{301}$ as the product of two factors (neither of which is 1);

(b) express $\dfrac{7^{49}+1}{7^7+1}$ as the product of two factors (neither of which is 1), each written in terms of various powers of 7 which you should not attempt to calculate explicitly.

2 Differentiate, with respect to x,

$$(ax^2+bx+c)\ln\left(x+\sqrt{1+x^2}\right) + \left(dx+e\right)\sqrt{1+x^2}\,,$$

where a, b, c, d and e are constants. You should simplify your answer as far as possible.

Hence integrate:

(i) $\ln\left(x+\sqrt{1+x^2}\right)$;

(ii) $\sqrt{1+x^2}$;

(iii) $x\ln\left(x+\sqrt{1+x^2}\right)$.

3 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that (for example) $\lfloor 2.9 \rfloor = 2$, $\lfloor 2 \rfloor = 2$ and $\lfloor -1.5 \rfloor = -2$.

On separate diagrams draw the graphs, for $-\pi \leqslant x \leqslant \pi$, of:

(i) $y = \lfloor x \rfloor$; (ii) $y = \sin\lfloor x \rfloor$; (iii) $y = \lfloor \sin x \rfloor$; (iv) $y = \lfloor 2\sin x \rfloor$.

In each case, you should indicate clearly the value of y at points where the graph is discontinuous.

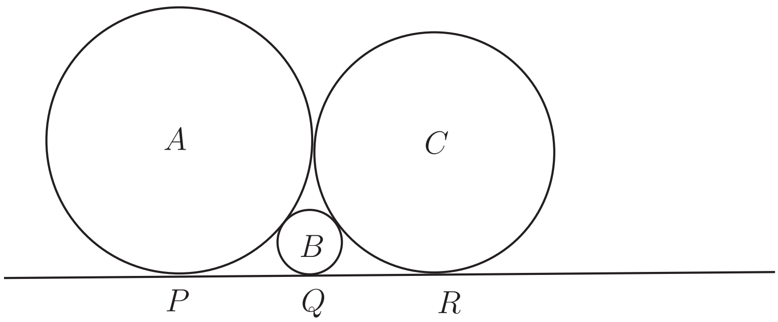
4 **(i)** Differentiate $\dfrac{z}{(1+z^2)^{\frac{1}{2}}}$ with respect to z.

(ii) The *signed curvature* κ of the curve $y = \mathrm{f}(x)$ is defined by

$$\kappa = \frac{\mathrm{f}''(x)}{\big(1 + (\mathrm{f}'(x))^2\big)^{\frac{3}{2}}}\,.$$

Use this definition to determine all curves for which the signed curvature is a non-zero constant. For these curves, what is the geometrical significance of κ?

5 **(i)**



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

6 The sides OA and CB of the quadrilateral $OABC$ are parallel. The point X lies on OA, between O and A. The position vectors of A, B, C and X relative to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively.

(i) Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{x} = m\mathbf{a}\,,$$

where k and m are scalars, and state the range of values that each of k and m can take.

(ii) The lines OB and AC intersect at D, the lines XD and BC intersect at Y and the lines OY and AB intersect at Z. Show that the position vector of Z relative to O can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk+1}\,.$$

(iii) The lines DZ and OA intersect at T. Show that

$$OT \times OA = OX \times TA \qquad \text{and} \qquad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA}\,,$$

where, for example, OT denotes the length of the line joining O and T.

7 The set S consists of all the positive integers that leave a remainder of 1 upon division by 4. The set T consists of all the positive integers that leave a remainder of 3 upon division by 4.

(i) Describe in words the sets $S \cup T$ and $S \cap T$.

(ii) Prove that the product of any two integers in S is also in S. Determine whether the product of any two integers in T is also in T.

(iii) Given an integer in T that is not a prime number, prove that at least one of its prime factors is in T.

(iv) For any set X of positive integers, an integer in X (other than 1) is said to be X-*prime* if it cannot be expressed as the product of two or more integers *all in* X (and all different from 1).

(a) Show that every integer in T is either T-prime or is the product of an odd number of T-prime integers.

(b) Find an example of an integer in S that can be expressed as the product of S-prime integers in two distinct ways. [Note: $s_1 s_2$ and $s_2 s_1$ are not counted as distinct ways of expressing the product of s_1 and s_2.]

8 Given an infinite sequence of numbers u_0, u_1, u_2, $\ldots$, we define the *generating function*, f, for the sequence by

$$\mathrm{f}(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots .$$

Issues of convergence can be ignored in this question.

(i) Using the binomial series, show that the sequence given by $u_n = n$ has generating function $x(1-x)^{-2}$, and find the sequence that has generating function $x(1-x)^{-3}$.

Hence, or otherwise, find the generating function for the sequence $u_n = n^2$. You should simplify your answer.

(ii) **(a)** The sequence u_0, u_1, u_2, $\ldots$ is determined by $u_n = ku_{n-1}$ $(n \geqslant 1)$, where k is independent of n, and $u_0 = a$. By summing the identity $u_n x^n \equiv k u_{n-1} x^n$, or otherwise, show that the generating function, f, satisfies

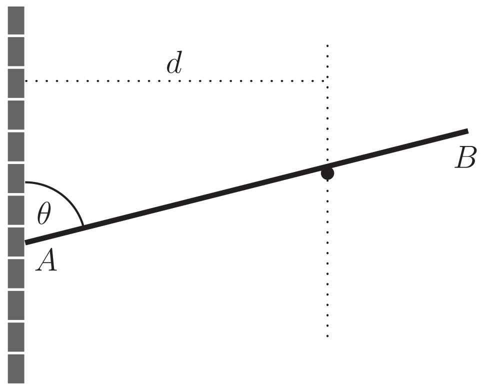
$$\mathrm{f}(x) = a + kx\mathrm{f}(x) .$$

Write down an expression for $\mathrm{f}(x)$.

(b) The sequence u_0, u_1, u_2, $\ldots$ is determined by $u_n = u_{n-1} + u_{n-2}$ $(n \geqslant 2)$ and $u_0 = 0$, $u_1 = 1$. Obtain the generating function.

Section B: Mechanics

9 A horizontal rail is fixed parallel to a vertical wall and at a distance d from the wall. A uniform rod AB of length $2a$ rests in equilibrium on the rail with the end A in contact with the wall. The rod lies in a vertical plane perpendicular to the wall. It is inclined at an angle θ to the vertical (where $0 < \theta < \frac{1}{2}\pi$) and $a \sin\theta < d$, as shown in the diagram.



The coefficient of friction between the rod and the wall is μ, and the coefficient of friction between the rod and the rail is λ.

(i) Show that in limiting equilibrium, with the rod on the point of slipping at both the wall and the rail, the angle θ satisfies

$$d \operatorname{cosec}^2\theta = a\big((\lambda+\mu)\cos\theta + (1-\lambda\mu)\sin\theta\big)\,.$$

(ii) Derive the corresponding result if, instead, $a\sin\theta > d$.

10 Four particles A, B, C and D are initially at rest on a smooth horizontal table. They lie equally spaced a small distance apart, in the order $ABCD$, in a straight line. Their masses are λm, m, m and m, respectively, where $\lambda > 1$.

Particles A and D are simultaneously projected, both at speed u, so that they collide with B and C (respectively). In the following collision between B and C, particle B is brought to rest. The coefficient of restitution in each collision is e.

(i) Show that $e = \dfrac{\lambda - 1}{3\lambda + 1}$ and deduce that $e < \frac{1}{3}$.

(ii) Given also that C and D move towards each other with the same speed, find the value of λ and of e.

11 The point O is at the top of a vertical tower of height h which stands in the middle of a large horizontal plain. A projectile P is fired from O at a fixed speed u and at an angle α above the horizontal.

(i) Show that the distance x from the base of the tower when P hits the plain satisfies

$$\frac{gx^2}{u^2} = h(1+\cos 2\alpha) + x\sin 2\alpha\,.$$

(ii) Show that the greatest value of x as α varies occurs when $x = h\tan 2\alpha$ and find the corresponding value of $\cos 2\alpha$ in terms of g, h and u.

(iii) Show further that the greatest achievable distance between O and the landing point is $\dfrac{u^2}{g} + h\,$.

Section C: Probability and Statistics

12 **(i)** Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.

(ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.

(iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n+1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2, and hence obtain a generalisation of the results of parts (i) and (ii).

13 An internet tester sends n e-mails simultaneously at time $t = 0$. Their arrival times at their destinations are independent random variables each having probability density function $\lambda e^{-\lambda t}$ ($0 \leqslant t < \infty$, $\lambda > 0$).

(i) The random variable T is the time of arrival of the e-mail that arrives first at its destination. Show that the probability density function of T is

$$n\lambda e^{-n\lambda t},$$

and find the expected value of T.

(ii) Write down the probability that the second e-mail to arrive at its destination arrives later than time t and hence derive the density function for the time of arrival of the second e-mail. Show that the expected time of arrival of the second e-mail is

$$\frac{1}{\lambda}\left(\frac{1}{n-1} + \frac{1}{n}\right).$$

Section A: Pure Mathematics

1 **(i)** Sketch the curve $y = \mathrm{e}^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $\mathrm{e}^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n\mathrm{e}^x \to 0$ as $x \to -\infty$ for any integer n.

(ii) Sketch the curve $y = \mathrm{e}^{x^2}(2x^4 - 5x^2 + 2)$.

