C4 DIFFERENTIATION Worksheet A

1 A curve is given by the parametric equations

$$x = t^2 + 1, \quad y = \frac{4}{t}.$$

a Write down the coordinates of the point on the curve where $t = 2$.

b Find the value of t at the point on the curve with coordinates $(\frac{5}{4}, -8)$.

2 A curve is given by the parametric equations

$$x = 1 + \sin t, \quad y = 2\cos t, \quad 0 \le t < 2\pi.$$

a Write down the coordinates of the point on the curve where $t = \frac{\pi}{2}$.

b Find the value of t at the point on the curve with coordinates $(\frac{3}{2}, -\sqrt{3})$.

3 Find a cartesian equation for each curve, given its parametric equations.

a $x = 3t, \quad y = t^2$ **b** $x = 2t, \quad y = \frac{1}{t}$ **c** $x = t^3, \quad y = 2t^2$

d $x = 1 - t^2, \quad y = 4 - t$ **e** $x = 2t - 1, \quad y = \frac{2}{t^2}$ **f** $x = \frac{1}{t-1}, \quad y = \frac{1}{2-t}$

4 A curve has parametric equations

$$x = 2t + 1, \quad y = t^2.$$

a Find a cartesian equation for the curve.

b Hence, sketch the curve.

5 Find a cartesian equation for each curve, given its parametric equations.

a $x = \cos\theta, \quad y = \sin\theta$ **b** $x = \sin\theta, \quad y = \cos 2\theta$ **c** $x = 3 + 2\cos\theta, \quad y = 1 + 2\sin\theta$

d $x = 2\sec\theta, \quad y = 4\tan\theta$ **e** $x = \sin\theta, \quad y = \sin^2 2\theta$ **f** $x = \cos\theta, \quad y = \tan^2\theta$

6 A circle has parametric equations

$$x = 1 + 3\cos\theta, \quad y = 4 + 3\sin\theta, \quad 0 \le \theta < 2\pi.$$

a Find a cartesian equation for the circle.

b Write down the coordinates of the centre and the radius of the circle.

c Sketch the circle and label the points on the circle where θ takes each of the following values:

$$0, \ \frac{\pi}{4}, \ \frac{\pi}{2}, \ \frac{3\pi}{4}, \ \pi, \ \frac{5\pi}{4}, \ \frac{3\pi}{2}, \ \frac{7\pi}{4}.$$

7 Write down parametric equations for a circle

a centre (0, 0), radius 5,

b centre (6, −1), radius 2,

c centre (a, b), radius r, where a, b and r are constants and $r > 0$.

8 For each curve given by parametric equations, find a cartesian equation and hence, sketch the curve, showing the coordinates of any points where it meets the coordinate axes.

a $x = 2t, \quad y = 4t(t - 1)$ **b** $x = 1 - \sin\theta, \quad y = 2 - \cos\theta, \quad 0 \le \theta < 2\pi$

c $x = t - 3, \quad y = 4 - t^2$ **d** $x = t + 1, \quad y = \frac{2}{t}$

1 A curve is given by the parametric equations

$$x = 2 + t, \quad y = t^2 - 1.$$

a Write down expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

b Hence, show that $\frac{dy}{dx} = 2t$.

2 Find and simplify an expression for $\frac{dy}{dx}$ in terms of the parameter t in each case.

a $x = t^2, \quad y = 3t$

b $x = t^2 - 1, \quad y = 2t^3 + t^2$

c $x = 2 \sin t, \quad y = 6 \cos t$

d $x = 3t - 1, \quad y = 2 - \frac{1}{t}$

e $x = \cos 2t, \quad y = \sin t$

f $x = e^{t+1}, \quad y = e^{2t-1}$

g $x = \sin^2 t, \quad y = \cos^3 t$

h $x = 3 \sec t, \quad y = 5 \tan t$

i $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}$

3 Find, in the form $y = mx + c$, an equation for the tangent to the given curve at the point with the given value of the parameter t.