2 **(i)** Show that $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and find a similar expression for $\sin 15^\circ$.

(ii) Show that $\cos\alpha$ is a root of the equation

$$4x^3 - 3x - \cos 3\alpha = 0\,,$$

and find the other two roots in terms of $\cos\alpha$ and $\sin\alpha$.

(iii) Use parts (i) and (ii) to solve the equation $y^3 - 3y - \sqrt{2} = 0$, giving your answers in surd form.

3 A prison consists of a square courtyard of side b bounded by a perimeter wall and a square building of side a placed centrally within the courtyard. The sides of the building are parallel to the perimeter walls.

Guards can stand either at the middle of a perimeter wall or in a corner of the courtyard. If the guards wish to see as great a length of the perimeter wall as possible, determine which of these positions is preferable. You should consider separately the cases $b < 3a$ and $b > 3a$.

4 The midpoint of a rod of length $2b$ slides on the curve $y = \frac{1}{4}x^2$, $x \geqslant 0$, in such a way that the rod is always tangent, at its midpoint, to the curve.

(i) Show that the curve traced out by one end of the rod can be written in the form

$$x = 2\tan\theta - b\cos\theta$$
$$y = \tan^2\theta - b\sin\theta$$

for some suitably chosen angle θ which satisfies $0 \leqslant \theta < \frac{1}{2}\pi$.

(ii) When one end of the rod is at a point A on the y-axis, the midpoint is at point P and $\theta = \alpha$. Let R be the region bounded by the following:

the curve $y = \frac{1}{4}x^2$ between the origin and P;

the y-axis between A and the origin;

the half-rod AP.

Show that the area of R is $\frac{2}{3}\tan^3\alpha$.

5 **(i)** The function f is defined, for $x > 0$, by

$$\mathrm{f}(x) = \int_1^3 (t-1)^{x-1}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{f}(x)$.

(ii) The function g is defined, for $-\infty < x < \infty$, by

$$\mathrm{g}(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-2xt+x^2}}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{g}(x)$.

6 The vertices of a plane quadrilateral are labelled A, B, A' and B', in clockwise order. A point O lies in the same plane and within the quadrilateral. The angles AOB and $A'OB'$ are right angles, and $OA = OB$ and $OA' = OB'$.

(i) Use position vectors relative to O to show that the midpoints of AB, BA', $A'B'$ and $B'A$ are the vertices of a square.

(ii) Given that the lengths of OA and OA' are fixed (and the conditions of the first paragraph still hold), find the value of angle BOA' for which the area of the square is greatest.

7 Let

$$\mathrm{f}(x) = 3ax^2 - 6x^3$$

and, for each real number a, let $\mathrm{M}(a)$ be the greatest value of $\mathrm{f}(x)$ in the interval $-\frac{1}{3} \leqslant x \leqslant 1$.

Determine $\mathrm{M}(a)$ for $a \geqslant 0$. [The formula for $\mathrm{M}(a)$ is different in different ranges of a; you will need to identify three ranges.]

8 Show that:

(i) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$;

(ii) if N is a positive integer, m is a non-negative integer and k is a positive odd integer, then $(N-m)^k + m^k$ is divisible by N.

Let $S = 1^k + 2^k + 3^k + \cdots + n^k$, where k is a positive odd integer. Show that if n is odd then S is divisible by n and that if n is even then S is divisible by $\frac{1}{2}n$.

Show further that S is divisible by $1 + 2 + 3 + \cdots + n$.

Section B: Mechanics

9 A short-barrelled machine gun stands on horizontal ground. The gun fires bullets, from ground level, at speed u continuously from $t = 0$ to $t = \dfrac{\pi}{6\lambda}$, where λ is a positive constant, but does not fire outside this time period. During this time period, the angle of elevation α of the barrel decreases from $\frac{1}{3}\pi$ to $\frac{1}{6}\pi$ and is given at time t by

$$\alpha = \tfrac{1}{3}\pi - \lambda t\,.$$

(i) Let $k = \dfrac{g}{2\lambda u}$. Show that, in the case $\frac{1}{2} \leqslant k \leqslant \frac{1}{2}\sqrt{3}$, the last bullet to hit the ground does so at a distance

$$\frac{2ku^2\sqrt{1-k^2}}{g}$$

from the gun.

(ii) What is the corresponding result if $k < \frac{1}{2}$?

10 A bus has the shape of a cuboid of length a and height h. It is travelling northwards on a journey of fixed distance at constant speed u (chosen by the driver). The maximum speed of the bus is w. Rain is falling from the southerly direction at speed v in straight lines inclined to the horizontal at angle θ, where $0 < \theta < \frac{1}{2}\pi$.

(i) By considering first the case $u = 0$, show that for $u > 0$ the total amount of rain that hits the roof and the back or front of the bus in unit time is proportional to

$$h\big|v\cos\theta - u\big| + av\sin\theta\,.$$

(ii) Show that, in order to encounter as little rain as possible on the journey, the driver should choose $u = w$ if either $w < v\cos\theta$ or $a\sin\theta > h\cos\theta$. How should the speed be chosen if $w > v\cos\theta$ and $a\sin\theta < h\cos\theta$? Comment on the case $a\sin\theta = h\cos\theta$.

(iii) How should the driver choose u on the return journey?

11 Two long circular cylinders of equal radius lie in equilibrium on an inclined plane, in contact with one another and with their axes horizontal. The weights of the upper and lower cylinders are W_1 and W_2, respectively, where $W_1 > W_2$. The coefficients of friction between the inclined plane and the upper and lower cylinders are μ_1 and μ_2, respectively, and the coefficient of friction between the two cylinders is μ. The angle of inclination of the plane is α (which is positive).

(i) Let F be the magnitude of the frictional force between the two cylinders, and let F_1 and F_2 be the magnitudes of the frictional forces between the upper cylinder and the plane, and the lower cylinder and the plane, respectively. Show that $F = F_1 = F_2$.

(ii) Show that

$$\mu \geqslant \frac{W_1 + W_2}{W_1 - W_2},$$

and that

$$\tan\alpha \leqslant \frac{2\mu_1 W_1}{(1+\mu_1)(W_1 + W_2)}.$$

Section C: Probability and Statistics

12 The number X of casualties arriving at a hospital each day follows a Poisson distribution with mean 8; that is,

$$\mathrm{P}(X = n) = \frac{\mathrm{e}^{-8} 8^n}{n!}, \quad n = 0,\ 1,\ 2,\ \dots\ .$$

Casualties require surgery with probability $\frac{1}{4}$. The number of casualties arriving on any given day is independent of the number arriving on any other day and the casualties require surgery independently of one another.

(i) What is the probability that, on a day when exactly n casualties arrive, exactly r of them require surgery?

(ii) Prove (algebraically) that the number requiring surgery each day also follows a Poisson distribution, and state its mean.

(iii) Given that in a particular randomly chosen week a total of 12 casualties require surgery on Monday and Tuesday, what is the probability that 8 casualties require surgery on Monday? You should give your answer as a fraction in its lowest terms.

13 A fair die with faces numbered $1, \dots, 6$ is thrown repeatedly. The events A, B, C, D and E are defined as follows.

A: the first 6 arises on the nth throw.
B: at least one 5 arises before the first 6.
C: at least one 4 arises before the first 6.
D: exactly one 5 arises before the first 6.
E: exactly one 4 arises before the first 6.

Evaluate the following probabilities:

(i) $\mathrm{P}(A)$ **(ii)** $\mathrm{P}(B)$ **(iii)** $\mathrm{P}(B \cap C)$ **(iv)** $\mathrm{P}(D)$ **(v)** $\mathrm{P}(D \cup E)$.

For some parts of this question, you may want to make use of the binomial expansion in the form:

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2}x^2 + \cdots + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots\ .$$

Section A: Pure Mathematics

1 *All numbers referred to in this question are non-negative integers.*

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

2 **(i)** Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c\,,$ where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}\,$.

(iv) The curve B is given by $y = \big|\ln|x^2-4|\big|\,$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

3 The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2 \, \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x \right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.

4 An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the centre of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leqslant \theta < \pi$.

Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.

In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

5 **(i)** Let $\mathrm{f}(x) = (x + 2a)^3 - 27a^2 x$, where $a \geqslant 0$. By sketching $\mathrm{f}(x)$, show that $\mathrm{f}(x) \geqslant 0$ for $x \geqslant 0$.

(ii) Use part (i) to find the greatest value of xy^2 in the region of the x-y plane given by $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \leqslant 3$. For what values of x and y is this greatest value achieved?

(iii) Use part (i) to show that $(p + q + r)^3 \geqslant 27pqr$ for any non-negative numbers p, q and r. If $(p + q + r)^3 = 27pqr$, what relationship must p, q and r satisfy?

6 **(i)** The sequence of numbers $u_0, u_1, \ldots$ is given by $u_0 = u$ and, for $n \geqslant 0$,

$$u_{n+1} = 4u_n(1 - u_n)\,. \qquad (*)$$

In the case $u = \sin^2\theta$ for some given angle θ, write down and simplify expressions for u_1 and u_2 in terms of θ. Conjecture an expression for u_n and prove your conjecture.

(ii) The sequence of numbers $v_0, v_1, \ldots$ is given by $v_0 = v$ and, for $n \geqslant 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r\,,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2\,.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geqslant 0$, $v_{n+1} = -v_n^2 + 2v_n + 2\,.$

7 In the triangle OAB, the point D divides the side BO in the ratio $r:1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s:1$ (so that $OE = sEA$), where r and s are both positive.

(i) The lines AD and BE intersect at G. Show that

$$\mathbf{g} = \frac{rs}{1+r+rs}\,\mathbf{a} + \frac{1}{1+r+rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t:1$, find an expression for t in terms of r and s.

8 Let L_a denote the line joining the points $(a, 0)$ and $(0, 1-a)$, where $0 < a < 1$. The line L_b is defined similarly.

(i) Determine the point of intersection of L_a and L_b, where $a \neq b$.

(ii) Show that this point of intersection, in the limit as $b \to a$, lies on the curve C given by

$$y = (1 - \sqrt{x})^2 \qquad (0 < x < 1)\,.$$

(iii) Show that every tangent to C is of the form L_a for some a.

Section B: Mechanics

9 A particle of mass m is projected due east at speed U from a point on horizontal ground at an angle θ above the horizontal, where $0 < \theta < 90°$. In addition to the gravitational force mg, it experiences a horizontal force of magnitude mkg, where k is a positive constant, acting due west in the plane of motion of the particle.

(i) Determine expressions in terms of U, θ and g for the time, T_H, at which the particle reaches its greatest height and the time, T_L, at which it lands.

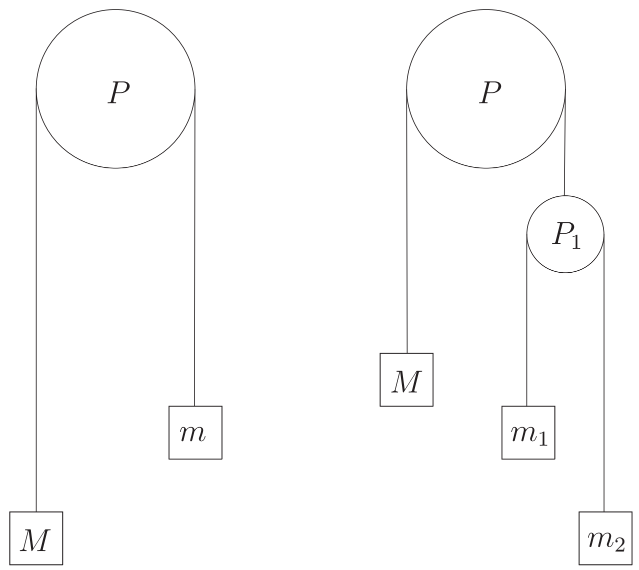
(ii) Let $T = U\cos\theta/(kg)$. By considering the relative magnitudes of T_H, T_L and T, or otherwise, sketch the trajectory of the particle in the cases $k\tan\theta < \frac{1}{2}$, $\frac{1}{2} < k\tan\theta < 1$, and $k\tan\theta > 1$. What happens when $k\tan\theta = 1$?

10 **(i)** A uniform spherical ball of mass M and radius R is released from rest with its centre a distance $H + R$ above horizontal ground. The coefficient of restitution between the ball and the ground is e. Show that, after bouncing, the centre of the ball reaches a height $R + He^2$ above the ground.

(ii) A second uniform spherical ball, of mass m and radius r, is now released from rest together with the first ball (whose centre is again a distance $H + R$ above the ground when it is released). The two balls are initially one on top of the other, with the second ball (of mass m) above the first. The two balls separate slightly during their fall, with their centres remaining in the same vertical line, so that they collide immediately after the first ball has bounced on the ground. The coefficient of restitution between the balls is also e. The centre of the second ball attains a height h above the ground.

Given that $R = 0.2$, $r = 0.05$, $H = 1.8$, $h = 4.5$ and $e = \frac{2}{3}$, determine the value of M/m.

11 The diagrams below show two separate systems of particles, strings and pulleys. In both systems, the pulleys are smooth and light, the strings are light and inextensible, the particles move vertically and the pulleys labelled with P are fixed. The masses of the particles are as indicated on the diagrams.



System I System II

(i) For system I show that the acceleration, a_1, of the particle of mass M, measured in the downwards direction, is given by

$$a_1 = \frac{M-m}{M+m}\,g\,,$$

where g is the acceleration due to gravity. Give an expression for the force on the pulley due to the tension in the string.

(ii) For system II show that the acceleration, a_2, of the particle of mass M, measured in the downwards direction, is given by

$$a_2 = \frac{M-4\mu}{M+4\mu}\,g\,,$$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2}$.

In the case $m = m_1 + m_2$, show that $a_1 = a_2$ if and only if $m_1 = m_2$.

Section C: Probability and Statistics

12 A game in a casino is played with a fair coin and an unbiased cubical die whose faces are labelled $1, 1, 1, 2, 2$ and 3. In each round of the game, the die is rolled once and the coin is tossed once. The outcome of the round is a random variable X. The value, x, of X is determined as follows. If the result of the toss is heads then $x = |ks - 1|$, and if the result of the toss is tails then $x = |k - s|$, where s is the number on the die and k is a given number.

(i) Show that $\mathrm{E}(X^2) = k + 13(k-1)^2/6$.

(ii) Given that both $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$ are positive integers, and that k is a single-digit positive integer, determine the value of k, and write down the probability distribution of X.

(iii) A gambler pays £1 to play the game, which consists of two rounds. The gambler is paid:

£w, where w is an integer, if the sum of the outcomes of the two rounds exceeds 25;

£1 if the sum of the outcomes equals 25;

nothing if the sum of the outcomes is less that 25.

Find, in terms of w, an expression for the amount the gambler expects to be paid in a game, and deduce the maximum possible value of w, given that the casino's owners choose w so that the game is in their favour.

13 A continuous random variable X has a *triangular* distribution, which means that it has a probability density function of the form

$$\mathrm{f}(x) = \begin{cases} \mathrm{g}(x) & \text{for } a < x \leqslant c \\ \mathrm{h}(x) & \text{for } c \leqslant x < b \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{g}(x)$ is an increasing linear function with $\mathrm{g}(a) = 0$, $\mathrm{h}(x)$ is a decreasing linear function with $\mathrm{h}(b) = 0$, and $\mathrm{g}(c) = \mathrm{h}(c)$.

Show that $\mathrm{g}(x) = \dfrac{2(x-a)}{(b-a)(c-a)}$ and find a similar expression for $\mathrm{h}(x)$.

(i) Show that the mean of the distribution is $\frac{1}{3}(a+b+c)$.

(ii) Find the median of the distribution in the different cases that arise.

Section A: Pure Mathematics

1 **(i)** Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} - 22 = 0\,;$

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0\,.$

2 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $\mathrm{f}(x) = \dfrac{\lfloor x \rfloor}{x}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leqslant x \leqslant 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $\mathrm{f}(x) = \frac{7}{12}$.

Solve also the equations $\mathrm{f}(x) = \frac{17}{24}$ and $\mathrm{f}(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $\mathrm{f}(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $\mathrm{f}(x) = c$ to have exactly n roots, where $n \geqslant 1$.

3 For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points P_1, P_2, $\ldots$ are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

4 **(i)** Show that, for $n > 0$,

$$\int_0^{\frac{1}{4}\pi} \tan^n x \, \sec^2 x \, \mathrm{d}x = \frac{1}{n+1} \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} \sec^n x \, \tan x \, \mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n} .$$

(ii) Evaluate the following integrals:

$$\int_0^{\frac{1}{4}\pi} x \, \sec^4 x \, \tan x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \, \tan x \, \mathrm{d}x .$$

5 The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \qquad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45°$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

6 **(i)** By considering the coefficient of x^r in the series for $(1 + x)(1 + x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \qquad (1 \leqslant r \leqslant n).$$

(ii) The sequence of numbers B_0, B_1, B_2, ... is defined by

$$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \qquad \text{and} \qquad B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}.$$

Show that $B_{n+2} - B_{n+1} = B_n$ $(n = 0, 1, 2, \ldots)$.

(iii) What is the relation between the sequence B_0, B_1, B_2, ... and the Fibonacci sequence F_0, F_1, F_2, ... defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$?

7 **(i)** Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

8 **(i)** The functions $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are defined by

$$\begin{aligned}
&\mathrm{a}(x) = x^2 \quad (-\infty < x < \infty),\\
&\mathrm{b}(x) = \ln x \quad (x > 0),\\
&\mathrm{c}(x) = 2x \quad (-\infty < x < \infty),\\
&\mathrm{d}(x) = \sqrt{x} \quad (x \geqslant 0)\,.
\end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

$$\mathrm{cb}, \qquad \mathrm{ab}, \qquad \mathrm{da}, \qquad \mathrm{ad}.$$

(ii) The functions f and g are defined by

$$\begin{aligned}
&\mathrm{f}(x) = \sqrt{x^2 - 1} \quad (|x| \geqslant 1),\\
&\mathrm{g}(x) = \sqrt{x^2 + 1} \quad (-\infty < x < \infty).
\end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned}
&\mathrm{h}(x) = x + \sqrt{x^2 - 1} \quad (x \geqslant 1),\\
&\mathrm{k}(x) = x - \sqrt{x^2 - 1} \quad (|x| \geqslant 1),
\end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

Section B: Mechanics

9 Two particles, A and B, are projected simultaneously towards each other from two points which are a distance d apart in a horizontal plane. Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has mass M and is projected at speed v at angle β above the horizontal. The trajectories of the two particles lie in the same vertical plane.