a $x = t^3, \quad y = 3t^2, \qquad t = 1$

b $x = 1 - t^2, \quad y = 2t - t^2, \qquad t = 2$

c $x = 2 \sin t, \quad y = 1 - 4 \cos t, \qquad t = \frac{\pi}{3}$

d $x = \ln (4 - t), \quad y = t^2 - 5, \qquad t = 3$

4 Show that the normal to the curve with parametric equations

$$x = \sec \theta, \quad y = 2 \tan \theta, \quad 0 \le \theta < \tfrac{\pi}{2},$$

at the point where $\theta = \frac{\pi}{3}$, has the equation

$$\sqrt{3}\, x + 4y = 10\sqrt{3}\,.$$

5 A curve is given by the parametric equations

$$x = \frac{1}{t}, \quad y = \frac{1}{t+2}.$$

a Show that $\frac{dy}{dx} = \left(\frac{t}{t+2}\right)^2$.

b Find an equation for the normal to the curve at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

6 A curve has parametric equations

$$x = \sin 2t, \quad y = \sin^2 t, \quad 0 \le t < \pi.$$

a Show that $\frac{dy}{dx} = \frac{1}{2} \tan 2t$.

b Find an equation for the tangent to the curve at the point where $t = \frac{\pi}{6}$.

7 A curve has parametric equations

$$x = 3 \cos \theta, \quad y = 4 \sin \theta, \quad 0 \le \theta < 2\pi.$$

a Show that the tangent to the curve at the point $(3 \cos \alpha, 4 \sin \alpha)$ has the equation

$$3y \sin \alpha + 4x \cos \alpha = 12.$$

b Hence find an equation for the tangent to the curve at the point $(-\frac{3}{2}, 2\sqrt{3}\,)$.

8 A curve is given by the parametric equations

$$x = t^2, \quad y = t(t - 2), \quad t \geq 0.$$

a Find the coordinates of any points where the curve meets the coordinate axes.

b Find $\frac{dy}{dx}$ in terms of x

i by first finding $\frac{dy}{dx}$ in terms of t,

ii by first finding a cartesian equation for the curve.

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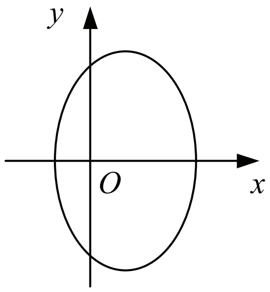

The diagram shows the ellipse with parametric equations

$$x = 1 - 2\cos\theta, \quad y = 3\sin\theta, \quad 0 \leq \theta < 2\pi.$$

a Find $\frac{dy}{dx}$ in terms of θ.

b Find the coordinates of the points where the tangent to the curve is

i parallel to the x-axis,

ii parallel to the y-axis.

10 A curve is given by the parametric equations

$$x = \sin\theta, \quad y = \sin 2\theta, \quad 0 \leq \theta \leq \tfrac{\pi}{2}.$$

a Find the coordinates of any points where the curve meets the coordinate axes.

b Find an equation for the tangent to the curve that is parallel to the x-axis.

c Find a cartesian equation for the curve in the form $y = \mathrm{f}(x)$.

11 A curve has parametric equations

$$x = \sin^2 t, \quad y = \tan t, \quad -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}.$$

a Show that the tangent to the curve at the point where $t = \frac{\pi}{4}$ passes through the origin.

b Find a cartesian equation for the curve in the form $y^2 = \mathrm{f}(x)$.

12 A curve is given by the parametric equations

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t \neq 0.$$

a Find an equation for the tangent to the curve at the point P where $t = 3$.

b Show that the tangent to the curve at P does not meet the curve again.

c Show that the cartesian equation of the curve can be written in the form

$$x^2 - y^2 = k,$$

where k is a constant to be found.

1 Differentiate with respect to x

a $4y$ **b** y^3 **c** $\sin 2y$ **d** $3e^{y^2}$

2 Find $\frac{dy}{dx}$ in terms of x and y in each case.

a $x^2 + y^2 = 2$ **b** $2x - y + y^2 = 0$ **c** $y^4 = x^2 - 6x + 2$

d $x^2 + y^2 + 3x - 4y = 9$ **e** $x^2 - 2y^2 + x + 3y - 4 = 0$ **f** $\sin x + \cos y = 0$

g $2e^{3x} + e^{-2y} + 7 = 0$ **h** $\tan x + \operatorname{cosec} 2y = 1$ **i** $\ln (x - 2) = \ln (2y + 1)$