The particles collide directly when each is at its point of greatest height above the plane.

(i) Given that both A and B return to their starting points, and that momentum is conserved in the collision, show that

$$m\cot\alpha = M\cot\beta\,.$$

(ii) Show further that the collision occurs at a point which is a horizontal distance b from the point of projection of A where

$$b = \frac{Md}{m+M}\,,$$

and find, in terms of b and α, the height above the horizontal plane at which the collision occurs.

10 Two parallel vertical barriers are fixed a distance d apart on horizontal ice. A small ice hockey puck moves on the ice backwards and forwards between the barriers, in the direction perpendicular to the barriers, colliding with each in turn. The coefficient of friction between the puck and the ice is μ and the coefficient of restitution between the puck and each of the barriers is r.

(i) The puck starts at one of the barriers, moving with speed v towards the other barrier. Show that

$$v_{i+1}^2 - r^2 v_i^2 = -2r^2\mu g d$$

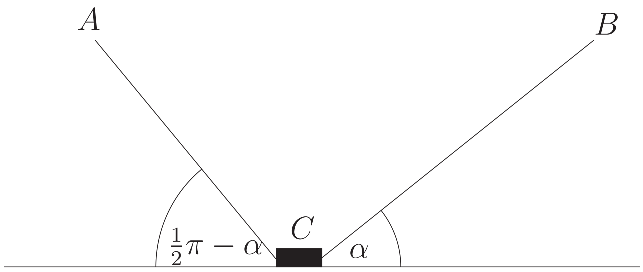
where v_i is the speed of the puck just after its ith collision.

(ii) The puck comes to rest against one of the barriers after traversing the gap between them n times. In the case $r \neq 1$, express n in terms of r and k, where $k = \dfrac{v^2}{2\mu g d}$. If $r = \mathrm{e}^{-1}$ (where e is the base of natural logarithms) show that

$$n = \tfrac{1}{2}\ln\big(1 + k(\mathrm{e}^2 - 1)\big)\,.$$

Give an expression for n in the case $r = 1$.

11



The diagram shows a small block C of weight W initially at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is μ. Two light strings, AC and BC, are attached to the block, making angles $\frac{1}{2}\pi - \alpha$ and α to the horizontal, respectively. The tensions in AC and BC are $T\sin\beta$ and $T\cos\beta$ respectively, where $0 < \alpha + \beta < \frac{1}{2}\pi$.

(i) In the case $W > T\sin(\alpha+\beta)$, show that the block will remain at rest provided

$$W\sin\lambda \geqslant T\cos(\alpha+\beta-\lambda)\,,$$

where λ is the acute angle such that $\tan\lambda = \mu$.

(ii) In the case $W = T\tan\phi$, where $2\phi = \alpha + \beta$, show that the block will start to move in a direction that makes an angle ϕ with the horizontal.

Section C: Probability and Statistics

12 Each day, I have to take k different types of medicine, one tablet of each. The tablets are identical in appearance. When I go on holiday for n days, I put n tablets of each type in a container and on each day of the holiday I select k tablets at random from the container.

(i) In the case $k = 3$, show that the probability that I will select one tablet of each type on the first day of a three-day holiday is $\frac{9}{28}$.

Write down the probability that I will be left with one tablet of each type on the last day (irrespective of the tablets I select on the first day).

(ii) In the case $k = 3$, find the probability that I will select one tablet of each type on the first day of an n-day holiday.

(iii) In the case $k = 2$, find the probability that I will select one tablet of each type on each day of an n-day holiday, and use Stirling's approximation

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n$$

to show that this probability is approximately $2^{-n}\sqrt{n\pi}$.

13 From the integers $1, 2, \ldots, 52$, I choose seven (distinct) integers at random, all choices being equally likely. From these seven, I discard any pair that sum to 53. Let X be the random variable the value of which is the number of discarded pairs. Find the probability distribution of X and show that $\mathrm{E}(X) = \frac{7}{17}$.

Note: $7 \times 17 \times 47 = 5593$.

Section A: Pure Mathematics

1 The line L has equation $y = c - mx$, with $m > 0$ and $c > 0$. It passes through the point $R(a, b)$ and cuts the axes at the points $P(p, 0)$ and $Q(0, q)$, where a, b, p and q are all positive.

(i) Find p and q in terms of a, b and m.

(ii) As L varies with R remaining fixed, show that the minimum value of the sum of the distances of P and Q from the origin is $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$, and find in a similar form the minimum distance between P and Q. (You may assume that any stationary values of these distances are minima.)

2 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

3 **(i)** Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and add to your diagram the tangent to the curve at the origin and the chord joining the origin to the point $(b, \sin b)$, where $0 < b < \frac{1}{2}\pi$.

By considering areas, show that

$$1 - \tfrac{1}{2}b^2 < \cos b < 1 - \tfrac{1}{2}b \sin b\,.$$

(ii) By considering the curve $y = a^x$, where $a > 1$, show that

$$\frac{2(a-1)}{a+1} < \ln a < -1 + \sqrt{2a-1}\,.$$

[**Hint**: You may wish to write a^x as $\mathrm{e}^{x \ln a}$.]

4 The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N.

(i) Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right).$$

(ii) In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

5 **(i)** Show that

$$\int_0^{\frac{1}{4}\pi} \sin(2x) \ln(\cos x)\, \mathrm{d}x = \frac{1}{4}(\ln 2 - 1)\,,$$

(ii) and that

$$\int_0^{\frac{1}{4}\pi} \cos(2x) \ln(\cos x)\, \mathrm{d}x = \frac{1}{8}(\pi - \ln 4 - 2)\,.$$

(iii) Hence evaluate

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \big(\cos(2x) + \sin(2x)\big)\, \ln\big(\cos x + \sin x\big)\, \mathrm{d}x\,.$$

6 A thin circular path with diameter AB is laid on horizontal ground. A vertical flagpole is erected with its base at a point D on the diameter AB. The angles of elevation of the top of the flagpole from A and B are α and β respectively (both are acute). The point C lies on the circular path with DC perpendicular to AB and the angle of elevation of the top of the flagpole from C is ϕ.

(i) Show that $\cot\alpha\cot\beta = \cot^2\phi$.

(ii) Show that, for any p and q,

$$\cos p\cos q\sin^2\tfrac{1}{2}(p+q) - \sin p\sin q\cos^2\tfrac{1}{2}(p+q) = \tfrac{1}{2}\cos(p+q) - \tfrac{1}{2}\cos(p+q)\cos(p-q).$$

(iii) Deduce that, if p and q are positive and $p+q \leqslant \frac{1}{2}\pi$, then

$$\cot p\cot q \geqslant \cot^2\tfrac{1}{2}(p+q)$$

and hence show that $\phi \leqslant \frac{1}{2}(\alpha+\beta)$ when $\alpha+\beta \leqslant \frac{1}{2}\pi$.

7 A sequence of numbers t_0, t_1, t_2, ... satisfies

$$t_{n+2} = pt_{n+1} + qt_n \qquad (n \geqslant 0),$$

where p and q are real. Throughout this question, x, y and z are non-zero real numbers.

(i) Show that, if $t_n = x$ for all values of n, then $p+q=1$ and x can be any (non-zero) real number.

(ii) Show that, if $t_{2n} = x$ and $t_{2n+1} = y$ for all values of n, then $q \pm p = 1$. Deduce that either $x = y$ or $x = -y$, unless p and q take certain values that you should identify.

(iii) Show that, if $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$ for all values of n, then

$$p^3 + q^3 + 3pq - 1 = 0\,.$$

Deduce that either $p+q=1$ or $(p-q)^2 + (p+1)^2 + (q+1)^2 = 0$. Hence show that either $x = y = z$ or $x+y+z=0$.

8 **(i)** Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad\qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad\qquad (x \neq 0).$$

Section B: Mechanics

9 A tall shot-putter projects a small shot from a point $2.5\,\text{m}$ above the ground, which is horizontal. The speed of projection is $10\,\text{m}\,\text{s}^{-1}$ and the angle of projection is θ above the horizontal.

(i) Taking the acceleration due to gravity to be $10\,\text{m}\,\text{s}^{-2}$, show that the time, in seconds, that elapses before the shot hits the ground is

$$\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right),$$

where $c = \cos 2\theta$.

(ii) Find an expression for the range in terms of c and show that it is greatest when $c = \frac{1}{5}$.

(iii) Show that the extra distance attained by projecting the shot at this angle rather than at an angle of $45°$ is $5(\sqrt{6}-\sqrt{2}-1)\,\text{m}$.

10 I stand at the top of a vertical well. The depth of the well, from the top to the surface of the water, is D. I drop a stone from the top of the well and measure the time that elapses between the release of the stone and the moment when I hear the splash of the stone entering the water.

In order to gauge the depth of the well, I climb a distance δ down into the well and drop a stone from my new position. The time until I hear the splash is t less than the previous time.

(i) Show that

$$t = \sqrt{\frac{2D}{g}} - \sqrt{\frac{2(D-\delta)}{g}} + \frac{\delta}{u},$$

where u is the (constant) speed of sound.

(ii) Hence show that

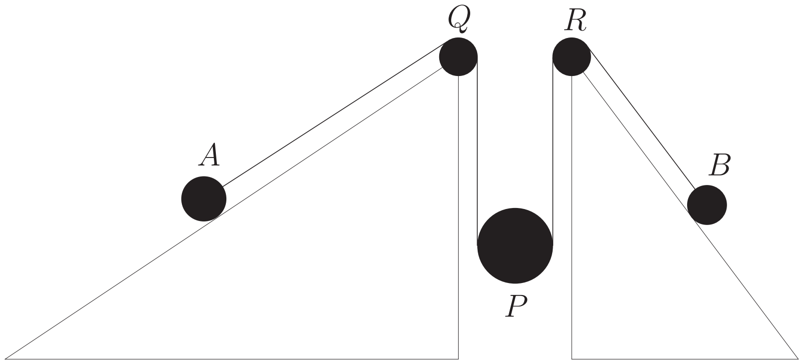
$$D = \tfrac{1}{2}gT^2,$$

where $T = \dfrac{1}{2}\beta + \dfrac{\delta}{\beta g}$ and $\beta = t - \dfrac{\delta}{u}$.

(iii) Taking $u = 300\,\text{m}\,\text{s}^{-1}$ and $g = 10\,\text{m}\,\text{s}^{-2}$, show that if $t = \frac{1}{5}\,\text{s}$ and $\delta = 10\,\text{m}$, the well is approximately $185\,\text{m}$ deep.

11 The diagram shows two particles, A of mass $5m$ and B of mass $3m$, connected by a light inextensible string which passes over two smooth, light, fixed pulleys, Q and R, and under a smooth pulley P which has mass M and is free to move vertically.

Particles A and B lie on fixed rough planes inclined to the horizontal at angles of $\arctan\frac{7}{24}$ and $\arctan\frac{4}{3}$ respectively. The segments AQ and RB of the string are parallel to their respective planes, and segments QP and PR are vertical. The coefficient of friction between each particle and its plane is μ.



(i) Given that the system is in equilibrium, with both A and B on the point of moving up their planes, determine the value of μ and show that $M = 6m$.

(ii) In the case when $M = 9m$, determine the initial accelerations of A, B and P in terms of g.

Section C: Probability and Statistics

12 Fire extinguishers may become faulty at any time after manufacture and are tested annually on the anniversary of manufacture.

The time T years after manufacture until a fire extinguisher becomes faulty is modelled by the continuous probability density function

$$\mathrm{f}(t) = \begin{cases} \dfrac{2t}{(1+t^2)^2} & \text{for } t \geqslant 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

A faulty fire extinguisher will fail an annual test with probability p, in which case it is destroyed immediately. A non-faulty fire extinguisher will always pass the test. All of the annual tests are independent.

(i) Show that the probability that a randomly chosen fire extinguisher will be destroyed exactly three years after its manufacture is $p(5p^2 - 13p + 9)/10$.

(ii) Find the probability that a randomly chosen fire extinguisher that was destroyed exactly three years after its manufacture was faulty 18 months after its manufacture.

13 I choose at random an integer in the range 10000 to 99999, all choices being equally likely. Given that my choice does not contain the digits 0, 6, 7, 8 or 9, show that the expected number of different digits in my choice is 3.3616.

Section A: Pure Mathematics

1 **(i)** Show that the gradient of the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $b\neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p,q) lies on both the straight line $ax+by=1$ and the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $ab\neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p=\pm q$.

Show further that either $(a-b)^2=1$ or $(a+b)^2=1$.

(ii) Show that if the straight line $ax+by=1$, where $ab\neq 0$, is a normal to the curve $\dfrac{a}{x}-\dfrac{b}{y}=1$, then $a^2-b^2=\frac{1}{2}$.

2 The number E is defined by $E=\displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x=\mathrm{e}-1-E\,,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x\,.$

3 Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

(i) By differentiating $(*)$, or otherwise, show that

$$\cot\tfrac{1}{9}\pi-\cot\tfrac{2}{9}\pi+\cot\tfrac{4}{9}\pi=\sqrt{3}\,.$$

(ii) By setting $\theta=\frac{1}{6}\pi-\phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi-\theta)\cot(\tfrac{1}{3}\pi+\theta)=\cot 3\theta\,.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi-\operatorname{cosec}\tfrac{5}{9}\pi+\operatorname{cosec}\tfrac{7}{9}\pi=2\sqrt{3}\,.$$

4 The distinct points P and Q, with coordinates $(ap^2,2ap)$ and $(aq^2,2aq)$ respectively, lie on the curve $y^2=4ax$. The tangents to the curve at P and Q meet at the point T.

(i) Show that T has coordinates $\big(apq,a(p+q)\big)$. You may assume that $p\neq 0$ and $q\neq 0$.

(ii) The point F has coordinates $(a,0)$ and ϕ is the angle TFP. Show that

$$\cos\phi=\frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

5 **(i)** Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

(ii) Let

$$I = \int_0^\pi \big|\sin x - kx\big|\,\mathrm{d}x\,.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

(iii) Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum.

(iv) Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1 - x + 2x^2}{(1-x)^3}$$

is $r^2 + 1$ and hence find the sum of the series

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \cdots\,.$$

(ii) Find the sum of the series

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots\,.$$

7 In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h)\,,$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right)\,,$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

8 **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n) = (1,0)$ and $(m,n) = (1,-1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

Section B: Mechanics

9 A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}\,.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

10 A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic.

(i) Explain why the speeds of A and B immediately before the first collision are the same.

(ii) The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M-m)u^2}{(M+m)^2 g}\,.$$

11 A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A.

(i) Show that

$$3\sin\alpha = 4\sin\beta\,,$$

where α and β are the angles PAB and PBA, respectively.

(ii) Given that $\cos\beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan\frac{1}{7}$.

Section C: Probability and Statistics

12 I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

Section A: Pure Mathematics

1 **(i)** Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

(ii) Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

2 The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

3 **(i)** Show that

$$\sin(x+y) - \sin(x-y) = 2\cos x \, \sin y$$

and deduce that

$$\sin A - \sin B = 2\cos \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

(ii) Show also that

$$\cos A - \cos B = -2\sin \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

(iii) The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

4 **(i)** Use the substitution $x = \dfrac{1}{t^2-1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\,\right) + c\,.$$

[Note You may use without proof the result $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$ **]**

(ii) The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360^o about the x-axis. Show that the volume enclosed is $2\pi \ln \frac{5}{4}$.

5 By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\dbinom{n}{0}+\dbinom{n}{1}+\dbinom{n}{2}+\cdots+\dbinom{n}{n}=2^n\,;$

(ii) $\dbinom{n}{1}+2\dbinom{n}{2}+3\dbinom{n}{3}+\cdots+n\dbinom{n}{n}=n2^{n-1}\,;$

(iii) $\dbinom{n}{0}+\dfrac{1}{2}\dbinom{n}{1}+\dfrac{1}{3}\dbinom{n}{2}+\cdots+\dfrac{1}{n+1}\dbinom{n}{n}=\dfrac{1}{n+1}\left(2^{n+1}-1\right)\,;$

(iv) $\dbinom{n}{1}+2^2\dbinom{n}{2}+3^2\dbinom{n}{3}+\cdots+n^2\dbinom{n}{n}=n\,(n+1)\,2^{n-2}\,.$

6 **(i)** Show that, if $y=\mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+y=0\,. \qquad (*)$$

(ii) In order to find other solutions of this differential equation, now let $y=u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+(x-2)\frac{\mathrm{d}u}{\mathrm{d}x}=0\,. \qquad (**)$$

(iii) By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x}=v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y=Ax+B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

7 Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$.

(i) Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

(ii) Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}\,,$$

lies on the lines OC and AB.

(iii) The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

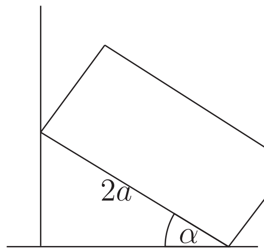
(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4\,.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

Section B: Mechanics

9



The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane.

(i) Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

(ii) The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

$$a\cos(2\lambda + \alpha) = b\sin\alpha\,,$$

where $\tan\lambda = \mu$.

(iii) Show also that if the lamina is square, then $\lambda = \frac{1}{4}\pi - \alpha$.

10 A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \left(\mathrm{e}^t\cos t\right)\mathbf{i} + \left(\mathrm{e}^t\sin t\right)\mathbf{j}\,.$$

(i) Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement.

(ii) Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

(iii) A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

11 Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions.

(i) Find the speeds of the particles after they collide for the first time and

(ii) show that they will both change direction if $2em > M - m$.

(iii) After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why

$$mv - MV = u(M - m),$$

and find v and V in terms of m, M, e, u and n.

Section C: Probability and Statistics

12 **(i)** A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ for this case, and show that

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} \mathrm{P}\left(X \geqslant n\right).$$

(ii) I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

(iii) Hence show that $\mathrm{E}(X) \geqslant 3$.

13 The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour.

(i) Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$p\mathrm{e}^{2\lambda} - \mathrm{e}^{\lambda} + 1 = 0.$$

(ii) Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

(iii) The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

(iv) Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Section A: Pure Mathematics

1 A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

2 A curve has the equation

$$y^3 = x^3 + a^3 + b^3 ,$$

where a and b are positive constants.