3 Differentiate with respect to x

a xy **b** x^2y^3 **c** $\sin x \tan y$ **d** $(x - 2y)^3$

4 Find $\frac{dy}{dx}$ in terms of x and y in each case.

a $x^2y = 2$ **b** $x^2 + 3xy - y^2 = 0$ **c** $4x^2 - 2xy + 3y^2 = 8$

d $\cos 2x \sec 3y + 1 = 0$ **e** $y = (x + y)^2$ **f** $xe^y - y = 5$

g $2xy^2 - x^3y = 0$ **h** $y^2 + x \ln y = 3$ **i** $x \sin y + x^2 \cos y = 1$

5 Find an equation for the tangent to each curve at the given point on the curve.

a $x^2 + y^2 - 3y - 2 = 0,$ $(2, 1)$ **b** $2x^2 - xy + y^2 = 28,$ $(3, 5)$

c $4 \sin y - \sec x = 0,$ $(\frac{\pi}{3}, \frac{\pi}{6})$ **d** $2 \tan x \cos y = 1,$ $(\frac{\pi}{4}, \frac{\pi}{3})$

6 A curve has the equation $x^2 + 2y^2 - x + 4y = 6$.

a Show that $\frac{dy}{dx} = \frac{1-2x}{4(y+1)}$.

b Find an equation for the normal to the curve at the point $(1, -3)$.

7 A curve has the equation $x^2 + 4xy - 3y^2 = 36$.

a Find an equation for the tangent to the curve at the point P (4, 2).

Given that the tangent to the curve at the point Q on the curve is parallel to the tangent at P,

b find the coordinates of Q.

8 A curve has the equation $y = a^x$, where a is a positive constant.

By first taking logarithms, find an expression for $\frac{dy}{dx}$ in terms of a and x.

9 Differentiate with respect to x

a 3^x **b** 6^{2x} **c** 5^{1-x} **d** 2^{x^3}

10 A biological culture is growing exponentially such that the number of bacteria present, N, at time t minutes is given by

$$N = 800(1.04)^t.$$

Find the rate at which the number of bacteria is increasing when there are 4000 bacteria present.

1 Given that $y = x^2 + 3x + 5$,

and that $x = (t - 4)^3$,

a find expressions for

i $\frac{dy}{dx}$ in terms of x, **ii** $\frac{dx}{dt}$ in terms of t,

b find the value of $\frac{dy}{dt}$ when

i $t = 5$, **ii** $x = 8$.

2 The variables x and y are related by the equation $y = x\sqrt{2x - 3}$.

Given that x is increasing at the rate of 0.3 units per second when $x = 6$, find the rate at which y is increasing at this instant.

3 The radius of a circle is increasing at a constant rate of 0.2 cm s^{-1}.

a Show that the perimeter of the circle is increasing at the rate of 0.4π cm s^{-1}.

b Find the rate at which the area of the circle is increasing when the radius is 10 cm.

c Find the radius of the circle when its area is increasing at the rate of 20 cm^2 s^{-1}.

4 The area of a circle is decreasing at a constant rate of 0.5 cm^2 s^{-1}.

a Find the rate at which the radius of the circle is decreasing when the radius is 8 cm.

b Find the rate at which the perimeter of the circle is decreasing when the radius is 8 cm.

5 The volume of a cube is increasing at a constant rate of 3.5 cm^3 s^{-1}. Find

a the rate at which the length of one side of the cube is increasing when the volume is 200 cm^3,

b the volume of the cube when the length of one side is increasing at the rate of 2 mm s^{-1}.

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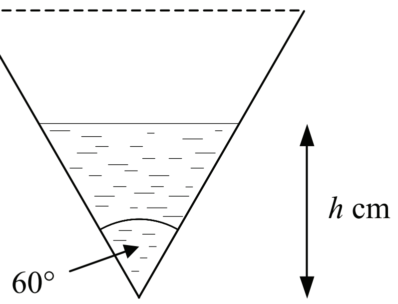

The diagram shows the cross-section of a right-circular paper cone being used as a filter funnel. The volume of liquid in the funnel is V cm^3 when the depth of the liquid is h cm.

Given that the angle between the sides of the funnel in the cross-section is 60° as shown,

a show that $V = \frac{1}{9}\pi h^3$.

Given also that at time t seconds after liquid is put in the funnel

$$V = 600e^{-0.0005t},$$

b show that after two minutes, the depth of liquid in the funnel is approximately 11.7 cm,

c find the rate at which the depth of liquid is decreasing after two minutes.