(i) Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2 y - a^2 x = a^3 + b^3 .$$

(ii) In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0 .$$

(iii) Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3 .$$

3 **(i)** By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a-x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.

Find the number of distinct real solutions of the equation

$$x = \big((1+a)x - a\big)^{\frac{1}{3}}$$

in the cases that arise according to the value of a.

(ii) Find the number of distinct real solutions of the equation

$$x = (b+x)^{\frac{1}{2}}$$

in the cases that arise according to the value of b.

4 **(i)** The sides of a triangle have lengths $p-q$, p and $p+q$, where $p > q > 0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta) = \cos\alpha + \cos\beta\,.$$

(ii) In the case $\alpha = 2\beta$, show that $\cos\beta = \frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

5 A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

$$V = \tfrac{1}{3}\pi r^2 h\,, \qquad A = \pi r \ell\,.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $\ell = \sqrt{3}\,r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

6 **(i)** Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

(ii) Show by means of a substitution that

$$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

(iii) Evaluate:

(a) $\displaystyle\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x$;

(b) $\displaystyle\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x$.

7 Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx\,\mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi}-1\right).$$

(i) Expand $\cos(A+B)+\cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x\,\mathrm{d}x = \tfrac{19}{650}\left(\mathrm{e}^{2\pi}-1\right).$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x\,\mathrm{d}x$.

8 **(i)** The equation of the circle C is

$$(x-2t)^2 + (y-t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y = 0$.

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Section B: Mechanics

9 Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection *above* the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection *below* the horizontal is β. The particles collide at time T after projection.

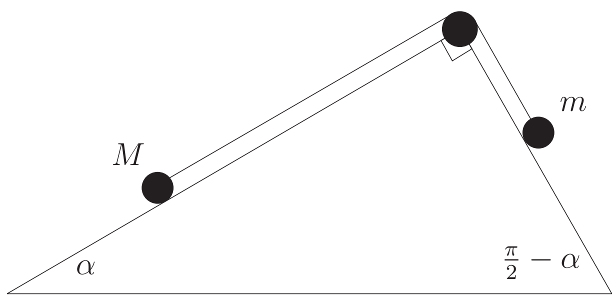
(i) Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2-1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0\,.$$

(ii) Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \leqslant (1+k)V^2\,.$$

10 A triangular wedge is fixed to a horizontal surface. The base angles of the wedge are α and $\frac{\pi}{2}-\alpha$. Two particles, of masses M and m, lie on different faces of the wedge, and are connected by a light inextensible string which passes over a smooth pulley at the apex of the wedge, as shown in the diagram. The contacts between the particles and the wedge are smooth.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ the particle of mass M will slide down the face of the wedge.

(ii) Given that $\tan\alpha = \dfrac{2m}{M}$, show that the magnitude of the acceleration of the particles is

$$\frac{g\sin\alpha}{\tan\alpha + 2}$$

and that this is maximised at $4m^3 = M^3$.

11 Two particles move on a smooth horizontal table and collide. The masses of the particles are m and M. Their velocities before the collision are $u\mathbf{i}$ and $v\mathbf{i}$, respectively, where $\mathbf{i}$ is a unit vector and $u > v$. Their velocities after the collision are $p\mathbf{i}$ and $q\mathbf{i}$, respectively. The coefficient of restitution between the two particles is e, where $e < 1$.

(i) Show that the loss of kinetic energy due to the collision is

$$\tfrac{1}{2}m(u-p)(u-v)(1-e)\,,$$

and deduce that $u \geqslant p$.

(ii) Given that each particle loses the same (non-zero) amount of kinetic energy in the collision, show that

$$u + v + p + q = 0\,,$$

and that, if $m \neq M$,

$$e = \frac{(M+3m)u + (3M+m)v}{(M-m)(u-v)}\,.$$

Section C: Probability and Statistics

12 Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

(ii) Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is

$$\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

13 I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)}.$$

(iii) Find $\mathrm{E}(K)$.

Section A: Pure Mathematics

1 **(i)** What does it mean to say that a number x is *irrational*?

(ii) Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p+q$ is irrational, then at least one of p and q is irrational.

(iii) Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

(iv) If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

2 The variables t and x are related by $t = x + \sqrt{x^2+2bx+c}$, where b and c are constants and $b^2 < c$.

(i) Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b} \,,$$

and hence integrate $\dfrac{1}{\sqrt{x^2+2bx+c}}$.

(ii) Verify by direct integration that your result holds also in the case $b^2 = c$ if $x+b > 0$ but that your result does not hold in the case $b^2 = c$ if $x+b < 0$.

3 **(i)** Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad\,. \qquad (*)$$

(ii) If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy\,.$

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx\,.$

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

(iii) Show similarly that the inequality

$$\frac{s}{t} + \frac{t}{r} + \frac{r}{s} + \frac{t}{s} + \frac{r}{t} + \frac{s}{r} \geqslant 6$$

holds for all positive r, s and t.

4 A function $\mathrm{f}(x)$ is said to be *convex* in the interval $a < x < b$ if $\mathrm{f}''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ is defined for $0 < x < 2\pi$ by

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ is convex.

(ii) The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

5 The polynomial $\mathrm{p}(x)$ is given by

$$\mathrm{p}(x) = x^n + \sum_{r=0}^{n-1} a_r x^r \,,$$

where a_0, a_1, ... , a_{n-1} are fixed real numbers and $n \geqslant 1$. Let M be the greatest value of $\big|\mathrm{p}(x)\big|$ for $|x| \leqslant 1$. Then *Chebyshev's theorem* states that $M \geqslant 2^{1-n}$.

(i) Prove Chebyshev's theorem in the case $n = 1$ and verify that Chebyshev's theorem holds in the following cases:

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}\,;$

(b) $\mathrm{p}(x) = x^3 - x\,.$

(ii) Use Chebyshev's theorem to show that the curve $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.

6 The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g is the inverse function to f, so that $\mathrm{g}(\mathrm{f}(x)) = x$.

(i) Sketch $\mathrm{f}(x)$ and $\mathrm{g}(x)$ on the same axes.

(ii) Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)}\,,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, and

(iii) explain this result by means of a diagram.

7 **(i)** The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by 60° clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$.

(ii) What is the new y-coordinate in the case of an anti-clockwise rotation by 60° ?

(iii) An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$ and find expressions for h_2 and h_3.

(iv) Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

8 **(i)** The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0\,. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

(ii) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0\,.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Section B: Mechanics

9 Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ.

(i) Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$.

(ii) Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

(iii) Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

10 On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2\sin^2\alpha > 2gh$.

(i) Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

(ii) Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'}\right) V\sin 2\alpha\,,$$

where $V' = \sqrt{V^2 - 2gh\,\mathrm{cosec}^2\alpha}$.

11 A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough rigid circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ.

(i) Given that the rod is in limiting equilibrium (i.e. on the point of slipping at both ends), show that

$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}\,,$$

where α is the angle between P_1O and the vertical $(0 < \alpha < 45°)$.

(ii) Let θ be the acute angle between the rod and the horizontal.

Show that $\theta = 2\lambda$, where λ is defined by $\tan\lambda = \mu$ and $0 < \lambda < 22.5°$.

Section C: Probability and Statistics

12 In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\,.$$

The independent random variables X_1 and X_2 each take values $1, 2, \ldots, N$, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geqslant X_2 \\ X_2 & \text{if } X_2 \geqslant X_1\,. \end{cases}$$

(i) Show that $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ for $r = 1, 2, \ldots, N$.

(ii) Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.

(iii) The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. Find an expression for m in terms of N and give an explicit value for m in the case $N = 100$.

(iv) Show that when N is very large,

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}\,.$$

13 Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.

(ii) Find the probability that exactly two husbands sit next to their wives.

(iii) Find the probability that no husband sits next to his wife.

Section A: Pure Mathematics

1 A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits $0, 1, 2, 3$ and 4.

(ii) Show that $\frac{1}{6}k\,(k+1)\,(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\displaystyle\sum_{r=0}^{n} r^2 \equiv \tfrac{1}{6}n\,(n+1)\,(2n+1)$.

2 **(i)** Given that $A = \arctan\frac{1}{2}$ and that $B = \arctan\frac{1}{3}$ (where A and B are acute) show, by considering $\tan(A+B)$, that $A + B = \frac{1}{4}\pi$.

The non-zero integers p and q satisfy

$$\arctan\frac{1}{p} + \arctan\frac{1}{q} = \frac{\pi}{4}\,.$$

Show that $(p-1)\,(q-1) = 2$ and hence determine p and q.

(ii) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan\frac{1}{r} + \arctan\frac{s}{s+t} = \frac{\pi}{4}\,,$$

then there are only two possible values for t, and give r in terms of s in each case.

3 **(i)** Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$.

(ii) Hence or otherwise evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^4\theta \, \mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^4\theta \, \mathrm{d}\theta\,.$$

(iii) Evaluate also

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \, \mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^6\theta \, \mathrm{d}\theta\,.$$

4 **(i)** Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x+b+c)\,\mathrm{Q}(x)$, where $\mathrm{Q}(x)$ is a quadratic expression.

(ii) Show that $2\mathrm{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

(iii) It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac - b^2\right)k = bc - a^2$$

and determine a similar expression involving k^2.

(iv) Hence show that

$$\left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0\,.$

(v) Deduce that either $k = 1$ or the two equations are identical.

5 *Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.*

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$.

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

6 **(i)** Given that $x^2 - y^2 = (x-y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = (\sqrt{m} - \sqrt{n})^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x-y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m-n)^4$.

7 **(i)** The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.

The line L_2 has vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Show that the distance D between a point on L_1 and a point on L_2 can be expressed in the form

$$D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36\,.$$

Hence determine the minimum distance between these two lines and find the coordinates of the points on the two lines that are the minimum distance apart.

(ii) The line L_3 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The line L_4 has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 4k \\ 1-k \\ -3k \end{pmatrix}$.

Determine the minimum distance between these two lines, explaining geometrically the two different cases that arise according to the value of k.

8 A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16)\ , \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2\,,\,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

Section B: Mechanics

9 A particle of weight W is placed on a rough plane inclined at an angle of θ to the horizontal. The coefficient of friction between the particle and the plane is μ. A horizontal force X acting on the particle is just sufficient to prevent the particle from sliding down the plane; when a horizontal force kX acts on the particle, the particle is about to slide up the plane. Both horizontal forces act in the vertical plane containing the line of greatest slope.

(i) Prove that

$$(k-1)\left(1+\mu^2\right)\sin\theta\cos\theta = \mu\,(k+1)$$

(ii) Hence show that $k \geqslant \dfrac{(1+\mu)^2}{(1-\mu)^2}$.

10 The Norman army is advancing with constant speed u towards the Saxon army, which is at rest. When the armies are d apart, a Saxon horseman rides from the Saxon army directly towards the Norman army at constant speed x. Simultaneously a Norman horseman rides from the Norman army directly towards the Saxon army at constant speed y, where $y > u$. The horsemen ride their horses so that $y - 2x < u < 2y - x$.

When each horseman reaches the opposing army, he immediately rides straight back to his own army without changing his speed.

(i) Represent this information on a displacement-time graph, and

(ii) show that the two horsemen pass each other at distances

$$\frac{xd}{x+y} \quad \text{and} \quad \frac{xd(2y-x-u)}{(u+x)(x+y)}$$

from the Saxon army.

(iii) Explain briefly what will happen in the cases (i) $u > 2y - x$ and (ii) $u < y - 2x$.

11 A smooth, straight, narrow tube of length L is fixed at an angle of $30°$ to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motion until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the tube.

(i) Show that D satisfies the equation

$$4gD^2 - 2\sqrt{3}\left(u^2 - Lg\right)D - 3L\left(u^2 - gL\right) = 0$$

and

(ii) hence that

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\,(u^2 - gL)}.$$

(iii) The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

Section C: Probability and Statistics

12 **(i)** A bag contains N sweets (where $N \geqslant 2$), of which a are red. Two sweets are drawn from the bag without replacement. Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

(ii) There are two bags, each containing N sweets (where $N \geqslant 2$). The first bag contains a red sweets, and the second bag contains b red sweets. There is also a biased coin, showing Heads with probability p and Tails with probability q, where $p + q = 1$.

The coin is tossed. If it shows Heads then a sweet is chosen from the first bag and transferred to the second bag; if it shows Tails then a sweet is chosen from the second bag and transferred to the first bag. The coin is then tossed a second time: if it shows Heads then a sweet is chosen from the first bag, and if it shows Tails then a sweet is chosen from the second bag.

Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

13 A bag contains eleven small discs, which are identical except that six of the discs are blank and five of the discs are numbered, using the numbers 1, 2, 3, 4 and 5. The bag is shaken, and four discs are taken one at a time without replacement.

Calculate the probability that:

(i) all four discs taken are numbered;

(ii) all four discs taken are numbered, given that the disc numbered "3" is taken first;

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first;

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken;

(v) exactly two numbered discs are taken, given that a numbered disc is taken first;

(vi) exactly two numbered discs are taken, given that a numbered disc is taken.

14 The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x+1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

It is known that $\mathrm{P}(X \geqslant 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathrm{P}\,(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathrm{P}\,(X = 1 \,|\, X \leqslant 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).

Section A: Pure Mathematics

1 **(i)** Find the integer, n, that satisfies $n^2 < 33127 < (n+1)^2$.

(ii) Find also a small integer m such that $(n+m)^2 - 33127$ is a perfect square.

(iii) Hence express 33127 in the form pq, where p and q are integers greater than 1.

(iv) By considering the possible factorisations of 33127, show that there are exactly two values of m for which $(n+m)^2 - 33127$ is a perfect square, and find the other value.

2 A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leqslant 14\pi a^2$ and determine the minimum value of A.

3 In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2+bx+c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

4 **(i)** By sketching on the same axes the graphs of $y = \sin x$ and $y = x$, show that, for $x > 0$:

(a) $x > \sin x$;

(b) $\dfrac{\sin x}{x} \approx 1$ for small x.

(ii) A regular polygon has n sides, and perimeter P. Show that the area of the polygon is

$$\frac{P^2}{4n\tan\left(\frac{\pi}{n}\right)} .$$

(iii) Show by differentiation (treating n as a continuous variable) that the area of the polygon increases as n increases with P fixed.

(iv) Show also that, for large n, the ratio of the area of the polygon to the area of the smallest circle which can be drawn around the polygon is approximately 1.

5 **(i)** Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K ,$$

where K is a constant.

(ii) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12} + \ln\frac{2}{3}$.

6 **(i)** Show that, if $(a\,,b)$ is **any** point on the curve $x^2 - 2y^2 = 1$, then $(3a+4b\,,2a+3b)$ also lies on the curve.

(ii) Determine the smallest positive integers M and N such that, if $(a\,,b)$ is **any** point on the curve $Mx^2 - Ny^2 = 1$, then $(5a+6b\,,4a+5b)$ also lies on the curve.

(iii) Given that the point $(a\,,b)$ lies on the curve $x^2 - 3y^2 = 1$, find positive integers P, Q, R and S such that the point $(Pa+Qb\,,Ra+Sb)$ also lies on the curve.

7 **(i)** Sketch on the same axes the functions $\operatorname{cosec} x$ and $2x/\pi$, for $0 < x < \pi$. Deduce that the equation $x \sin x = \pi/2$ has exactly two roots in the interval $0 < x < \pi$.

(ii) Show that

$$\int_{\pi/2}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \, \mathrm{d}x = 2\sin\alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha\cos\alpha - 1$$

where α is the larger of the roots referred to above.

(iii) Show that the region bounded by the positive x-axis, the y-axis and the curve

$$y = \Big| |\mathrm{e}^x - 1| - 1 \Big|$$

has area $\ln 4 - 1$.

8 *Note that the volume of a tetrahedron is equal to $\frac{1}{3}$ $\times$ the area of the base $\times$ the height.*

The points O, A, B and C have coordinates $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

(iii) Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Section B: Mechanics

9 A block of mass 4 kg is at rest on a smooth, horizontal table. A smooth pulley P is fixed to one edge of the table and a smooth pulley Q is fixed to the opposite edge. The two pulleys and the block lie in a straight line.

Two horizontal strings are attached to the block. One string runs over pulley P; a particle of mass x kg hangs at the end of this string. The other string runs over pulley Q; a particle of mass y kg hangs at the end of this string, where $x > y$ and $x + y = 6$.

The system is released from rest with the strings taut. When the 4 kg block has moved a distance d, the string connecting it to the particle of mass x kg is cut.

(i) Show that the time taken by the block from the start of the motion until it first returns to rest (assuming that it does not reach the edge of the table) is $\sqrt{d/(5g)}\,\mathrm{f}(y)$, where

$$\mathrm{f}(y) = \frac{10}{\sqrt{6-2y}} + \left(1 + \frac{4}{y}\right)\sqrt{6-2y}.$$

(ii) Calculate the value of y for which $\mathrm{f}'(y) = 0$.

10 **(i)** A particle P is projected in the x-y plane, where the y-axis is vertical and the x-axis is horizontal. The particle is projected with speed V from the origin at an angle of 45° above the positive x-axis. Determine the equation of the trajectory of P.

(ii) The point of projection (the origin) is on the floor of a barn. The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$ and α is an acute angle. Show that, if the particle just touches the roof, then $V(-1+\tan\alpha) = -2\sqrt{bg}$; you should justify the choice of the negative root. If this condition is satisfied, find, in terms of α, V and g, the time after projection at which touching takes place.

(iii) A particle Q can slide along a smooth rail fixed, in the x-y plane, to the under-side of the roof. It is projected from the point $(0, b)$ with speed U at the same time as P is projected from the origin. Given that the particles just touch in the course of their motions, show that

$$2\sqrt{2}\,U\cos\alpha = V\left(2 + \sin\alpha\cos\alpha - \sin^2\alpha\right).$$

11 Particles A_1, A_2, A_3, $\ldots$, A_n (where $n \geqslant 2$) lie at rest in that order in a smooth straight horizontal trough. The mass of A_{n-1} is m and the mass of A_n is λm, where $\lambda > 1$. Another particle, A_0, of mass m, slides along the trough with speed u towards the particles and collides with A_1. Momentum and energy are conserved in all collisions.

(i) Show that it is not possible for there to be exactly one particle moving after all collisions have taken place.

(ii) Show that it is not possible for A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iii) Show that it is not possible for A_{n-2}, A_{n-1} and A_n to be the only particles moving after all collisions have taken place.

(iv) Given that there are exactly two particles moving after all collisions have taken place, find the speeds of these particles in terms of u and λ.

Section C: Probability and Statistics

12 Oxtown and Camville are connected by three roads, which are at risk of being blocked by flooding. On two of the three roads there are two sections which may be blocked. On the third road there is only one section which may be blocked. The probability that each section is blocked is p. Each section is blocked independently of the other four sections.

(i) Show that the probability that Oxtown is cut off from Camville is $p^3\left(2-p\right)^2$.

(ii) I want to travel from Oxtown to Camville. I choose one of the three roads at random and find that my road is not blocked. Find the probability that I would not have reached Camville if I had chosen either of the other two roads. You should factorise your answer as fully as possible.

(iii) Comment briefly on the value of this probability in the limit $p \to 1$.

13 A very generous shop-owner is hiding small diamonds in chocolate bars. Each diamond is hidden independently of any other diamond, and on average there is one diamond per kilogram of chocolate.

(i) I go to the shop and roll a fair six-sided die once. I decide that if I roll a score of N, I will buy $100N$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

Show also that the expected number of diamonds I find is 0.35.

(ii) Instead, I decide to roll a fair six-sided die repeatedly until I score a 6. If I roll my first 6 on my Tth throw, I will buy $100T$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6-5\mathrm{e}^{-0.1}}$$

Calculate also the expected number of diamonds that I find. (You may find it useful to consider the the binomial expansion of $(1-x)^{-2}$.)

14 **(i)** A bag of sweets contains one red sweet and n blue sweets. I take a sweet from the bag, note its colour, return it to the bag, then shake the bag. I repeat this until the sweet I take is the red one. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

(ii) Instead, I take sweets from the bag, without replacing them in the bag, until I take the red sweet. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

Section A: Pure Mathematics

1 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

2 The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q.

(i) Show that R has coordinates $(pq, p+q)$.

(ii) The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1,0)$ lies on the line PQ, show that S has coordinates $(p^2+q^2+1,\ p+q)$, and that the quadrilateral $PSQR$ is a rectangle.

3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that, if $c \neq 1$, the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if $c^2 = -\dfrac{4ab}{(a-b)^2}$. Show that this condition can be written $c^2 = 1 - \left(\dfrac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leqslant 1$.

4 **(i)** Given that $\cos\theta = \dfrac{3}{5}$ and that $\dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\dfrac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \dfrac{11}{2}$ and that $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

5 **(i)** Evaluate the integral

$$\int_0^1 (x+1)^{k-1}\ \mathrm{d}x$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

(ii) Evaluate the integral

$$\int_0^1 x\,(x+1)^m\ \mathrm{d}x$$

in the different cases that arise according to the value of m.

6 **(i)** The point A has coordinates $(5\,,16)$ and the point B has coordinates $(-4\,,4)$. The variable point P has coordinates $(x\,,y)$ and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100\ .$$

(ii) The point C has coordinates $(a\,,0)$ and the point D has coordinates $(b\,,0)$, where $a \neq b$. The variable point Q moves on a path such that

$$QC = k \times QD\ ,$$

where $k > 1$. Given that the path of Q is the same as the path of P, show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}\ .$$

Show further that $(a+7)(b+7) = 100$.

7 The notation $\prod_{r=1}^{n} \mathrm{f}(r)$ denotes the product $\mathrm{f}(1) \times \mathrm{f}(2) \times \mathrm{f}(3) \times \cdots \times \mathrm{f}(n)$.

Simplify the following products as far as possible:

(i) $\prod_{r=1}^{n} \left(\dfrac{r+1}{r}\right)$;

(ii) $\prod_{r=2}^{n} \left(\dfrac{r^2-1}{r^2}\right)$;

(iii) $\prod_{r=1}^{n} \left(\cos\dfrac{2\pi}{n} + \sin\dfrac{2\pi}{n}\cot\dfrac{(2r-1)\,\pi}{n}\right)$, where n is even.

8 Show that, if $y^2 = x^k \mathrm{f}(x)$, then $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}\dfrac{\mathrm{df}}{\mathrm{d}x}$.

(i) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

(ii) Find the solution of the differential equation

$$2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln(x) - xy^2$$

for which $y = 1$ when $x = 1$.

Section B: Mechanics

9 A non-uniform rod AB has weight W and length $3l$. When the rod is suspended horizontally in equilibrium by vertical strings attached to the ends A and B, the tension in the string attached to A is T.

(i) When instead the rod is held in equilibrium in a horizontal position by means of a smooth pivot at a distance l from A and a vertical string attached to B, the tension in the string is T. Show that $5T = 2W$.

(ii) When instead the end B of the rod rests on rough horizontal ground and the rod is held in equilibrium at an angle θ to the horizontal by means of a string that is perpendicular to the rod and attached to A, the tension in the string is $\frac{1}{2}T$. Calculate θ and find the smallest value of the coefficient of friction between the rod and the ground that will prevent slipping.

10 Three collinear, non-touching particles A, B and C have masses a, b and c, respectively, and are at rest on a smooth horizontal surface. The particle A is given an initial velocity u towards B. These particles collide, giving B a velocity v towards C. These two particles then collide, giving C a velocity w.

The coefficient of restitution is e in both collisions. Determine an expression for v, and show that

$$w = \frac{abu\,(1+e)^2}{(a+b)\,(b+c)}\,.$$

Determine the final velocities of each of the three particles in the cases:

(i) $\dfrac{a}{b} = \dfrac{b}{c} = e\,;$

(ii) $\dfrac{b}{a} = \dfrac{c}{b} = e\,.$

11 A particle moves so that $\mathbf{r}$, its displacement from a fixed origin at time t, is given by

$$\mathbf{r} = (\sin 2t)\,\mathbf{i} + (2\cos t)\,\mathbf{j}\,,$$

where $0 \leqslant t < 2\pi$.

(i) Show that the particle passes through the origin exactly twice.

(ii) Determine the times when the velocity of the particle is perpendicular to its displacement.

(iii) Show that, when the particle is not at the origin, its velocity is never parallel to its displacement.

(iv) Determine the maximum distance of the particle from the origin, and sketch the path of the particle.

Section C: Probability and Statistics

12 **(i)** The probability that a hobbit smokes a pipe is 0.7 and the probability that a hobbit wears a hat is 0.4. The probability that a hobbit smokes a pipe but does not wear a hat is p. Determine the range of values of p consistent with this information.

(ii) The probability that a wizard wears a hat is 0.7; the probability that a wizard wears a cloak is 0.8; and the probability that a wizard wears a ring is 0.4. The probability that a wizard does not wear a hat, does not wear a cloak and does not wear a ring is 0.05. The probability that a wizard wears a hat, a cloak and also a ring is 0.1. Determine the probability that a wizard wears exactly two of a hat, a cloak, and a ring.

The probability that a wizard wears a hat but not a ring, **given** that he wears a cloak, is q. Determine the range of values of q consistent with this information.

13 The random variable X has mean μ and standard deviation σ. The distribution of X is symmetrical about μ and satisfies:

$$\mathrm{P}\left(X \leqslant \mu + \sigma\right) = a \qquad \text{and} \qquad \mathrm{P}\left(X \leqslant \mu + \tfrac{1}{2}\sigma\right) = b\,,$$

where a and b are fixed numbers. Do not assume that X is Normally distributed.

(i) Determine expressions (in terms of a and b) for

$$\mathrm{P}\left(\mu - \tfrac{1}{2}\sigma \leqslant X \leqslant \mu + \sigma\right) \qquad \text{and} \qquad \mathrm{P}\left(X \leqslant \mu + \tfrac{1}{2}\sigma \;\middle|\; X \geqslant \mu - \tfrac{1}{2}\sigma\right).$$

(ii) My local supermarket sells cartons of skimmed milk and cartons of full-fat milk: 60% of the cartons it sells contain skimmed milk, and the rest contain full-fat milk.

The volume of skimmed milk in a carton is modelled by X ml, with $\mu = 500$ and $\sigma = 10$. The volume of full-fat milk in a carton is modelled by X ml, with $\mu = 495$ and $\sigma = 10$.

(a) Today, I bought one carton of milk, chosen at random, from this supermarket. When I get home, I find that it contains less than 505 ml. Determine an expression (in terms of a and b) for the probability that this carton of milk contains more than 500 ml.

(b) Over the years, I have bought a very large number of cartons of milk, all chosen at random, from this supermarket. 70% of the cartons I have bought have contained at most 505 ml of milk. Of all the cartons that have contained at least 495 ml of milk, one third of them have contained full-fat milk. Use this information to estimate the values of a and b.

14 The random variable X can take the value $X = -1$, and also any value in the range $0 \leqslant X < \infty$. The distribution of X is given by

$$\mathrm{P}(X = -1) = m\,, \qquad \mathrm{P}(0 \leqslant X \leqslant x) = k(1 - \mathrm{e}^{-x})\,,$$

for any non-negative number x, where k and m are constants, and $m < \frac{1}{2}$.

(i) Find k in terms of m.

(ii) Show that $\mathrm{E}(X) = 1 - 2m$.

(iii) Find, in terms of m, $\mathrm{Var}\,(X)$ and the median value of X.

(iv) Given that

$$\int_0^\infty y^2 \mathrm{e}^{-y^2} \mathrm{d}y = \tfrac{1}{4}\sqrt{\pi}\,,$$

find $\mathrm{E}\big(|X|^{\frac{1}{2}}\big)$ in terms of m.

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